Dirichlet forms, Poincaré inequalities, and the Sobolev spaces of Korevaar-Schoen*

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Abstract

We answer a question of Jost on the validity of Poincaré inequalities for metric space-valued functions in a Dirichlet domain. We also investigate the relationship between Dirichlet functions and elements of the Sobolev-type space of functions introduced by Korevaar and Schoen.

1 Introduction

There has been an extensive study done of Sobolev type functions on metric spaces and harmonic analysis from the point of view of the underlying geometry of the metric space; see [C], [H], [HaK], [KMc], [HeK1], [HeK2], [HKM], [KS], [KKMa], [KiMa], [HKST], [Sh1], [Sh2], [KaSh], [KiSh], and the references therein. Concurrently a lot of literature on the study of Dirichlet forms from the point of view of probability theory has appeared; see [J], [St1], [St2], [St3], [M], [BM1], [BM2], [BB], [FOT], and the references therein. The paper [J] studies Dirichlet forms on general metric spaces and establishes regularity results for corresponding harmonic functions. While the approach of Dirichlet forms yields only classes corresponding to the Sobolev space $W^{1,2}(\mathbb{R}^n)$, the advantage of the approach of [J] to harmonic analysis is that it is applicable in an even more general setting than that studied in [C], [Sh1], [Sh2], and [KiSh]. In fact, there are fractal spaces without many rectifiable curves but supporting nontrivial Dirichlet forms satisfying a Poincaré inequality; see [BB]. In such spaces the theory developed in [J], [St1], [St2], and [St3] is more useful. The key inequality behind the theory developed in these papers is the Poincaré inequality. The aim of this paper is to answer a question posed in [J] regarding the validity of a Poincaré inequality for metric space-valued Dirichlet forms given that a corresponding Poincaré inequality holds for extended real-valued Dirichlet functions. We use a technique developed in [HKST] to answer this question. The main result of this paper is Theorem 3.4.

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While the axiomatic definition of Dirichlet forms given in [BD], [St1], and [FOT] deals only with real-valued functions on metric spaces ([BD] deals only with complex-valued functions on the complex plane, but the results there are generalizable), [J] considers functions between general metric spaces. We will consider the general definition of [J]. In the approach taken in [J], Jost fixes the domain and considers Dirichlet functions into all metric spaces simultaneously. For ease of presentation we will break this definition down and consider a Dirichlet domain to consist of Dirichlet functions between two metric spaces.

This paper is designed as follows. The second section recapitulates the definition of Dirichlet energy forms, see also [J]. Some claims made in this section about the corresponding bilinear Dirichlet form are proved in the appendix to this paper. We answer a question of Jost [J, p. 12] in the third section using a method developed in [HKST]. In the fourth section we study the connection between the Sobolev spaces of Korevaar and Schoen [KS] and the domains of Dirichlet forms.

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2 Definitions

We follow the approach of Jost in defining Dirichlet forms using the energy forms; see [J, Section 2]. Given a metric measure space X endowed with a metric d and a non-trivial Borel-regular measure μ, the space $L^2(X, \mu)$ is the collection of all extended real-valued functions f for which $\|f\|^2_{L^2} := \int_X |f|^2\, d\mu$ is finite. This space becomes a Hilbert space when endowed with the inner product $(f, g) := \int_X fg\, d\mu$.

We first define a Dirichlet domain of extended real-valued functions on X and then, given a second metric space Y, we use this definition to define a Dirichlet domain of functions from X to Y.

Definition 2.1. We say that a map $\mathcal{E} : L^2(X, \mu) \to \mathbb{R}^+ \cup \{\infty\}$ is a *Dirichlet energy form* if the following five conditions are satisfied:

1. *Quadratic contraction property* - If $\mathcal{E}(f) < \infty$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is an L-Lipschitz map then $\mathcal{E}(\varphi \circ f) \le L^2\mathcal{E}(f)$.

2. *Closedness* - If $\{u_n\}$ is a Cauchy sequence in $L^2(X)$ so that $\mathcal{E}(u_n) < \infty$ for all n and $\mathcal{E}(u_n - u_m) \to 0$ as $n, m \to \infty$, then $\mathcal{E}(u_n - u) \to 0$ as $n \to \infty$. Here u is the L^2-limit of the Cauchy sequence $\{u_n\}$.

3. *Density condition* - Denoting by $D(\mathcal{E})$ the collection of all functions f in $L^2(X, \mu)$ for which $\mathcal{E}(f) < \infty$, we have that $D(\mathcal{E})$ is dense in $L^2(X, \mu)$ in its norm and is dense in the space $\mathrm{Lip}_0(X)$ of Lipschitz functions with compact support equipped with the supremum norm.

4. *Minkowski inequality* - For every pair of functions u, v in $L^2(X)$,

$$\sqrt{\mathcal{E}(u + v)} \le \sqrt{\mathcal{E}(u)} + \sqrt{\mathcal{E}(v)}.$$

5. *Parallelogram rule* - For every pair of functions $u, v \in L^2(X)$,

$$\mathcal{E}(u+v) + \mathcal{E}(u-v) = 2\left(\mathcal{E}(u) + \mathcal{E}(v)\right).$$

The set $D(\mathcal{E})$ is said to be the *Dirichlet domain* corresponding to $\mathcal{E}$.

Because of the Minkowski inequality and the quadratic contraction property, $D(\mathcal{E})$ is a vector space, and indeed is a normed space when endowed with the norm

$$\|u\|_{\mathcal{E}} := \|u\|_{L^2(X)} + \sqrt{\mathcal{E}(u)}.$$

Conditions 2 and 4 together imply that if $\{u_n\}$ is a sequence satisfying the hypotheses of condition 2 and u is the corresponding L^2-limit of this sequence, then $\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}(u_n) < \infty$. Moreover, as a consequence of the closedness condition 2, $D(\mathcal{E})$ is a Banach space under this norm. See [J], [St1], [St2], [M], and [FOT] for further properties of Dirichlet forms.

Let now Y also denote a metric space (with metric d_Y), $y_0 \in Y$, and let $L^2(X; Y, y_0)$ be the collection of all maps $f : X \to Y$ for which the "norm" $\|f\|_{L^2} := \int_X d_Y(f(x), y_0)^2\, d\mu(x)$ is finite. Without loss of generality we can assume that Y is a Banach space, since every metric space can be isometrically embedded in a Banach space; see for example, [HKST, Section 1.2]. The advantage in assuming that Y is a Banach space lies in the fact that in this case $L^2(X; Y, y_0)$ is also a normed vector space.

Using the fact that $L^2(X)$ is a Hilbert space, Beurling and Deny prove in [BD] the existence of a unique measure-valued bilinear form associated with the Dirichlet form

$$\eta : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathcal{M}(X)$$

(where $\mathcal{M}(X)$ is the collection of all finite signed Radon measures on X) which satisfies

$$\frac{1}{4}\left(\mathcal{E}(u+v) - \mathcal{E}(u-v)\right) = \int_X d\eta(u,v).$$

The measure $\eta(u,u)$ associated with the Dirichlet function u plays the role of the point-wise derivative of u in the general setting.

Definition 2.2. A map $\mathcal{E} : \bigcup_{y_0\in Y} L^2(X; Y, y_0) \to \mathbb{R}^+ \cup \{\infty\}$ is said to be a *Dirichlet energy form* if there is a corresponding Dirichlet energy form $\mathcal{E}_{\mathbb{R}} : L^2(X,\mu) \to \mathbb{R}^+ \cup \{\infty\}$ so that the following conditions are satisfied:

1. Quadratic contraction property - whenever $\mathcal{E}(f) < \infty$ and $\varphi : Y \to \mathbb{R}$ is an L-Lipschitz map,

   $$\mathcal{E}_{\mathbb{R}}(\varphi \circ f) \le L^2 \mathcal{E}(f).$$

   Moreover, if $\mathcal{E}(f)$ is finite and $\psi : Y \to Y$ is an L-Lipschitz map, then

   $$\mathcal{E}(\psi \circ f) \le L^2 \mathcal{E}(f).$$

2. Closedness - if $\{u_n\}$ is a Cauchy sequence in $L^2(X;Y,y_0)$ converging to u so that $\mathcal{E}(u_n)<\infty$ and for every Lipschitz map $\varphi:Y\to\mathbb{R}$ the sequence $\{\varphi\circ u_n\}$ satisfies the hypotheses of condition 2 of Definition 2.1 for the functional $\mathcal{E}_{\mathbb{R}}$, then $\mathcal{E}(u)<\infty$.

3. If $\mathcal{E}_{\mathbb{R}}(\varphi\circ f)<\infty$ for every Lipschitz map $\varphi:Y\to\mathbb{R}$ then $\mathcal{E}(f)<\infty$.

4. Minkowski inequality - if u,v are two functions in $\bigcup_{y_0\in Y}L^2(X;Y,y_0)$, then

$$\sqrt{\mathcal{E}(u+v)}\le\sqrt{\mathcal{E}(u)}+\sqrt{\mathcal{E}(v)}.$$

5. Parallelogram rule - for every pair of functions u,v in $\bigcup_{y_0\in Y}L^2(X;Y,y_0)$,

$$\mathcal{E}(u+v)+\mathcal{E}(u-v)=2\left(\mathcal{E}(u)+\mathcal{E}(v)\right).$$

6. There is a symmetric bilinear Radon measure-valued form

$$\eta:D(\mathcal{E})\times D(\mathcal{E})\to\mathcal{M}(X)$$

(where $\mathcal{M}(X)$ is the collection of all finite signed Radon measures on X) which satisfies

$$\frac{1}{4}\left(\mathcal{E}(u+v)-\mathcal{E}(u-v)\right)=\int_X d\eta(u,v).$$

This bilinear form satisfies the following conditions. First, if $u,v\in D(\mathcal{E})$, then the support of the measure $\eta(u,v)$ is disjoint from any open set on which u or v is constant. Next, if ψ is an L-Lipschitz map from $\mathbb{R}$ to Y and f is a function in $L^2(X)$ such that $\mathcal{E}_{\mathbb{R}}(f)<\infty$, then

$$d\eta(\psi\circ f,\psi\circ f)\le L^2 d\eta_{\mathbb{R}}(f,f), \tag{2.3}$$

and moreover, if φ is an L-Lipschitz map from Y to $\mathbb{R}$ and $\mathcal{E}(u)<\infty$, then

$$d\eta_{\mathbb{R}}(\varphi\circ u,\varphi\circ u)\le L^2 d\eta(u,u). \tag{2.4}$$

Here $\eta_{\mathbb{R}}$ is the energy measure associated with the form $\mathcal{E}_{\mathbb{R}}$ and inequalities (2.3) and (2.4) are assumed to hold on the level of measures (this property is called *strong locality* in [J] and [M]).

The set $D(\mathcal{E}):=\{f\in\bigcup_{y_0\in Y}L^2(X;Y,y_0)\ :\ \mathcal{E}(f)<\infty\}$ is called the *Dirichlet domain* of $\mathcal{E}$.

Because of Condition 1, whenever k is a real number and $f\in D(\mathcal{E})$, we have $\mathcal{E}(kf)=k^2\mathcal{E}(f)$ and $\mathcal{E}(f+k)=\mathcal{E}(f)$; see [J]. Condition 6 cannot be proven for the metric space-valued Dirichlet functions using the method in [BD] since in general $L^2(X;Y,y_0)$ is not a Hilbert space. Therefore we assume this condition as part of the definition. In most examples, such measure-valued forms can be easily seen to exist by the very nature of the construction of the energy form $\mathcal{E}$. We leave it to the reader to verify that condition 6 implies conditions 4, 5, and the first part of condition 1. For the sake of transparency we include these conditions explicitly in the above definition.

In order to obtain a bilinear form from the energy form, [J] considers a polarization of the energy form

$$E(u,v) := \frac{1}{4}\left(\mathcal{E}(u+v) - \mathcal{E}(u-v)\right).$$

By conditions 4 and 5 such polarization is a bilinear form on $\bigcup_{y_0\in Y} L^2(X;Y,y_0)$; see the Appendix. While [J] does not explicitly state conditions 4 and 5 as part of the definition of the energy form, they are sufficient to yield a bilinear form via the above polarization. Furthermore, conditions 4 and 5 hold true if the polarization is bilinear.

The classical example of a Dirichlet form is given by the gradient operator: $D(\mathcal{E}) = W^{1,2}(\mathbb{R}^n)$, and $\mathcal{E}(f) = \int_{\mathbb{R}^n} |\nabla f|^2$ for functions f in $W^{1,2}(\mathbb{R}^n)$. In this case the associated measure-valued form is given by $d\eta(u,v)(x) := \nabla u(x)\cdot\nabla v(x)\,dx$. A generalization of the classical Sobolev space $W^{1,2}(\mathbb{R}^n)$ is the Newtonian space $N^{1,2}(X)$ for a metric measure space X; see [HKST], [C], or [Sh1].[1] Cheeger shows in [C, Theorem 4.38] that a form of Rademacher's theorem holds on metric measure spaces equipped with a doubling measure supporting a $(1,2)$-Poincaré inequality (see [HeK1] or [C] for the definition of Poincaré inequality used in this result). He in fact constructs a derivative operator:

$$d : N^{1,2}(X) \to L^2(X;\mathbb{R}^k)$$

for some positive integer k. Specifically, there is a measurable partition $X = \bigcup_\alpha X_\alpha \cup Z$, $\mu(Z) = 0$, and a collection of Lipschitz maps $F_\alpha : X \to \mathbb{R}^k$, so that whenever $f : X \to \mathbb{R}$ is a Lipschitz map the equation

$$\lim_{y\to x}\left|\frac{f(x) - f(y) - df(x)\cdot(F_\alpha(x) - F_\alpha(y))}{d(y,x)}\right| = 0 \tag{2.5}$$

holds for almost every $x \in X_\alpha$.

Lipschitz functions with bounded support are dense in $N^{1,2}(X)$; see [Sh1]. In [C, Theorem 4.47] it is shown that the operator d admits a unique extension to all functions in $N^{1,2}(X)$ so that whenever $\{f_n\}$ is a Cauchy sequence of Lipschitz functions in $N^{1,2}(X)$ converging to f then $|df_n - df| = |d(f_n - f)|$ converges to 0 in the L^2-norm. A scaled version of this derivative operator yields a Dirichlet type form similar to the classical case: $D(\mathcal{E}) = N^{1,2}(X)$ and, for functions $f \in N^{1,2}(X)$, we define

$$\mathcal{E}(f) = \int_X |df|^2\,d\mu. \tag{2.6}$$

The inner product structure $\langle\cdot,\cdot\rangle$ of $\mathbb{R}^k$ permits us to construct the corresponding bilinear form

$$E(u,v) = \int_X \langle du\,,\,dv\rangle\,d\mu. \tag{2.7}$$

[1] The paper [C] gives an alternate definition of a Sobolev type space on metric measure spaces, but it is shown in [Sh1] that this definition yields the same space as the Newtonian space studied in [Sh1] and [HKST].

It is easy to verify that the form defined by $\mathcal{E}(u) = E(u,u)$ does satisfy the conditions given in the definition 2.1 of a Dirichlet form. Thus the Newtonian space $N^{1,2}(X)$ is an example of a Dirichlet domain. Analogous definitions of Newtonian spaces exist for maps between two abstract metric measure spaces supporting a doubling measure and a Poincaré inequality, see [C] and [HKST]. Such Newtonian spaces also give rise to Dirichlet forms.

3 Independence of Poincaré inequalities on target spaces

In this section we use a technique developed in [HKST] to answer a question of Jost [J, p. 12] regarding Poincaré inequalities.

Let E denote a Dirichlet form on a metric measure space X equipped with a non-trivial Borel measure μ. The following definitions are based on [J].

Given a Dirichlet form $\mathcal{E}$ satisfying Definition 2.1 we say that a subspace Γ of $D(\mathcal{E}) \cap C_0^0(X)$ is a *μ-separating core* if Γ is dense in $C_0^0(X)$ with respect to the supremum norm, is dense in $D(\mathcal{E})$ with respect to the norm $\|f\|_1 := \mathcal{E}(f)^{1/2} + \|f\|_{L^2}$, and has the following separation property: for every $x, y \in X$ with $x \neq y$ there is a function $\varphi \in \Gamma$ so that $\varphi(x) \neq \varphi(y)$ and $d\eta_{\mathbb{R}}(\varphi,\varphi) \le d\mu$ on the level of measures, i.e., $\eta_{\mathbb{R}}(\varphi,\varphi)(E) \le \mu(E)$ for all Borel subsets E of X. Here $C_0^0(X)$ denotes the space of all continuous functions with compact support. See [J] and [St2] for more on separating cores.

For the remainder of the paper we assume that the Dirichlet form $\mathcal{E}$ has a μ-separating core Γ. As in [J, Section 4] and [BM1] we construct an induced metric d_E on X as follows:

$$d_E(x,y) = \sup\{\varphi(x) - \varphi(y) \ : \ \varphi \in \Gamma, d\eta_{\mathbb{R}}(\varphi,\varphi) \le d\mu\} \tag{3.1}$$

for $x, y \in X$. Also as in [J] and [BM1], we assume that the topology induced by the metric d_E coincides with the topology induced by the underlying metric of X and that the measure μ on X is a doubling measure for balls in the d_E-metric, that is, there exists a constant $C \ge 1$ so that for every $x \in X$ and every $r > 0$,

$$\mu(B_E(x,2r)) = \mu(2B_E(x,r)) \le C\mu(B_E(x,r)).$$

Here $B_E(y,R) := \{z \in X \ : \ d_E(y,z) < R\}$.

Definition 3.2. Let X be a metric measure space equipped with a metric d and measure μ, V be a Banach space, and $\mathcal{E}$ be a Dirichlet form acting on functions from X into V. We say that X supports a *weak* $(2,2)$-*Poincaré inequality for the form $\mathcal{E}$ with respect to V* (or merely, *weak Poincaré inequality with respect to V*), if there are constants $\lambda \ge 1$ and $C > 0$ so that for every function $f \in D(\mathcal{E})$ and every d_E-ball $B_E \subset X$ we have

$$\int_{B_E} \|f - f_{B_E}\|^2 \, d\mu \le C \operatorname{diam}_E(B_E)^2 \int_{\lambda B_E} d\eta(f,f), \tag{3.3}$$

where

$$f_{B_E} := \frac{1}{\mu(B_E)} \int_{B_E} f \, d\mu =: \fint_{B_E} f \, d\mu,$$

$\| \cdot \|$ is the norm on the Banach space V, and η is the energy density measure associated with the Dirichlet form; see condition 6 in Definition 2.2. If $V = \mathbb{R}$, then we say that X supports a *weak Poincaré inequality for* $\mathcal{E}$. The strong form of the inequality corresponds to the choice $\lambda = 1$.

We note here that for Borel sets A the mean value f_A satisfies the equation $\Lambda f_A = \frac{1}{\mu(A)} \int_A \Lambda f \, d\mu$ for every $\Lambda \in V^*$; see [HKST, Section 2] and the references therein for more on integration of Banach space-valued functions.

Let $\mathcal{E}$ be a Dirichlet form acting on L^2-maps from X to a collection $\mathcal{V}$ of non-trivial Banach spaces V so that there is a common Dirichlet form $\mathcal{E}_{\mathbb{R}}$ satisfying conditions 1 and 2 of Definition 2.2 for *each* Dirichlet domain corresponding to the Banach spaces V in $\mathcal{V}$.

Theorem 3.4. *Let X be a metric space equipped with a non-trivial Borel measure μ, and let $\mathcal{E}$ be a Dirichlet form on X as above. Then the following are equivalent:*

(1) X supports a weak Poincaré inequality for the form $\mathcal{E}_{\mathbb{R}}$.

(2) X supports a weak Poincaré inequality for $\mathcal{E}$ with respect to some Banach space V in $\mathcal{V}$.

(3) For every Banach space V in $\mathcal{V}$, X supports a weak Poincare inequality for $\mathcal{E}$ with respect to V.

We remind the reader that we assume the measure μ is a doubling measure with respect to the induced metric d_E.

Recall again that if Y is an arbitrary metric space, it can be isometrically embedded into some Banach space. Hence the above theorem yields a positive answer to the question posed by Jost in [J, p. 12]: the metric measure space X supports a weak Poincaré inequality for real-valued functions in $D(\mathcal{E})$ if and only if it supports a weak Poincaré inequality (as in inequality (3.3)) for every metric space target Y.

The proof of Theorem 3.4 follows closely the argument in [HKST, Theorem 4.3].

Proof. The implication (3) $\Rightarrow$ (2) is trivial and the implication (2) $\Rightarrow$ (1) follows from inequality (2.3) once we embed $\mathbb{R}$ isometrically into the (non-trivial) Banach space V. To complete the proof, we will show the implication (1) $\Rightarrow$ (3).

Let $V \in \mathcal{V}$, $f \in D(\mathcal{E}) \cap L^2(X; V, v_0)$, and let $\Lambda \in V^*$ with $\|\Lambda\|_{V^*} \leq 1$. Then $\Lambda : V \to \mathbb{R}$ is a $\|\Lambda\|_{V^*}$-Lipschitz map, and we have the strong locality inequality (2.4). By the quadratic contraction property, $\Lambda \circ f$ is in $D(\mathcal{E}_{\mathbb{R}})$. Hence

$$\fint_{B_E} |\Lambda \circ f - (\Lambda \circ f)_{B_E}|^2 \, d\mu \leq C \|\Lambda\|_{V^*}^2 \operatorname{diam}_E(B_E)^2 \frac{\eta(f,f)(\lambda B_E)}{\mu(\lambda B_E)}$$

for every $\Lambda \in V^*$ and every d_E-ball $B_E \subset X$. If x, y are Lebesgue points of $\Lambda \circ f$, then letting $B_0 = B(x, 2d_E(x,y))$ and $B_i = B(x, 2^{-i}d_E(x,y))$, $B_{-i} = B(y, 2^{-i}d_E(x,y))$ for $i \in \mathbb{N}$ (the balls in the d_E-metric), by the standard telescoping argument we see that

$$|\Lambda \circ f(x) - \Lambda \circ f(y)| \leq \sum_{i \in \mathbb{Z}} |(\Lambda \circ f)_{B_i} - (\Lambda \circ f)_{B_{i+1}}| \tag{3.5}$$

$$\leq C\|\Lambda\|_{V^*}^2 d_E(x,y) \left(M_{2\lambda d_E(x,y)} \eta(f,f)(x)^{1/2} + \right. \tag{3.6}$$

$$\left. + M_{2\lambda d_E(x,y)} \eta(f,f)(y)^{1/2} \right), \tag{3.7}$$

where

$$M_R \eta(f,f)(x) = \sup_{0<r<R} \frac{\eta(f,f)(B_E(x,r))}{\mu(B_E(x,r))}. \tag{3.8}$$

(Note that by assumption the d_E-ball $B_E(x,r)$ is an open subset of the metric space (X,d); since μ is a Borel measure, $B_E(x,r)$ is μ-measurable, and so the Lebesgue differentiation theorem holds.)

By Proposition 3.10 below, it now suffices to show that for μ-almost every x, y in X we have the inequality

$$\|f(x) - f(y)\| \leq C d_E(x,y) \left(M_{\lambda_1 d_E(x,y)} \eta(f,f)(x)^{1/2} + M_{\lambda_1 d_E(x,y)} \eta(f,f)(y)^{1/2} \right) \tag{3.9}$$

for some $\lambda_1 \geq 1$ independent of x, y and f. This can be done as in [HKST, Theorem 4.3]; the proof will be included here for the sake of completeness.

By inequality (3.5), for every $\Lambda \in V^*$ there is a set $Z_\Lambda \subset X$ of measure zero so that for every pair of points $x, y \in X \setminus Z_\Lambda$ inequality (3.5) holds for the function $\Lambda \circ f$. Since the measure on (X, d_E) is doubling, X is a separable space. Therefore there exists a set $Z_0 \subset X$ of measure zero so that $f(X \setminus Z_0)$ is separable (this is a consequence of Lusin's theorem together with the fact that continuous images of separable metric spaces are separable). Hence the difference set $\{f(x) - f(y) : x, y \in X \setminus Z_0\}$ is also separable, and has a countable dense subset $\{v_i\}_{i \in \mathbb{N}}$ of nonzero elements. By the Hahn-Banach theorem, for each integer i there exists $\Lambda_i \in V^*$ so that $\|\Lambda_i\|_{V^*} = 1$ and $\Lambda_i(v_i) = \|v_i\|$. We abbreviate $Z_i = Z_{\Lambda_i}$. Note that the measure of the set $Z_0 \cup \bigcup_i Z_i$ is zero, and for every pair x, y in the complement of this zero-measure set, inequality (3.5) holds for the functions $\Lambda_k \circ f$, $k \in \mathbb{N}$. Now for every pair x, y in $X \setminus (Z_0 \cup \bigcup_i Z_i)$ we can approximate $\|f(x) - f(y)\|$ by a sub-sequence $\{\Lambda_{i_k} \circ f(x) - \Lambda_{i_k} \circ f(y)\}$ to obtain the inequality (3.9). $\square$

The following proposition, used in the above proof, is similar to Proposition 4.6 in [HKST].

Proposition 3.10. *Let (X, d_E) be a metric space equipped with a doubling measure μ, let V be a Banach space, and let $f \in D(\mathcal{E}) \cap L^2(X;V)$. If for μ-almost every $x, y \in X$ the function f satisfies*

$$\|f(x) - f(y)\| \leq C d_E(x,y) \left(M_{\lambda_1 d_E(x,y)} \eta(f,f)(x)^{1/2} + M_{\lambda_1 d_E(x,y)} \eta(f,f)(y)^{1/2} \right) \tag{3.11}$$

then f satisfies the Poincaré inequality (3.3) with $\lambda = 2\lambda_1$.

Proof. By inequality (3.11), if $B_E \subset X$ is a d_E-ball, then

$$\begin{aligned}
\fint_{B_E} \|f(x) - f_{B_E}\| \, d\mu(x) &= \fint_{B_E} \left\| \fint_{B_E} f(x) - f(y) \, d\mu(y) \right\| d\mu(x) \\
&\leq \fint_{B_E} \fint_{B_E} \|f(x) - f(y)\| \, d\mu(y) \, d\mu(x) \\
&\leq C(\mathrm{diam}_E(B_E)) \fint_{B_E} M_{\lambda_1 \, \mathrm{diam}(B_E)} \eta(f,f)(z)^{1/2} \, d\mu(z).
\end{aligned}$$

An application of Lemma 3.15 yields the required result. ☐

Lemma 3.12. *Let η be a Borel measure on the metric measure space (X, d_E, μ) so that $0 < \eta(X) < \infty$. Then the restricted Hardy-Littlewood maximal function*

$$M_R\eta(x) = \sup_{0<r<R} \frac{\eta(B_E(x,r))}{\mu(B_E(x,r))} \tag{3.13}$$

satisfies the weak $(1,1)$-inequality: there exists a constant $C > 0$ so that

$$\mu(\{x \in B_E \ : \ M_R\eta(x) > \tau\}) \leq \frac{C}{\tau}\eta(2B_E) \tag{3.14}$$

for all $\tau > 0$ and all d_E-balls B_E of radius greater than or equal to R.

Using Lemma 3.12, we can prove the following lemma as in [HaK, 14.11].

Lemma 3.15. *Under the assumptions of Lemma 3.12,*

$$\fint_{B_E} M_R\eta(x)^{1/2} \, d\mu(x) \leq C\sqrt{\frac{\eta(2B_E)}{\mu(B_E)}}. \tag{3.16}$$

Proof of Lemma 3.12. The proof follows the classical proof of the Hardy-Littlewood theorem. Let

$$A = \{x \in B_E \ : \ M_R\eta(x) > \tau\}.$$

For each $x \in A$ there is a radius $r_x > 0$ so that $r_x \leq R$ and $\frac{\eta(B_E(x,r_x))}{\mu(B_E(x,r_x))} > \tau$. The collections of these d_E-balls $B_E(x, r_x)$, $x \in A$, covers the set A. By a general covering theorem (see [HaK, Theorem 14.12]), we can find a countable sub-cover $\{5B_i\}$ of A so that $\{B_i\}$ is a pairwise disjoint collection. By the doubling property of μ with respect to the metric d_E and by the fact that $B_E(x, r_x) \subset 2B_E$,

$$\mu(A) \leq \sum_i \mu(5B_i) \leq C \sum_i \mu(B_i) \leq \frac{C}{\tau} \sum_i \eta(B_i) \leq \frac{C}{\tau}\eta\left(\cup_i B_i\right) \leq \frac{C}{\tau}\eta(2B_E).$$

Thus Lemma 3.12 is proved. ☐

Proof of Lemma 3.15. Using the Cavalieri principle and Lemma 3.12, we have

$$\begin{aligned}\int_{B_E} M_R\eta(x)^{1/2}\, d\mu(x) &= \int_0^\infty \mu(\{x \in B_E \,:\, M_R\eta(x) > \tau^2\})\, d\tau \\ &\le \int_0^\beta \mu(B_E)\, d\tau + \int_\beta^\infty \frac{C\eta(2B_E)}{\tau^2}\, d\tau \\ &= \mu(B_E)\beta + \frac{C}{\beta}\eta(2B_E)\end{aligned}$$

for $\beta > 0$. Choosing $\beta = \sqrt{\frac{\eta(2B_E)}{\mu(B_E)}}$, we see from above that

$$\int_{B_E} M_R\eta(x)^{1/2}\, d\mu(x) \le C\sqrt{\mu(B_E)\eta(2B_E)},$$

and this yields the required inequality. $\square$

By means of Theorem 3.4 one can strengthen the principal results of the paper [J]. The assumption of a metric space-valued Poincaré inequality can be replaced with the corresponding real-valued inequality, which is in principle more easily verifiable. Recall that [J] proves the Harnack inequality and Hölder continuity of metric space-valued Dirichlet energy minimizers, under the additional assumption that the target space has non-positive curvature.

4 Dirichlet domains and the Sobolev spaces of Korevaar and Schoen

In this final section we will assume that the Dirichlet form $\mathcal{E}$ satisfies the following condition in addition to conditions 1 through 6 of Definition 2.2.

7. We assume that the μ-separating core Γ consists of all Lipschitz functions $\varphi : X \to \mathbb{R}$ with bounded support and that for all such Lipschitz functions φ,
$$C^2 \left(\operatorname{Lip}\varphi(x)\right)^2\, d\mu(x) \ge d\eta_{\mathbb{R}}(\varphi,\varphi) \ge C^{-2}\left(\operatorname{Lip}\varphi(x)\right)^2\, d\mu(x),$$
where
$$\operatorname{Lip}\varphi(x) := \lim_{r\to 0}\sup_{y\in B(x,r)} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)}$$
and the constant $C \ge 1$ is independent of x and φ.

We also assume henceforth that X is quasi-convex, that is, there is a constant $C_0 > 0$ so that for every pair of points x, y in X there is a rectifiable curve connecting x and y with length at most $Cd(x,y)$. For a relationship between Poincaré inequalities and quasi-convexity see [HaK, Proposition 4.4].

By condition 7 we can replace the d_E-balls in the Poincaré inequality with the underlying metric balls and the corresponding d_E-diameter of these balls by their regular diameter, and furthermore, the assumption in [J] and [BM1] regarding the topology induced by d_E is automatically satisfied (see the discussion in Section 3). This follows from the next lemma.

Lemma 4.1. *If the Dirichlet form $\mathcal{E}$ satisfies condition 7, then the metric d_E is bilipschitz equivalent to the underlying metric d.*

Proof. Let x and y be two points in X. Then we can construct a 1-Lipschitz map $\varphi : X \to \mathbb{R}$ so that $\varphi(x) = d(x,y)$, $\varphi(y) = 0$, and the support of φ lies in the closure of the ball $B(x, d(x,y))$. By the preceding assumptions, the modified function $C^{-1}\varphi$ is an admissible function for calculating $d_E(x,y)$, and therefore we see that $d_E(x,y) \geq C^{-1}d(x,y)$. Now suppose that $\varphi \in \Gamma$ and $d\eta(\varphi,\varphi) \leq d\mu$. Then φ is Lipschitz with $\operatorname{Lip}\varphi(x) \leq C$, and hence the global Lipschitz constant of φ is bounded above by a constant $L > 0$ depending only on C and the quasi-convexity constant C_0. Thus we have $|\varphi(x) - \varphi(y)| \leq Ld(x,y)$, and hence $d_E(x,y) \leq Ld(x,y)$. Thus the induced metric d_E is bilipschitz equivalent with the underlying metric d. $\square$

As stated in the second section, the Newtonian space $N^{1,2}(X)$ of [Sh1] is an example of a Dirichlet domain. Another example is the Sobolev type space of Korevaar and Schoen [KS]. In the paper [St3], Sturm proves that the Korevaar-Schoen space $KS^{1,2}(X;Y)$ is a Dirichlet domain corresponding to a local Dirichlet form. Given the discussion in [St3], it is natural to ask whether $KS^{1,2}(X;Y)$ is the only Dirichlet domain (up to isomorphism) of functions on a metric space. In this section we explore the relationship between the Sobolev type spaces $KS^{1,2}(X;Y)$ defined by Korevaar and Schoen [KS] and the domain $D(\mathcal{E})$ of a general Dirichlet form

$$\mathcal{E} : \bigcup_{y_0 \in Y} L^2(X;Y,y_0) \to \mathbb{R}^+ \cup \{\infty\}.$$

As a consequence of Proposition 4.2 an extension of Sturm's result follows: the Dirichlet domain constructed from the space $KS^{1,2}(X;Y,y_0)$ cannot be extended as a Dirichlet domain to a larger space.

Recall that the space $KS^{1,2}(X;Y,y_0)$ is defined to be the collection of all functions $f \in L^2(X;Y,y_0)$ for which the approximating energies

$$e_\epsilon^2(x;f) := \fint_{B(x,\epsilon)} \frac{d_Y(f(x),f(y))^2}{\epsilon^2}\, d\mu_X(y)$$

converge to a finite energy

$$E_{KS}(f) := \sup_{\text{balls } B} \limsup_{\epsilon \to 0} \int_B e_\epsilon^2(x;f)\, d\mu_X(x) < \infty.$$

We show that, under some additional conditions on the Dirichlet energy form $\mathcal{E}$, the corresponding Dirichlet domain $D(\mathcal{E})$ embeds into the Korevaar-Schoen space $KS^{1,2}(X;Y)$ continuously. Since Y can be isometrically embedded into a Banach space V, again without loss of generality we consider only the case $Y = V$.

Recall that condition 7 is assumed for the following two propositions and hence a measure is doubling with respect to one of the metrics d and d_E if, and only if, it is doubling with respect to the other metric. Moreover, the Poincaré inequality remains valid if we replace the d_E-metric balls B_E with d-metric balls B.

Proposition 4.2. *Consider a metric measure space X supporting a doubling measure and a Poincaré inequality for the Dirichlet energy form $\mathcal{E}_{\mathbb{R}}$ related to the Dirichlet form $\mathcal{E} : L^2(X;Y,y_0) \to \mathbb{R}^+ \cup \{\infty\}$. Then $D(\mathcal{E})$ embeds via a Banach space isomorphism into $KS^{1,2}(X;Y,y_0)$.*

We follow the argument of [KMc, Section 4] in proving the above proposition.

Proof. By Theorem 3.4 X supports a Poincaré inequality for $\mathcal{E}$ with respect to V. Let $f \in D(\mathcal{E})$ and $x, y \in X$ be two Lebesgue points for f. Note that by the doubling property of the measure on X such Lebesgue points are of full measure in X. By the usual telescoping argument,

$$\begin{aligned}\|f(x) - f(y)\| &\le \sum_{i\in\mathbb{Z}} \|f_{B_i} - f_{B_{i+1}}\| \le \sum_{i\in\mathbb{Z}} \fint_{B_i} \|f - f_{B_i}\|\, d\mu \\ &\le \sum_{i\in\mathbb{Z}} \left(\fint_{B_i} \|f - f_{B_i}\|^2\, d\mu \right)^{1/2}\end{aligned}$$

where $\|\cdot\|$ denotes the norm on the Banach space V and $B_0 = B(x, d(x,y))$. Let $B_i = 2^{-i}B_0$ and $B_{-i} = 2^{-i}B(y, d(x,y))$ for $i \in \mathbb{N}$. By the Poincaré inequality for $\mathcal{E}$,

$$\|f(x) - f(y)\| \le C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \left(\frac{\eta(f,f)(\lambda B_i)}{\mu(\lambda B_i)} \right)^{1/2}.$$

As in [HaK, Section 5] and [KMc, Theorem 4.1], we define

$$J_r\eta(f,f)(x) := \sum_{i=0}^{\infty} 2^{-i} r \left(\frac{\eta(f,f)(B(x, 2^{-i}\lambda r))}{\mu(B(x, 2^{-i}\lambda r))} \right)^{1/2}$$

as a generalization of the Riesz potential. Then we have

$$\|f(x) - f(y)\| \le C\left(J_r\eta(f,f)(x) + J_r\eta(f,f)(y)\right)$$

whenever $r \ge d(x,y)$. Hence

$$\|f(x) - f(y)\|^2 \le C\left(J_r\eta(f,f)(x)^2 + J_r\eta(f,f)(y)^2\right).$$

Fix $\epsilon > 0$ and let $y \in B(x, \epsilon)$. Choosing $r = 2\epsilon$ we have

$$\frac{\|f(x) - f(y)\|^2}{\epsilon^2} \le \frac{C}{\epsilon^2}\left(J_{2\epsilon}\eta(f,f)(x)^2 + J_{2\epsilon}\eta(f,f)(y)^2\right),$$

and therefore

$$e_\epsilon^2(x;f) \le \frac{C}{\epsilon^2}\left(J_{2\epsilon}\eta(f,f)(x)^2 + \fint_{B(x,\epsilon)} J_{2\epsilon}\eta(f,f)(y)^2\, d\mu_X(y) \right).$$

By an adaptation of the proof of [HaK, Theorem 5.3] we know that

$$\int_{B(x_0,r)} J_{2r}\eta(f,f)(z)^2\,d\mu_X(z) \le Cr^2\eta(f,f)(B(x_0,2\lambda r)) \tag{4.3}$$

for every $x_0 \in X$. Thus

$$e^2_\epsilon(x;f) \le \frac{C}{\epsilon^2}J_{2\epsilon}\eta(f,f)(x)^2 + C\frac{\eta(f,f)(B(x,2\lambda\epsilon))}{\mu(B(x,2\lambda\epsilon))}$$

and hence (using (4.3) again)

$$\begin{aligned}\int_{B(x_0,\epsilon)} e^2_\epsilon(x;f)\,d\mu_X(x) &\le C\eta(f,f)(B(x_0,2\lambda\epsilon))\\ &\qquad + C\int_{B(x_0,\epsilon)}\frac{\eta(f,f)(B(x,2\lambda\epsilon))}{\mu(B(x,2\lambda\epsilon))}\,d\mu_X(x).\end{aligned}$$

From this we can conclude that

$$\int_{B(x_0,\epsilon)} e^2_\epsilon(x;f)\,d\mu_X(x) \le C\eta(f,f)(B(x_0,3\lambda\epsilon)),$$

and now by the doubling property of the measure μ and by a standard covering argument,

$$\int_X e^2_\epsilon(x;f)\,d\mu_X(x) \le C\eta(f,f)(X) = C\mathcal{E}(f).$$

Thus $E_{KS}(f) \le C\mathcal{E}(f)$ and we have the required embedding. $\square$

The next result provides us with a converse to the above proposition under an additional condition on the form $\mathcal{E}$.

Definition 4.4. We say that a Dirichlet form $\mathcal{E} : D(\mathcal{E})\times D(\mathcal{E}) \to \mathbb{R}_+$ has the *strong quadratic contraction property* if all real-valued Lipschitz functions with bounded support are in the domain $D(\mathcal{E}_\mathbb{R})$ and

$$\mathcal{E}(f) \le \sup_\varphi \mathcal{E}_\mathbb{R}(\varphi\circ f) \tag{4.5}$$

where the supremum is taken over all 1-Lipschitz functions φ from Y into $\mathbb{R}$ (in fact, equality holds in (4.5) by condition 1 of the definition of Dirichlet forms).

Observe that inequality (4.5) strengthens condition 3 in Definition 2.2. The classical examples $W^{1,2}(\mathbb{R}^n)$, $H^{1,2}(X)$, $M^{1,2}(X)$ of Dirichlet domains as well as the fractal constructions of Barlow and Bass [BB] all satisfy the above property, and hence it is a reasonable assumption.

Proposition 4.6. *Let $\mathcal{E}$ be a Dirichlet form on maps from X to V, where X is a proper metric measure space endowed with a doubling measure. If $\mathcal{E}$ has the strong quadratic contraction property and moreover X supports a Poincaré inequality for the Korevaar-Schoen measure E_{KS}, then $KS^{1,2}(X;V,v_0)$ embeds in $D(\mathcal{E})$.*

Proof. Let $f \in KS^{1,2}(X; V, v_0)$ and $\varphi : V \to \mathbb{R}$ be a 1-Lipschitz function. Then $|\varphi \circ f(x) - \varphi \circ f(y)| \le |f(x) - f(y)|$, and hence $\varphi \circ f$ belongs to $KS^{1,2}(X; \mathbb{R}) = KS^{1,2}(X)$ with $E_{KS}(\varphi \circ f) \le E_{KS}(f)$.

Given $\epsilon > 0$ we can cover X by a countable number of balls $B_i = B(x_i, \epsilon)$ of uniformly bounded overlap (since the measure on X is doubling). Fix such an $\epsilon > 0$ and define a locally Lipschitz approximation f_ϵ as follows:

$$f_\epsilon(x) := \sum_{i \in \mathbb{N}} \varphi_i(x)\, (\varphi \circ f)_{B_i}$$

where the functions φ_i, $i \in \mathbb{N}$, constitute a C/ϵ-Lipschitz partition of unity subordinate to the cover $\{B_i\}$. By [KMc, Lemma 4.6] we have

$$\begin{aligned} |f_\epsilon(x) - f_\epsilon(y)| &\le C d(x,y) \fint_{B(x,2\epsilon)} \left(e_{5\epsilon}^2(z; \varphi \circ f)\right)^{1/2} d\mu(z) \\ &\le C d(x,y) \left(\fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z; \varphi \circ f)\, d\mu(z) \right)^{1/2}. \end{aligned}$$

Hence

$$\operatorname{Lip} f_\epsilon(x) \le C \left(\fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z; \varphi \circ f)\, d\mu(z) \right)^{1/2}.$$

Therefore by the assumption on the energy density $\eta_{\mathbb{R}}$ (condition 7), if B is a ball of radius ϵ, then

$$\begin{aligned} \int_B d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon)(x) &\le C \int_B \fint_{B(x,2\epsilon)} e_{5\epsilon}^2(z; \varphi \circ f)\, d\mu(z)\, d\mu(x) \\ &\le C \int_B e_{5\epsilon}^2(x; \varphi \circ f)\, d\mu(x). \end{aligned}$$

Let B_0 be a ball in X. Then by a standard covering argument the above inequality yields

$$\int_{B_0} d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon) \le C \int_{B_0} e_{5\epsilon}^2(x; f)\, d\mu(x)$$

where $C > 0$ is independent of B_0.

Now suppose f is a Lipschitz function with compact support - that is, outside a compact set, f takes on the vector value 0. By the Poincaré inequality $E_{KS}(f) = 0$ implies that f is constant and hence $\mathcal{E}(f) = 0$. Therefore we may assume without loss of generality that $E_{KS}(f) > 0$. Since $d\eta_{\mathbb{R}}(f_\epsilon + a, f_\epsilon + a) = d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon)$ for every constant $a \in \mathbb{R}$, without loss of generality we can assume that the 1-Lipschitz function φ has the property that $\varphi(0) = 0$. Then each f_ϵ also is Lipschitz and has compact support. Furthermore, we can choose the above ball B_0 sufficiently large so that f and all of the functions f_ϵ, $0<\epsilon<1$, have support inside B_0. Then by the strong locality of $\eta_{\mathbb{R}}$,

$$\int_{B_0} d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon) = \int_X d\eta_{\mathbb{R}}(f_\epsilon, f_\epsilon).$$

Since $\limsup_{\epsilon\to 0}\int_{B_0} e^2_{5\epsilon}(x;\varphi\circ f)\le E_{KS}(f)<\infty$ and $E_{KS}(f)>0$, for sufficiently small $\epsilon>0$ we have $\int_{B_0} e^2_{5\epsilon}(x;\varphi\circ f)\le 2E_{KS}(f)$. Hence $\mathcal{E}_{\mathbb{R}}(f_\epsilon)\le CE_{KS}(f)$ for sufficiently small $\epsilon>0$. Therefore $\{f_\epsilon\}$ is a bounded sequence of functions in $D(\mathcal{E}_{\mathbb{R}})$ when $D(\mathcal{E}_{\mathbb{R}})$ is equipped with the norm

$$\|u\|_{\mathcal{E}_{\mathbb{R}}} := \mathcal{E}_{\mathbb{R}}(u)^{1/2} + \|u\|_{L^2}, \qquad u\in D(\mathcal{E}_{\mathbb{R}}),$$

and $f_\epsilon\to\varphi\circ f$ almost everywhere by the argument in the proof of [HKST, Lemma 8.2]. Since $D(\mathcal{E}_{\mathbb{R}})$ is a Hilbert space under the norm $\|\cdot\|_{\mathcal{E}_{\mathbb{R}}}$, we can use Mazur's lemma to conclude that $\varphi\circ f\in D(\mathcal{E}_{\mathbb{R}})$ if f is Lipschitz with bounded support, and in this case we also see that $\mathcal{E}_{\mathbb{R}}(\varphi\circ f)\le CE_{KS}(f)$. Now by inequality (4.5), we see that f is indeed in $D(\mathcal{E})$ and that its Dirichlet energy satisfies $\mathcal{E}(f)\le\sup_\varphi \mathcal{E}_{\mathbb{R}}(\varphi\circ f)\le CE_{KS}(f)$ (if we do not have inequality (4.5), then by condition 3 of the definition of Dirichlet forms we can still conclude that $f\in D(\mathcal{E})$, but we do not have control over the Dirichlet energy of f in terms of $E_{KS}(f)$).

As X supports a Poincaré inequality for the Korevaar-Schoen energy measure E_{KS}, Lipschitz functions of bounded support are dense in $KS^{1,2}(X;Y)$ (see [KS] or [KMc]). Hence all functions $f\in KS^{1,2}(X;V,v_0)$ are in $D(\mathcal{E})$ with $\mathcal{E}(f)\le CE_{KS}(f)$, and thus we have the embedding stated in the proposition. $\square$

5 Appendix

In this appendix we provide a proof of the claim in Section 2 that the polarization of an energy form is bilinear. We also look at more natural conditions that would guarantee the validity of condition 6 in the definition of Dirichlet forms.

Proposition 5.1. *Let $\mathcal{E}: L^2(X,\mu)\to\mathbb{R}^+\cup\{\infty\}$ be a Dirichlet form. Then the polarization*

$$E(u,v) := \frac{1}{4}(\mathcal{E}(u+v)-\mathcal{E}(u-v)) \tag{5.2}$$

is a bilinear form on $L^2(X,\mu)$.

Proof. Recall that $\mathcal{E}$ satisfies

$$\mathcal{E}(u+v)+\mathcal{E}(u-v) = 2\left(\mathcal{E}(u)+\mathcal{E}(v)\right), \tag{5.3}$$

see condition 5 of Definition 2.1. Let u,v,w be three functions in $D(\mathcal{E})$. By repeated use of equation (5.3), we find that

$$\begin{aligned} E(u,v)+E(w,v) &= \frac{\mathcal{E}(u+v)+\mathcal{E}(w+v)}{4} - \frac{\mathcal{E}(u-v)+\mathcal{E}(w-v)}{4} \\ &= \frac{\mathcal{E}(u+w+2v)+\mathcal{E}(u-w)}{8} - \frac{\mathcal{E}(u+w-2v)+\mathcal{E}(u-w)}{8} \\ &= \frac{\mathcal{E}(u+w+2v)-\mathcal{E}(u+w-2v)}{8} = \frac{E(u+w,2v)}{2}. \end{aligned}$$

Note also that $4E(0,v) = \mathcal{E}(v) - \mathcal{E}(v) = 0$. Hence

$$E(f,v) = E(f,v) + E(0,v) = \frac{E(f+0,2v)}{2}$$

for $f \in D(\mathcal{E})$ and so

$$E(u,v) + E(w,v) = E(u+w,v). \tag{5.4}$$

If α is a non-negative integer, then $E(\alpha u,v) = \alpha E(u,v)$ by equation (5.4). If $\alpha = 1/q$ where q is a positive integer, then

$$E(u,v) = E(q\frac{1}{q}u,v) = qE(\alpha u,v),$$

and again we have $E(\alpha u,v) = \alpha E(u,v)$. Hence $E(\alpha u,v) = \alpha E(u,v)$ whenever $\alpha \in \mathbb{Q}^+$. Now if $\alpha \in \mathbb{R}^+ \setminus \mathbb{Q}^+$, we can choose a sequence $\{\alpha_i\} \subset \mathbb{Q}^+$ so that $\alpha_i \to \alpha$, and by using the above equation for each α_i we see that $E(\alpha_i u,v) = \alpha_i E(u,v) \to \alpha E(u,v)$. By the Minkowski inequality,

$$\sqrt{\mathcal{E}(\alpha u+v)} = \sqrt{\mathcal{E}(\alpha u - \alpha_i u + \alpha_i u + v)} \leq \sqrt{\mathcal{E}(\alpha u - \alpha_i u)} + \sqrt{\mathcal{E}(\alpha_i u + v)}.$$

Hence

$$|\sqrt{\mathcal{E}(\alpha u+v)} - \sqrt{\mathcal{E}(\alpha_i u+v)}| \leq \sqrt{\mathcal{E}(\alpha u - \alpha_i u)} \leq |\alpha - \alpha_i|\sqrt{\mathcal{E}(u)} \to 0$$

where the last inequality is due to the strong quadratic contraction property. Thus $\mathcal{E}(\alpha_i u+v) \to \mathcal{E}(\alpha u+v)$, and similarly $\mathcal{E}(\alpha_i u-v) \to \mathcal{E}(\alpha u-v)$. Hence $E(\alpha_i u,v) \to E(\alpha u,v)$, and by the above discussion for rational α we see that the equality

$$E(\alpha u,v) = \alpha E(u,v) \tag{5.5}$$

holds for all non-negative real numbers α.

Finally, note that $0 = E(0,v) = E(u-u,v) = E(u,v) + E(-u,v)$ by equation (5.4) and so

$$E(-u,v) = -E(u,v). \tag{5.6}$$

It is easy to see that

$$E(u,v) = E(v,u). \tag{5.7}$$

By equations (5.4), (5.5), (5.6), and (5.7) we see that E is indeed a symmetric bilinear form. $\square$

It is also easily verified that the bilinearity of the polarization implies conditions 4 and 5 of the definition of Dirichlet energy forms.

We consider condition 6 in the definition of Dirichlet forms. Note that in the above proof of bilinearity of the polarization, we only needed conditions 1 through 5 of the definition of Dirichlet forms.

If $\mathcal{E}$ is a Dirichlet form on real-valued functions, then to guarantee the existence of the corresponding measure-valued form η it suffices to know that for every non-negative continuous function φ with bounded support we have

$$E(\varphi u, u) - \frac{1}{2}E(u^2, \varphi) \geq 0. \tag{5.8}$$

Under this assumption, the Riesz representation theorem applied to the positive definite linear operator $T_u(\varphi) = E(\varphi u, u) - \frac{1}{2}E(u^2, \varphi)$ yields a finite Radon measure $\eta(u, u)$ so that $T_u(\varphi) = \int_X \varphi \, d\eta(u, u)$. By covering X with a countable collection of balls with bounded overlap and considering a partition of unity ψ_k, $k \in \mathbb{N}$, we see that $E(u, u) = \sum_{k\in\mathbb{N}}(E(\psi_k u, u) - \frac{1}{2}E(u^2, \psi_k))$ is given by $E(u, u) = \eta(u, u)(X)$. (Here we use the fact that $E(v, w) = 0$ whenever w is constant.) A corresponding polarization yields the measure-valued bilinear form. Such polarization is bilinear because the form E is. To verify this, it suffices to verify that for every compactly supported Lipschitz function φ

$$\int_X \varphi \, d\eta(u + v, w) = \int_X \varphi \, d\eta(u, w) + \int_X \varphi \, d\eta(v, w).$$

Using the equation $\int_X \varphi \, d\eta(f, f) = E(\varphi u, u) - \frac{1}{2}E(u^2, \varphi)$ and using the bilinearity of E we can obtain the above equality.

The Dirichlet forms obtained in [BB] as well as the Dirichlet forms obtained by using the Cheeger derivatives (see the discussion in Section 2) satisfy the condition (5.8). It is not clear what its analogue for general metric space-valued forms should be. One possible analogue would be to require that for all non-negative continuous functions with bounded support $E(\varphi u, u) - \frac{1}{2}E(|u|^2, \varphi) \geq 0$ where, if the range of u lies in a Banach space, $|u(x)|$ is the Banach space norm of the element $u(x)$.

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