CONFORMAL METRICS AND SIZE OF THE BOUNDARY

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ABSTRACT. We establish a lower bound for the Hausdorff dimension of the boundary associated with a conformal deformation of the Euclidean metric on the unit ball in $\mathbb{R}^n$. The deformations we consider are motivated by quasiconformal maps. Our abstract approach leads to new results on the boundary behavior of these maps. We prove a conjecture by Hanson on the compression of sets and obtain an improved version of the so-called wall theorem. We also establish a Riesz-Privalov theorem in higher dimensions.

1. INTRODUCTION

In the recent paper [3] by Bonk, Koskela and Rohde it was shown that a large part of Geometric Function Theory relies on only two properties of the derivative $|f'|$ of a conformal map f of the unit disc $\mathbb{B}^2$ in the complex plane. Namely, the estimate $|f'(z)| \approx |f'(w)|$ whenever the points $z, w \in \mathbb{B}^2$ have hyperbolic distance at most 1 and the fact that

$$\int_{f^{-1}(B(w,r))} |f'|^2 \, dm_2 \leq \pi r^2 \text{ for } w \in f(\mathbb{B}^2) \text{ and } r > 0.$$

(For notation used in the introduction see the body of the paper, in particular Sec. 2).

More generally, we can consider a continuous density $\rho\colon \mathbb{B}^n \to (0, \infty)$ and define a metric d_ρ and a measure μ_ρ by setting

$$d_\rho(x, y) = \inf_\gamma \int_\gamma \rho(z) \, |dz| \quad \text{for} \quad x, y \in \mathbb{B}^n,$$

where the infimum is taken over all rectifiable curves γ that join x to y in $\mathbb{B}^n$, and

$$\mu_\rho(E) = \int_E \rho^n \, dm_n \quad \text{for a Borel set } E \subseteq \mathbb{B}^n.$$

The following conditions generalize the two properties of the density $\rho = |f'|$, where f is a conformal map. The first one is the Harnack type inequality

$$\mathrm{HI}(A): \; 1/A \leq \frac{\rho(x)}{\rho(y)} \leq A \text{ whenever } x, y \in B(z, \tfrac{1}{2}(1 - |z|)) \text{ for some } z \in \mathbb{B}^n,$$

and the second one a volume growth condition for the open balls in d_ρ-metric

$$\mathrm{VG}(B): \qquad \mu_\rho(B_\rho(x, r)) \leq B r^n \quad \text{for all} \quad x \in \mathbb{B}^n, \; r > 0.$$

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For densities ρ which satisfy these conditions for some constants $A \geq 1$ and $B > 0$, in [3] a detailed study of the metric space $(\mathbb{B}^n, d_\rho)$ was performed and counterparts for many properties of conformal maps were established. In the case that ρ arises as the derivative of a conformal map f, the space $(\mathbb{B}^n, d_\rho)$ looks like the image $f(\mathbb{B}^2)$ equipped with the internal metric. Another class of densities satisfying $\mathrm{HI}(A)$ and $\mathrm{VG}(B)$ consists of appropriately averaged Jacobians a_f of quasiconformal maps $f\colon \mathbb{B}^n \to \mathbb{R}^n$. In this case A and B depend only on the dimension n and the dilatation K of f (cf. Sec. 6). The study of $(\mathbb{B}^n, d_\rho)$ is not simply a repetition of the theory of quasiconformal maps as the class of the admissible densities ρ satisfying $\mathrm{HI}(A)$ and $\mathrm{VG}(B)$ is strictly larger than the collection of densities arising as averaged Jacobians (cf. [2]).

An interesting object is the boundary $\partial_\rho \mathbb{B}^n$ which we have to add to $(\mathbb{B}^n, d_\rho)$ in order to make this metric space complete. As a set the boundary $\partial_\rho \mathbb{B}^n$ that can be identified with the set of those points $\zeta \in \partial \mathbb{B}^n$ for which the ray $[0, \zeta)$ is rectifiable in the metric d_ρ. If ρ comes from a conformal map f, then $\partial_\rho \mathbb{B}^2$ consists of the points $\zeta \in \partial \mathbb{B}^2$ for which the image of the ray $[0, \zeta)$ is rectifiable. So $\partial_\rho \mathbb{B}^2$ corresponds to the rectifiably accessible boundary points of $f(\mathbb{B}^2)$.

Related to expansion and compression behavior of a quasiconformal map $f\colon \mathbb{B}^n \to \mathbb{R}^n$ on the boundary $\partial \mathbb{B}^n$ is the quest to find upper and lower bounds for the Hausdorff dimension $\dim_\rho(E)$ (with respect to the metric d_ρ) of subsets $E \subseteq \partial_\rho \mathbb{B}^n$. The "expansion behavior" is relatively well understood. One of the results in [3] states that, even though $\partial_\rho \mathbb{B}^n$ can be infinite dimensional relative to d_ρ, an essential part E of $\partial_\rho \mathbb{B}^n$ is of Hausdorff dimension $\dim_\rho(E)$ at most n. For the "compression behavior" only the weak result $\dim_\rho(\partial_\rho \mathbb{B}^n) \geq C(n, B) > 0$ was shown in [3].

The main purpose of the present paper is to establish methods which can be used to understand compression behavior of quasiconformal maps and more generally of densities ρ satisfying $\mathrm{HI}(A)$ and $\mathrm{VG}(B)$.

Our first result shows that these densities behave like quasiconformal maps and establishes the natural lower bound for $\dim_\rho(\partial_\rho \mathbb{B}^n)$.

Theorem 1.1. *Suppose $n \geq 2$ and $\rho\colon \mathbb{B}^n \to (0, \infty)$ is continuous and satisfies* $\mathrm{HI}(A)$ *and* $\mathrm{VG}(B)$ *for some constants $A \geq 1$ and $B > 0$. Then* $\dim_\rho(\partial_\rho \mathbb{B}^n) \geq n - 1$.

In fact, even more is true; the bound on the dimension holds locally as well. To study this local behavior we associate with a point $x \in \mathbb{B}^n$ a corresponding part S_x of the boundary $\partial \mathbb{B}^n$. The set S_x is the radial projection of a Whitney type ball centered at x from the origin on $\partial \mathbb{B}^n$. For simplicity we formulate the local version only for quasiconformal mappings and use the $(n-1)$-dimensional Hausdorff content $\mathcal{H}^{n-1,\infty}$ to measure the size of a set.

Theorem 1.2. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then*

$$\mathcal{H}^{n-1,\infty}(f(S_x)) \geq c\,\mathrm{dist}(f(x), \partial\Omega)^{n-1} \quad \text{for} \quad x \in \mathbb{B}^n,$$

where $c = c(n, K) > 0$.

The conclusion of Theorem 1.2 was conjectured by Hanson in [4].

Related to Theorem 1.2 we also obtain a new proof for the "wall" conjecture (cf. [5]) that was originally verified by Väisälä [9]. Actually, we will prove a stronger result (cf. Thm. 6.5).

Theorem 1.3. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then*

$$\mathcal{H}^{n-1,\infty}\big(\partial\Omega \cap B(y, 2\,\mathrm{dist}(y, \partial\Omega))\big) \geq c\,\mathrm{dist}(y, \partial\Omega)^{n-1} \quad \text{for} \quad y \in \Omega,$$

where $c = c(n, K) > 0$.

Furthermore, our methods allow us to relate the growth of the integral means of ρ^{n-1} to the size of $\partial_\rho \mathbb{B}^n$. This is similar to the classical Riesz-Privalov theorem which says that if $f\colon \mathbb{B}^2 \to \Omega$ is a conformal map onto a Jordan region $\Omega \subseteq \mathbb{C}$, then the length of $\partial\Omega$ and the H^1-norm of f' are equal. We again content ourselves with restricting to the quasiconformal setting. To state the result we need a version of the concept of porosity: If $\Omega \subseteq \mathbb{R}^n$ is a region and $\lambda \geq 1$, then the λ-porous part $\partial_\lambda\Omega$ of the boundary of Ω is the set of all boundary points y which are the limit of a sequence of points in Ω whose distance to y does not exceed the distance to the boundary by more than the factor λ (cf. Def. 7.5).

Our result generalizes the Riesz-Privalov theorem for quasiconformal maps $f\colon \mathbb{B}^n \to \Omega$ and compares the Hausdorff $(n-1)$-measure of an appropriate porous part of $\partial\Omega$ with some "norm" $||a_f||_{n-1}$ coming from the integral means of a_f^{n-1} for the averaged Jacobian a_f.

Theorem 1.4. *Suppose $n \geq 3$ and $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then there exists $\lambda = \lambda(n, K) \geq 1$ and $c = c(n, K) \geq 1$ such that*

$$(1/c)||a_f||_{n-1}^{n-1} \leq \mathcal{H}^{n-1}(\partial_\lambda\Omega) \leq c||a_f||_{n-1}^{n-1}.$$

A similar statement is true in the planar case $n = 2$ when we replace $\partial_\lambda\Omega$ by $\partial\Omega$. Theorem 1.4 is a consequence of more general estimates for $n \geq 3$. Namely, $\mathcal{H}^{n-1}(\partial_\lambda\Omega)$ can be controlled from above by $||a_f||_{n-1}^{n-1}$ for any $\lambda \geq 1$. The converse estimate holds for the entire boundary $\partial\Omega$, but it may well happen that $\mathcal{H}^{n-1}(\partial\Omega) = \infty$ when $||a_f||_{n-1}^{n-1} < \infty$. On the other hand, we can allow for a nontrivial exceptional set. Indeed, if $n \geq 3$ and $M \subseteq \partial\mathbb{B}^n$ satisfies $\mathcal{H}^{n-2}(M) = 0$, then $||a_f||_{n-1}^{n-1} \leq C(K, n)\mathcal{H}^{n-1}(f(\partial\mathbb{B}^n \setminus M))$.

An outline of the content of this paper is as follows. In Sec. 2 we set up notation, and prove and cite some auxiliary results. In Sec. 3 we discuss Hausdorff measures and introduce an important concept to measure the size of a set in the boundary which is based on "shadowing" instead of covering a set. Sec. 4 is devoted to studying how a set can separate a set on the boundary from the origin. These considerations lead to Prop. 4.8

which is the main result of this section. Sec. 5 discusses modulus estimates. The principal result Prop. 5.5 is an upper inequality for the modulus of curve families in $\partial\mathbb{B}^n$ that we call the "Main Modulus Estimate". It is of prime importance for the results in this paper. An immediate consequence is Thm. 1.1, whose proof is given at the end of Sec. 5. In Sec. 6 we prove Thm. 6.3 and the Hanson conjecture Thm. 1.2 (cf. Cor. 6.4). Moreover, we prove Thm. 6.5 which implies Thm. 1.3. Finally, in Sec. 7 we look at analogs of the Riesz-Privalov Theorem for quasiconformal maps. In dimension 2 we obtain Thm. 7.3. In higher dimensions we have to utilize the concept of the λ-porous part of the boundary and we prove Thm. 7.6 and Thm. 7.8. An immediate consequence of these results is Thm. 1.4. We give an example (cf. 7.10) which shows that our results are optimal.

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2. Notation and auxiliary results

In general, we will denote by $C(A, B, \dots)$, $c_1(A, B, \dots)$, etc., positive constants that can be chosen to depend only on some parameters $A, B, \dots$. So $C = C(A, B, \dots)$ is short hand for the fact that C is a positive constant that depends only on $A, B, \dots$, while an inequality $a \leq c(A, B, \dots)$ for some quantity a means that it is bounded by a number depending only on the specified parameters. If we write $a \lesssim b$, $a \gtrsim b$, $a \approx b$ for some quantities a and b, then we mean that there exists a positive constant (the constant of "comparability") depending only on some specified parameters such that $a \leq cb$, $ca \geq b$, and $(1/c)a \leq b \leq ca$, respectively. If in these inequalities no parameters for c are specified, then it is understood that we can take a fixed numerical constant for c.

We use the notation $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. If M is any set, then $\#M \in \mathbb{N}_0 \cup \{\infty\}$ is the number of points in M. We consider the Euclidean space $\mathbb{R}^n$, and unless otherwise stated $n \geq 2$. We denote by $|x|$ the Euclidean norm of a vector $x \in \mathbb{R}^n$. So $|x - y|$ is the Euclidean distance of two points $x, y \in \mathbb{R}^n$. Moreover, we denote by $\operatorname{diam}(E)$ the diameter of a set $E \subseteq \mathbb{R}^n$, by $\operatorname{dist}(E, F)$ the distance of two sets $E, F \subseteq \mathbb{R}^n$ and by $B(x, r)$ and $\bar{B}(x, r)$ the open and closed Euclidean ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, respectively. The unit ball in $\mathbb{R}^n$ is $\mathbb{B}^n$. If $E \subseteq \mathbb{R}^n$ is a set, then ∂E and $\bar{E}$ denote the boundary and the closure of E, respectively. The spherical metric σ on the unit sphere $\partial\mathbb{B}^n$ is the metric induced by the restriction of the Euclidean Riemannian metric of $\mathbb{R}^n$ to $\partial\mathbb{B}^n$. The sets $\Sigma(\zeta, r) \subseteq \partial\mathbb{B}^n$ and $\bar{\Sigma}(\zeta, r) \subseteq \partial\mathbb{B}^n$ are the open and closed spherical balls with center $\zeta \in \partial\mathbb{B}^n$ and radius $r > 0$, respectively. If $x, y \in \mathbb{R}^n$, then $[x, y]$ will be the compact line segment with end points x and y. Similar notation will be used for open and half open line segments.

If $z \in \mathbb{B}^n$ we let

$$B_z = B(z, \tfrac{1}{2}(1-|z|)) \tag{2.1}$$

be the "Whitney" type ball in $\mathbb{B}^n$ centered at z. If $E \subseteq \mathbb{B}^n$ we denote by $S(E)$ the radial projection of E on $\partial\mathbb{B}^n$ from the origin. More precisely,

$$S(E) = \{\zeta \in \partial\mathbb{B}^n : [0,\zeta) \cap E \neq \emptyset\}.$$

The set $S(E)$ is the "shadow" of E on the boundary. Note that $S(E) = \partial\mathbb{B}^n$ if $0 \in E$. An important special case is the shadow of a Whitney type ball B_z and we introduce special notation for this situation. So for $z \in \mathbb{B}^n$ we let

$$S_z = S(B_z) = \{\zeta \in \partial\mathbb{B}^n : [0,\zeta) \cap B(z, \tfrac{1}{2}(1-|z|)) \neq \emptyset\}. \tag{2.2}$$

The set $S_z \subseteq \partial\mathbb{B}^n$ can be written as a spherical ball with radius comparable to $1-|z|$. To be specific, for $z \in \mathbb{B}^n$ we have

$$S_z = \partial\mathbb{B}^n = \overline{\Sigma}(\zeta,\pi) \text{ for arbitrary } \zeta \in \partial\mathbb{B}^n \quad \text{if} \quad 0 \leq |z| < \tfrac{1}{3},$$

and

$$S_z = \Sigma\left(\zeta, \arcsin\left(\tfrac{1-|z|}{2|z|}\right)\right) \quad \text{if} \quad \tfrac{1}{3} \leq |z| < 1,$$

where $\zeta = S(z)$ is the projection of z from 0 on $\partial\mathbb{B}^n$. If $B \subseteq \mathbb{R}^n$ is a ball of radius r and $\lambda > 0$, then λB is the ball with the same center as B and radius λr, closed or open according to which of these properties B has. If we want to use the same notation for spherical balls, we have to be careful, because the center of a spherical ball cannot uniquely be recovered from the ball if its radius is too large. If $\lambda \geq 1$ this ambiguity does not matter, and $\lambda\Sigma(a,r) = \Sigma(a,\lambda r)$ is well defined for such λ.

We will need the following elementary fact whose proof we leave to the reader.

Lemma 2.1. *If $x, y \in \mathbb{B}^n$ and $S_y \subseteq S_x$, then $[0,z] \cap B_x \neq \emptyset$ for every $z \in B_y$.*

A curve in $\Omega \subseteq \mathbb{R}^n$ is a continuous mapping $\gamma\colon I \to \Omega$, where $I \subseteq \mathbb{R}$ is an interval. We denote the length of γ by $\mathrm{length}(\gamma)$, and (abusing notation) the image set $\gamma(I)$ also by γ.

If $\rho\colon \mathbb{R}^n \to [0,\infty]$ is a nonnegative Borel measurable function (which we will often call densities), then the ρ-length of a locally rectifiable curve is defined as

$$\mathrm{length}_\rho(\gamma) = \int_\gamma \rho(z)\,|dz|,$$

where integration is with respect to Euclidean length.

Suppose in addition that $\rho\colon \mathbb{B}^n \to (0,\infty)$ is continuous. Then, as indicated in the introduction, we can define a metric d_ρ by setting for $x, y \in \mathbb{B}^n$

$$d_\rho(x,y) = \inf \mathrm{length}_\rho(\gamma),$$

where the infimum is taken over all rectifiable curves in $\mathbb{B}^n$ connecting x and y. It is easy to see that the topology on $\mathbb{B}^n$ induced by d_ρ agrees with the standard topology. Notions related to the metric d_ρ will have the usual notation used for the Euclidean metric with the additional subscript ρ. For example, $B_\rho(x,r)$ denotes the open ρ-ball centered at $x \in \mathbb{B}^n$ with radius $r > 0$.

Let

$$h(x) = \frac{2}{1-|x|^2} \quad \text{for} \quad x \in \mathbb{B}^n$$

be the density of the hyperbolic metric on $\mathbb{B}^n$ of constant negative sectional curvature -1. In accordance with our previous convention, d_h will be the hyperbolic metric and metric notation that refers to the hyperbolic metric will have the subscript h.

Recall that a boundary point x of a region $\Omega \subseteq \mathbb{R}^n$ is called accessible, if there exists a curve $\gamma\colon [a,b] \to \mathbb{R}^n$ such that $\gamma([a,b)) \subseteq \Omega$ and $\gamma(b) = x$.

Lemma 2.2. *Suppose that $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density. Then for all $a \in \mathbb{B}^n$ and $r > 0$, the set $\Omega = B_\rho(a,r) \subseteq \mathbb{B}^n$ is a region and every (Euclidean) boundary point of Ω which lies in $\mathbb{B}^n$ is accessible.*

Proof. The set Ω is open and every point in Ω can be connected with a by a path lying in Ω. Hence Ω is a region.

To prove the second part of the lemma, let $x \in \partial\Omega \cap \mathbb{B}^n$ be arbitrary. Note that $a \neq x$. We can choose $\varepsilon > 0$ small enough such that $a \notin \bar{B}(x,\varepsilon) \subseteq \mathbb{B}^n$ and $\frac{1}{2}\rho(x) \leq \rho(y) \leq 2\rho(x)$ for $y \in \bar{B}(x,\varepsilon)$.

Since x is a boundary point of $\Omega = B_\rho(a,r)$, by definition of the metric d_ρ there exist curves $\alpha_k\colon [0,s_k] \to \Omega$ for $k \in \mathbb{N}_0$ such that $\alpha_k(0) = a$, $|\alpha_k(s_k) - x| \leq 2^{-(k+1)}\varepsilon$ and $\operatorname{length}_\rho(\alpha_k) < r$ for $k \in \mathbb{N}_0$.

For $k \in \mathbb{N}_0$ we can choose points $0 < t_{k,0} < t_{k,1} < \ldots < t_{k,k} < s_k$ such that

$$|\alpha_k(t_{k,\nu}) - x| = 2^{-\nu}\varepsilon \text{ and } \alpha_k([t_{k,\nu}, s_k]) \subseteq \bar{B}(x, 2^{-\nu}\varepsilon) \tag{2.3}$$

for $\nu \in \{0, \ldots, k\}$.

By successively choosing subsequences of $\{\alpha_k\}$ and passing to a "diagonal" subsequence of $\{\alpha_k\}$ if necessary, we may in addition assume that

$$|\alpha_k(t_{k,\nu}) - \alpha_l(t_{l,\nu})| \leq 2^{-(\nu+3)}\varepsilon \quad \text{for} \quad \nu \in \mathbb{N}_0,\ k,l \geq \nu. \tag{2.4}$$

Since $\alpha_k(s_k) \in \bar{B}(x, 2^{-(k+1)}\varepsilon)$ and $|\alpha_k(t_{k,k}) - x| = 2^{-k}\varepsilon$ we have $\operatorname{length}(\alpha_k|[t_{k,k}, s_k]) \geq 2^{-(k+1)}\varepsilon$ and so $\operatorname{length}_\rho(\alpha_k|[t_{k,k}, s_k]) \geq 2^{-(k+2)}\varepsilon\rho(x)$. This implies

$$\begin{aligned} d_\rho(a, \alpha_k(t_{k,k})) &\leq \operatorname{length}_\rho(\alpha_k|[0, t_{k,k}]) < r - \operatorname{length}_\rho(\alpha_k|[t_{k,k}, s_k]) \\ &< r - 2^{-(k+2)}\varepsilon\rho(x). \end{aligned} \tag{2.5}$$

For $k \in \mathbb{N}_0$ let L_k be the line segment with end points $\alpha_k(t_{k,k})$ and $\alpha_{k+1}(t_{k+1,k})$. Then $L_k \subseteq \bar{B}(x, 2^{-k}\varepsilon)$ by (2.3), and by (2.4)

$$\operatorname{length}_\rho(L_k) \leq 2\rho(x) \cdot 2^{-(k+3)}\varepsilon = 2^{-(k+2)}\varepsilon\rho(x).$$

Together with (2.5) this implies $L_k \subseteq \Omega$.

Now let $\gamma\colon [0,1) \to \Omega$ be a curve whose image set is

$$\alpha_0([0, t_{0,0}]) \cup L_0 \cup \alpha_1([t_{1,0}, t_{1,1}]) \cup L_1 \cup \alpha_2([t_{2,1}, t_{2,2}]) \cup \ldots$$

Note that in this union the end point of a curve is the initial point of the curve following in the union. Moreover, $\gamma \subseteq \Omega$. Note $\gamma(0) = a$ and since

$$L_k \cup \alpha_k(t_{k,k-1}) \cup \ldots \subseteq \bar{B}(x, 2^{-k}\varepsilon) \quad \text{for} \quad k \in \mathbb{N}_0,$$

we have $\lim_{t\to 1} \gamma(t) = x$. This shows that x is an accessible boundary point of Ω. □

For the arguments in the following sections it is very important to fix once and for all some set $\mathcal{C}$ in $\mathbb{B}^n$ that is uniformly spread out with respect to the hyperbolic metric. In our notation we suppress the dependence of $\mathcal{C}$ on n. To obtain such a set, consider a Whitney cube decomposition $\mathcal{W}$ of $\mathbb{B}^n$. This means $\mathcal{W}$ is a countable collection of dyadic cubes with sides parallel to the coordinate planes which have pairwise disjoint interiors such that $\mathbb{B}^n = \bigcup_{Q\in\mathcal{W}} Q$ and

$$\operatorname{diam}(Q) \le \operatorname{dist}(Q, \partial\mathbb{B}^n) \le 4\operatorname{diam}(Q) \quad \text{for} \quad Q \in \mathcal{W}.$$

Now let $\mathcal{C}$ be the set of the centers x_Q of the cubes $Q \in \mathcal{W}$. Then the countable set $\mathcal{C}$ is a uniformly separated net in the hyperbolic metric, i.e.,

$$\begin{aligned} &\sup_{z\in\mathbb{B}^n} \operatorname{dist}_h(z, \mathcal{C}) \le c_1(n) \quad \text{for} \quad z \in \mathbb{B}^n, \text{ and} \\ &\qquad d_h(x, y) \ge c_2(n) > 0 \quad \text{for} \quad x, y \in \mathcal{C},\, x \ne y. \end{aligned} \tag{2.6}$$

Moreover,

$$1 \le \sum_{z\in\mathcal{C}} \chi_{B_z} \le \sum_{z\in\mathcal{C}} \chi_{\lambda B_z} \le c_3(n, \lambda) \quad \text{for} \quad 0 < \lambda < 2, \tag{2.7}$$

and

$$1 \le \sum_{z\in\mathcal{C},\, B_z\cap\partial B(0,t)\ne\emptyset} \chi_{S_z} \le c_4(n) \quad \text{for} \quad 0 \le t < 1. \tag{2.8}$$

Here χ_E denotes the characteristic function of a set. Property (2.7) says that the collection $\{B_z : z \in \mathcal{C}\}$ covers $\mathbb{B}^n$. Moreover, since for $0 < \lambda < 2$ we have that the hyperbolic diameter of λB_z for $z \in \mathbb{B}^n$ is uniformly bounded by a constant depending only on λ, the collection $\{\lambda B_z : z \in \mathcal{C}\}$ has bounded overlap. Property (2.8) says that if we project the balls $\{B_z : z \in \mathcal{C}\}$ which meet a fixed sphere $\partial B(0, t)$ on $\partial\mathbb{B}^n$, then the collection of spherical balls thus obtained covers $\partial\mathbb{B}^n$ with bounded overlap.

With a density $\rho\colon \mathbb{B}^n \to (0, \infty)$ we can associate a measure μ_ρ defined on Borel sets $E \subseteq \mathbb{B}^n$ by

$$\mu_\rho(E) = \int_E \rho^n \, dm_n,$$

where m_n denotes Lebesgue measure on $\mathbb{R}^n$. The spherical measure on any sphere $\partial B(a, r) \subseteq \mathbb{R}^n$ is denoted by σ_{n-1}.

The continuous densities $\rho\colon \mathbb{B}^n \to (0, \infty)$ that we consider in this paper satisfy in addition the conditions HI(A) and VG(B) (for some constants $A \ge 1$ and $B > 0$) stated in the introduction.

For these densities we will recall several results from [3]. The first one is the Gehring-Hayman Theorem [3, Thm. 3.1].

Theorem 2.3. (Gehring-Hayman Theorem) *Suppose* $\rho\colon \mathbb{B}^n \to (0,\infty)$ *is a continuous density on* $\mathbb{B}^n$ *satisfying* HI(A) *and* VG(B)*. Then there is a constant* $C = C(A,B,n) > 0$ *with the following property. If* α *is a hyperbolic geodesic in* $\mathbb{B}^n$ *with end points in* $\bar{\mathbb{B}}^n$ *and* γ *is any other curve in* $\mathbb{B}^n$ *with the same end points, then*

$$\operatorname{length}_\rho(\alpha) \le C \operatorname{length}_\rho(\gamma).$$

Recall that hyperbolic geodesics are subarcs of circles or lines perpendicular to $\partial\mathbb{B}^n$.

It is very convenient to have special notation for the ρ-size of the Whitney type ball B_x. If ρ is fixed we let

$$r_x = \rho(x)(1-|x|) \approx \operatorname{diam}_\rho(B_x) \quad \text{for} \quad x \in \mathbb{B}^n. \tag{2.9}$$

In [3] it was shown that the completion of the space $(\mathbb{B}^n, d_\rho)$ can be obtained by adding a boundary $\partial_\rho\mathbb{B}^n$ to $\mathbb{B}^n$. The ρ-boundary $\partial_\rho\mathbb{B}^n$ can be identified with the subset of $\partial\mathbb{B}^n$ consisting of all $\zeta \in \partial\mathbb{B}^n$ with $\operatorname{length}_\rho([0,\zeta]) < \infty$. The inequality $\operatorname{length}_\rho([0,\zeta]) < \infty$ is true for all points outside an exceptional set E of vanishing n-capacity. In particular, the Hausdorff dimension of E is zero. The metric d_ρ extends to $\mathbb{B}^n \cup \partial_\rho\mathbb{B}^n$. We will use the notation $B'_\rho(a,r)$ for the open ball with center $a \in \mathbb{B}^n \cup \partial_\rho\mathbb{B}^n$ and radius $r > 0$ as a subset of $\mathbb{B}^n \cup \partial_\rho\mathbb{B}^n$.

3. Hausdorff measures

For $M \subseteq \mathbb{R}^n$, $\alpha \in (0,\infty)$, and $\delta \in (0,\infty]$ let

$$\mathcal{H}^{\alpha,\delta}(M) = \inf\Big\{\sum_{k\in\mathbb{N}} r_k^\alpha : M \subseteq \bigcup_{k\in\mathbb{N}} B(x_k,r_k) \wedge \forall k \in \mathbb{N} : r_k \le \delta\Big\}. \tag{3.1}$$

Here and in the following we use the convention $\inf\emptyset = +\infty$. Note that $\mathcal{H}^{\alpha,\delta}(M)$ is nonincreasing in δ. In particular, we can define the α-Hausdorff measure $\mathcal{H}^\alpha(M) \in [0,\infty]$ by

$$\mathcal{H}^\alpha(M) := \lim_{\delta\to 0} \mathcal{H}^{\alpha,\delta}(M). \tag{3.2}$$

The number $\mathcal{H}^{\alpha,\infty}(M)$ is called the α-Hausdorff content of M. For $\delta \in (0,\infty]$ we have

$$\mathcal{H}^{\alpha,\infty}(M) \le \mathcal{H}^{\alpha,\delta}(M) \le \mathcal{H}^\alpha(M). \tag{3.3}$$

The Hausdorff dimension of M is defined by

$$\dim(M) = \inf\{\alpha \in (0,\infty) : \mathcal{H}^\alpha(M) = 0\}. \tag{3.4}$$

Suppose $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density on $\mathbb{B}^n$ satisfying HI(A) and VG(B). To measure the size of a set $M \subseteq \mathbb{B}^n \cup \partial_\rho\mathbb{B}^n$ we define $\mathcal{H}_\rho^{\alpha,\delta}(M)$ similar as in (3.1) by using ρ-balls instead of Euclidean balls. We define $\mathcal{H}_\rho^\alpha(M)$ as in (3.2) and $\dim_\rho(M)$ as in (3.4). Note that we get the following inequality which corresponds to (3.3)

$$\mathcal{H}_\rho^{\alpha,\infty}(M) \le \mathcal{H}_\rho^{\alpha,\delta}(M) \le \mathcal{H}_\rho^\alpha(M) \quad \text{for} \quad \delta \in (0,\infty].$$

This inequality shows that if we want to estimate the size of M from above, then we get the strongest statements if we use the ρ-Hausdorff measure $\mathcal{H}^{\alpha}_{\rho}(M)$. Similarly, we get the strongest statements for lower estimations if we use the ρ-Hausdorff content $\mathcal{H}^{\alpha,\infty}_{\rho}(M)$. It turns out that, if ρ comes from a quasiconformal mapping $f\colon \mathbb{B}^n \to \mathbb{R}^n$ (i.e., $\rho = a_f$, cf. Sec. 6), then the ρ-Hausdorff content is in general too large to obtain lower estimates on the Hausdorff content of image sets. Therefore, we will introduce a content for a set which is quantitatively smaller than the ρ-Hausdorff content. For its definition, recall the set $\mathcal{C}$ that we fixed in Sec. 2 and the notation $r_x = \rho(x)(1-|x|)$ for our fixed density ρ. Now let $M \subseteq \partial_\rho \mathbb{B}^n$, $\alpha \in (0,\infty)$, and $\delta \in (0,\infty]$. Define

$$\Phi^{\alpha,\delta}_{\rho}(M) = \inf\left\{ \sum_{x\in\mathcal{E}} r_x^{\alpha} : \mathcal{E} \subseteq \mathcal{C} \wedge M \subseteq \bigcup_{x\in\mathcal{E}} S_x \wedge \forall x \in \mathcal{E} : r_x \le \delta \right\}.$$

In contrast to Hausdorff contents where covers of the set were considered, in the definition of $\Phi^{\alpha,\delta}_{\rho}$ we consider families of Whitney type balls whose shadows cover M. We restrict ourselves to sets $M \subseteq \partial_\rho \mathbb{B}^n$ here, because $\Phi^{\alpha,\delta}_{\rho}$ will be useful only for estimating the size of sets in the boundary.

Proposition 3.1. *Suppose $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density satisfying* HI(A) *and* VG(B). *If $\alpha \in (0,\infty)$ and $\delta \in [0,\infty]$, then there exist constants $\lambda = \lambda(A,B,n) > 0$ and $C = C(A,B,n,\alpha) > 0$ such that*

$$\Phi^{\alpha,\lambda\delta}_{\rho}(M) \le C\mathcal{H}^{\alpha,\delta}_{\rho}(M) \quad \textit{for} \quad M \subseteq \partial_\rho \mathbb{B}^n.$$

The proof will easily follow from the following lemma where we make the assumptions of Prop. 3.1.

Lemma 3.2. *There exist constants $l_0 = l_0(n) \in \mathbb{N}$ and $C = C(A,B,n) > 0$ with the following property. If $a \in \mathbb{B}^n \cup \partial_\rho \mathbb{B}^n$, and $r > 0$, then there exists $x_1, \dots, x_l \in \mathcal{C}$ with $l \le l_0$ and such that $\partial_\rho \mathbb{B}^n \cap B'_\rho(a,r) \subseteq \bigcup_{\nu=1}^{l} S_{x_\nu}$ and $r_{x_\nu} \le Cr$ for $\nu \in \{1,\dots,l\}$.*

Proof. The constants of comparability in this proof will depend on A, B, and n.

Let $M := \partial_\rho \mathbb{B}^n \cap B'_\rho(a,r) \subseteq \partial\mathbb{B}^n$. There is nothing to prove if $M = \emptyset$. If M consists of a single point, ζ say, then $\mathrm{length}_\rho([0,\zeta)) < \infty$. This implies that r_x tends to zero for points $x \in \mathbb{B}^n$ for which $B_x \cap [0,\zeta) \ne \emptyset$ as x tends to ζ. So if $x \in \mathcal{C}$ with $B_x \cap [0,\zeta) \ne \emptyset$ is sufficiently close to ζ, then $\{\zeta\} = M \subseteq S_x$ and r_x is arbitrarily small.

So we may assume, that M contains at least two points. Hence $\mathrm{diam}(M) > 0$ and we can choose $\zeta_1, \zeta_2 \in M$ with $|\zeta_1 - \zeta_2| \ge \frac{1}{2}\mathrm{diam}(M)$. Moreover, there exists $z \in \mathbb{B}^n$ such that $M \subseteq S_z$ and $1-|z| \approx \mathrm{diam}(M)$. For some $l_0 = l_0(n) \in \mathbb{N}$, the set B_z can be covered by balls $B_{x_1}, \dots, B_{x_l}$, where $l \le l_0$ and $x_1, \dots, x_l \in \mathcal{C}$, and we can in addition assume that each of these balls B_{x_ν} meets B_z. Thus by HI(A)

$$r_{x_\nu} \approx r_z \quad \text{for} \quad \nu \in \{1,\dots,l\}, \tag{3.5}$$

and

$$M \subseteq S_z = S(B_z) \subseteq \bigcup_{\nu=1}^{l} S_{x_\nu}. \tag{3.6}$$

Let γ be the hyperbolic geodesic in $\mathbb{B}^n$ joining ζ_1 and ζ_2. By the Gehring-Hayman theorem this is essentially the curve of shortest ρ-length in $\mathbb{B}^n$ with end points ζ_1 and ζ_2. Note that even though $\zeta_1, \zeta_2 \in \partial_\rho \mathbb{B}^n$, the distance $d_\rho(\zeta_1, \zeta_2)$ is still equal to infimum over $\operatorname{length}_\rho(\alpha)$ for curves α in $\mathbb{B}^n$ connecting ζ_1 and ζ_2. Hence $d_\rho(\zeta_1, \zeta_2) \approx \operatorname{length}_\rho(\gamma)$. On the other hand, since $|\zeta_1 - \zeta_2| \geq \frac{1}{2}\operatorname{diam}(M) \approx 1 - |z|$ and $M \subseteq S_z$, the hyperbolic geodesic γ has a subcurve γ' with $\operatorname{length}(\gamma') \gtrsim 1 - |z|$ and $\sup_{y \in \gamma'} d_h(y, z) \lesssim 1$. It follows that

$$r_z \lesssim \operatorname{length}_\rho(\gamma') \leq \operatorname{length}_\rho(\gamma) \approx d_\rho(\zeta_1, \zeta_2) \leq 2r.$$

The lemma follows from this, (3.5) and (3.6). □

Proof of Prop. 3.1. Suppose $M \subseteq \partial_\rho \mathbb{B}^n$ and $\bigcup_{k \in \mathbb{N}} B'_\rho(x_k, r_k)$ is a cover of M with $r_k \leq \delta$ for $k \in \mathbb{N}$. For each ball $B'_\rho(x_k, r_k)$ choose points $x_{k,1}, \dots, x_{k,l_k} \in \mathcal{C}$ according to Lemma 3.2. Here $l_k \leq l_0(n)$. For $\lambda = \lambda(A, B, n) > 0$ we have that

$$r_{x_{k,\nu}} \leq \lambda r_k \leq \lambda\delta \quad \text{for} \quad k \in \mathbb{N},\, \nu \in \{1, \dots, l_k\}.$$

Moreover,

$$M \subseteq \partial_\rho \mathbb{B}^n \cap \bigcup_{k \in \mathbb{N}} B_\rho(x_k, r_k) \subseteq \bigcup_{k \in \mathbb{N}} \bigcup_{\nu=1}^{l_k} S_{x_{k,\nu}}.$$

Therefore,

$$\Phi_\rho^{\alpha, \lambda\delta}(M) \leq \sum_{k \in \mathbb{N}} \sum_{\nu=1}^{l_k} r_{x_{k,\nu}}^\alpha \leq l_0 \lambda^\alpha \sum_{k \in \mathbb{N}} r_k^\alpha.$$

Passing to the infimum over all covers of M in the last sum the claim follows with $C = C(A, B, n, \alpha) = l_0 \lambda^\alpha$. □

Define for $M \subseteq \partial_\rho \mathbb{B}^n$, $\alpha \in (0, \infty)$, and $\delta \in (0, \infty]$

$$\tilde{\Phi}_\rho^{\alpha,\delta}(M) = \inf\Big\{ \sum_{x \in \mathcal{E}} r_x^\alpha : \mathcal{E} \subseteq \mathcal{C} \wedge M \subseteq \bigcup_{x \in \mathcal{E}} S_x \wedge \forall x \in \mathcal{E} : 1 - |x| \leq \delta \Big\}.$$

Lemma 3.3. *Suppose $\rho \colon \mathbb{B}^n \to (0, \infty)$ is a continuous density satisfying* HI(A) *and* VG(B). *Then there exists constants $C = C(A, B, n) > 0$ and $\beta = \beta(B, n) > 0$ with the following property. If $\alpha \in (0, \infty)$ and $\delta \in (0, \infty]$, then*

$$\tilde{\Phi}_\rho^{\alpha,\delta}(M) \leq \Phi_\rho^{\alpha,\delta'}(M) \quad \textit{for} \quad M \subseteq \partial_\rho \mathbb{B}^n,$$

where $\delta' = C\rho(0)\delta^\beta$.

Proof. By [3, Thm. 5.1] we have

$$1 - |x| \leq \left(\frac{r_x}{C\rho(0)} \right)^{1/\beta} \quad \text{for} \quad x \in \mathbb{B}^n,$$

with $\beta = \beta(B, n) > 0$ and $C = C(A, B, n) > 0$. So if $x \in \mathcal{C}$ and $r_x \leq C\rho(0)\delta^\beta$, then $1 - |x| \leq \delta$. The claim follows. □

4. Shadows and separation

Throughout this section we make the standing assumption that $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density that satisfies HI(A) and VG(B) for some constants $A \geq 1$ and $B > 0$. All constants of comparability will depend only on A, B and n.

Lemma 4.1. *Suppose $\lambda > 0$, $\mathcal{E} \subseteq \mathcal{C}$, and $\sup_{x\in\mathcal{E}} r_x < \infty$. Then the family $\{B_\rho(x,\lambda r_x) : x \in \mathcal{E}\}$ is locally finite in $\mathbb{B}^n$, i.e., for every $z \in \mathbb{B}^n$ there exists a neighborhood U of z such that $U \cap B_\rho(x,\lambda r_x) \neq \emptyset$ for only finitely many $x \in \mathcal{E}$. Moreover, if $\Omega = \bigcup_{x\in\mathcal{E}} B_\rho(x,\lambda r_x)$, then every boundary point of Ω which lies in $\mathbb{B}^n$ is accessible.*

Proof. There exists $c_1 = c_1(n,A) > 0$ and $c_2 = c_2(n,A) > 0$ such that the sets $B_\rho(x,c_1 r_x)$, $x \in \mathcal{C}$, are pairwise disjoint, and $\mu_\rho(B_\rho(x,c_1 r_x)) \geq c_2 r_x^n$.

To show the local finiteness of our family at a point $z \in \mathbb{B}^n$, note that there are only finitely many members $x \in \mathcal{E}$ with $B_x \cap B_z \neq \emptyset$. So it is enough to show that the set $\mathcal{E}_z$ of all $x \in \mathcal{E}$ such that $B_x \cap B_z = \emptyset$ and $B_\rho(x,\lambda r_x) \cap U \neq \emptyset$ is finite, where $U = \frac{1}{2}B_z$.

Now if $x \in \mathcal{E}_z$, then $B_\rho(x,c_1 r_x) \subseteq B_\rho(z,R)$, where

$$R := \operatorname{diam}_\rho(\tfrac{1}{2}B_z) + (\lambda + c_1)\sup_{x\in\mathcal{E}} r_x < \infty.$$

On the other hand, for $x \in \mathcal{E}_z$ we have $\lambda r_x \geq \operatorname{dist}_\rho(\partial B_z, \frac{1}{2}B_z) \geq \frac{1}{4A} r_z$. From VG($B$) we now see

$$\begin{aligned}(\#\mathcal{E}_z)\frac{c_2}{(4A\lambda)^n} r_z^n &\leq c_2 \sum_{x\in\mathcal{E}_z} r_x^n \leq \mu_\rho\Bigg(\bigcup_{x\in\mathcal{E}_z} B_\rho(x,c_1 r_x)\Bigg) \\ &\leq \mu_\rho(B_\rho(z,R)) \leq BR^n.\end{aligned}$$

This gives an upper bound for $\#\mathcal{E}_z$ as desired.

The first part of the lemma implies $\partial\Omega \cap \mathbb{B}^n \subseteq \left(\bigcup_{x\in\mathcal{E}} \partial B_\rho(x,\lambda r_x)\right) \cap \mathbb{B}^n$. The second claim now follows from Lem. 2.2 $\square$

For a set $\Omega \subseteq \mathbb{B}^n$ denote by $R(\Omega)$ the set of all $\zeta \in \partial\mathbb{B}^n$ for which Ω separates a tail of $[0,\zeta)$, i.e., there exists $z \in [0,\zeta)$ such that every curve γ in $\mathbb{B}^n$ connecting the origin to a point of $[z,\zeta)$ has to meet Ω. Obviously, $R(\Omega_1) \subseteq R(\Omega_2) \subseteq \partial\mathbb{B}^n$, whenever $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{B}^n$.

Lemma 4.2. *Suppose Ω_1, Ω_2 are regions in $\mathbb{B}^n$ such that every boundary point of these regions in $\mathbb{B}^n$ is accessible. If $\Omega_1 \cap \Omega_2 = \emptyset$, then $R(\Omega_1) \subseteq R(\Omega_2)$ or $R(\Omega_2) \subseteq R(\Omega_1)$ or $R(\Omega_1) \cap R(\Omega_2) = \emptyset$.*

Proof. Suppose none of the three possibilities holds. Then there exist $\zeta_1 \in R(\Omega_1) \setminus R(\Omega_2)$, $\zeta_2 \in R(\Omega_2) \setminus R(\Omega_1)$, and $\zeta \in R(\Omega_1) \cap R(\Omega_2)$.

Moreover, we can choose $z_1 \in [0,\zeta_1)$, $z_2 \in [0,\zeta_2)$ and $z \in [0,\zeta)$ such that Ω_1 separates $[z_1,\zeta_1)$ and $[z,\zeta)$ from the origin, and Ω_2 separates $[z_2,\zeta_2)$ and $[z,\zeta)$ from the origin. Since $\zeta_1 \notin R(\Omega_2)$ and $\zeta_2 \notin R(\Omega_1)$, there exist curves γ_1 and γ_2 in $\mathbb{B}^n$ connecting the origin to $[z_1,\zeta_1)$ and $[z_2,\zeta_2)$, respectively, such that $\gamma_1 \cap \Omega_2 = \emptyset$ and $\gamma_2 \cap \Omega_1 = \emptyset$. Since Ω_1 separates $[z_1,\zeta_1)$ and Ω_2 separates $[z_2,\zeta_2)$ from the origin, we have $\gamma_1 \cap \Omega_1 \neq \emptyset$ and $\gamma_2 \cap \Omega_2 \neq \emptyset$.

Therefore, we can find curves $\alpha_1 \subseteq \gamma_1$, $\alpha_2 \subseteq \gamma_2$ such that α_1, α_2 start in Ω_1, Ω_2, respectively, end at the origin and do not meet Ω_2, Ω_1, respectively.

Since Ω_1 and Ω_2 are regions, this shows that we can travel from any point of Ω_1 to the origin along a curve without hitting Ω_2 and vice versa. Since Ω_1 and Ω_2 both separate $[z, \zeta)$ from the origin, there exists a first point $z_0 \in (\overline{\Omega}_1 \cup \overline{\Omega}_2) \cap [0, \zeta)$ as we travel from z to 0 along $[0, z]$.

W.l.o.g. we may assume $z_0 \in \overline{\Omega}_1$. Since $\Omega_1 \cap \Omega_2 = \emptyset$, $z_0 \notin \Omega_2$. Assume $z_0 \in \partial\Omega_1$. Since Ω_1 is a region, and every boundary point of Ω_1 in $\mathbb{B}^n$ is accessible, we can find a curve α connecting an end point of α_1 in Ω_1 with z_0 such that $\alpha \setminus \{z_0\} \subseteq \Omega_1$. This is also true if $z_0 \in \Omega_1$.

In any case, $[z, z_0] \cup \alpha \cup \alpha_1$ is a curve connecting $[z, \zeta)$ with the origin without hitting Ω_2. This is a contradiction, since Ω_2 separates $[z, \zeta)$ from the origin. □

Lemma 4.3. *Suppose Ω_1, Ω_2 are regions as in Lem 4.2. If $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\emptyset \neq R(\Omega_1) \subsetneqq R(\Omega_2)$, then any curve γ from the origin to a point in Ω_1 meets Ω_2.*

Proof. As in the first part of the proof of Lem. 4.2, we see that there exists a curve α_2 from the origin to a point in Ω_2 with $\alpha_2 \cap \Omega_1 = \emptyset$. Assume there is a curve γ as in the statement with $\gamma \cap \Omega_2 = \emptyset$. Then let $\alpha_1 = \gamma$, take a point $\zeta \in R(\Omega_1) \subseteq R(\Omega_2)$ and proceed as in the proof of Lem. 4.2 to get a contradiction. □

Lemma 4.4. *Suppose Ω_ν, $\nu \in \mathbb{N}$, are pairwise disjoint regions in $\mathbb{B}^n$ that have accessible boundary points in $\mathbb{B}^n$. If the family $\{\Omega_\nu\}$ is locally finite in $\mathbb{B}^n$, then every set $R(\Omega_\alpha) \neq \emptyset$ is contained in a unique set $R(\Omega_\beta)$ for which $R(\Omega_\beta)$ is maximal among the sets $R(\Omega_\nu)$, $\nu \in \mathbb{N}$.*

Proof. If $R(\Omega_\alpha) \neq \emptyset$ and both $R(\Omega_\alpha) \subseteq R(\Omega_{\beta_1})$ and $R(\Omega_\alpha) \subseteq R(\Omega_{\beta_2})$, where $R(\Omega_{\beta_1})$ and $R(\Omega_{\beta_2})$ are maximal, then $R(\Omega_{\beta_1}) \subseteq R(\Omega_{\beta_2})$ or $R(\Omega_{\beta_2}) \subseteq R(\Omega_{\beta_1})$ by Lem. 4.2. By maximality, $R(\Omega_{\beta_1}) = R(\Omega_{\beta_2})$. This shows the uniqueness.

For the existence we have to show that every inclusion chain

$$\emptyset \neq R(\Omega_{\alpha_1}) \subsetneqq R(\Omega_{\alpha_2}) \subsetneqq R(\Omega_{\alpha_3}) \subsetneqq \dots$$

is finite.

Suppose not and let $\zeta \in R(\Omega_{\alpha_1})$. Then there exists $z_1 \in \Omega_{\alpha_1} \cap [0, \zeta)$, for otherwise Ω_{α_1} would not separate any tail of $[0, \zeta)$ from the origin.

By Lem. 4.3 there exists $z_2 \in \Omega_2 \cap [0, z_1]$. Repeating this argument we obtain points $z_\nu \in \Omega_{\alpha_\nu}$ for $\nu \in \mathbb{N}$ such that $z_{\nu+1} \in [0, z_\nu]$, $\nu \in \mathbb{N}$. Then the sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converges and has a limit point $z \in \mathbb{B}^n$. This is impossible, since the regions Ω_ν are disjoint and form a locally finite family in $\mathbb{B}^n$. □

Lemma 4.5. *Suppose $\Omega \subseteq \mathbb{B}^n$ is a region and $\zeta \in \partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n$ lies in the boundary of $R(\Omega)$ (as a subset of $\partial\mathbb{B}^n$). Then $\zeta \in \overline{\Omega}$ and* $\operatorname{diam}_\rho(\Omega) = \infty$.

Proof. Suppose $\zeta \notin \overline{\Omega}$. Then there exists $\varepsilon > 0$ such that $B(\zeta, \varepsilon) \cap \Omega = \emptyset$. Since ζ lies in the boundary of $R(\Omega)$, there exist points $\zeta_1 \in R(\Omega) \cap B(\zeta, \varepsilon)$

and $\zeta_2 \in [\partial\mathbb{B}^n \setminus R(\Omega)] \cap B(\zeta, \varepsilon)$. Choose $z_2 \in [0, \zeta_2)$ close enough to ζ_2 such that $z_2 \in B(\zeta, \varepsilon)$. Since Ω separates no tail of $[0, \zeta_2)$, there exists a curve γ connecting the origin to $[z_2, \zeta_2)$ with $\gamma \cap \Omega = \emptyset$. But then Ω cannot separate any tail of $[0, \zeta_1)$ from the origin, since any line segment $[x, y]$ lies in $\mathbb{B}^n \setminus \Omega$ whenever $x, y \in B(\zeta, \varepsilon) \cap \mathbb{B}^n$. Choosing x on $[0, \zeta_1)$ sufficiently close to ζ_1 and for y the end point of γ on $[z_2, \zeta_2)$, we can connect any tail of $[0, \zeta_1)$ with 0 without hitting Ω. This contradicts $\zeta_1 \in R(\Omega)$.

Assume $\operatorname{diam}_\rho(\Omega) < \infty$. Then there exists $C_1 < \infty$ such that $d_\rho(0, x) \leq C_1$ for $x \in \Omega$. By the Gehring-Hayman Theorem, $[0, x]$ is essentially the curve of smallest ρ-length connecting 0 and x in $\mathbb{B}^n$. It follows that there exists a constant $C_2 < \infty$ such that $\operatorname{length}_\rho([0, x]) \leq C_2$ for $x \in \Omega$. Since $\zeta \in \overline{\Omega}$ we can choose a sequence (x_k) in Ω with $(x_k) \to \zeta$. A limiting argument using the continuity of ρ then implies

$$\operatorname{length}_\rho([0, \zeta)) = \lim_{r \to 1, r < 1} \operatorname{length}_\rho([0, r\zeta]) \leq \limsup_{k \to \infty} \operatorname{length}_\rho([0, x_k]) \leq C_2.$$

But $\zeta \notin \partial_\rho \mathbb{B}^n$ and so $\operatorname{length}_\rho([0, \zeta)) = \infty$. This is a contradiction. $\square$

Lemma 4.6. *Suppose $t \geq 1$. Then there exists $\lambda = \lambda(A, B, n, t) > 0$ with the following property. If $x \in \mathbb{B}^n$ and $\Omega = B_\rho(x, \lambda r_x)$, then $tS_x \subseteq R(\Omega)$.*

Proof. By [3, Thm. 6.3 and Prop. 6.2] there exists $c_1 = c_1(A, B, n) > 0$ such that the following separation property is true. If $\zeta \in \partial\mathbb{B}^n$, $y \in (0, \zeta)$, and γ is any curve in $\mathbb{B}^n$ connecting $[0, y)$ and (y, ζ), then $\gamma \cap B_\rho(y, c_1 r_y) \neq \emptyset$. In other words, $\zeta \in R(B_\rho(y, c_1 r_y))$ whenever $\zeta \in \partial\mathbb{B}^n$ and $y \in [0, \zeta)$.

Now if $\zeta \in tS_x$, then $\operatorname{dist}_h(x, [0, \zeta)) < c_2(t)$. Hence for each $\zeta \in tS_x$, we can find $y_\zeta \in [0, \zeta)$ with $d_h(x, y_\zeta) < c_2(t)$. Then $r_{y_\zeta} \leq c_3 r_x$ and $d_\rho(x, y_\zeta) \leq c_3 r_x$ for $\zeta \in tS_x$, where $c_3 = c_3(A, t) > 0$. Hence if $\lambda = \lambda(A, B, n, t) = c_3(1 + c_1)$, then $B_\rho(y_\zeta, c_1 r_{y_\zeta}) \subseteq B_\rho(x, \lambda r_x)$, and so

$$tS_x \subseteq \bigcup_{\zeta \in tS_x} R(B_\rho(y_\zeta, c_1 r_{y_\zeta})) \subseteq R(B_\rho(x, \lambda r_x)).$$

$\square$

Lemma 4.7. *Suppose $\zeta_1, \zeta_2 \in \partial\mathbb{B}^n$, $x \in [0, \zeta_1)$, $\varepsilon > 0$, and $|\zeta_1 - \zeta_2| \geq \varepsilon(1 - |x|)$. If γ is any curve in $\mathbb{B}^n$ connecting $[x, \zeta_1) \setminus B_x$ and $[0, \zeta_2)$, then*

$$\operatorname{length}_\rho(\gamma) \geq c r_x,$$

where $c = c(A, B, n, \varepsilon) > 0$.

Proof. Let $u \in [x, \zeta_1) \setminus B_x$ and $v \in [0, \zeta_2)$ be the end points of γ, and α be the hyperbolic geodesic joining u and v. Then by the Gehring-Hayman theorem

$$\operatorname{length}_\rho(\gamma) \geq c_1(A, B, n) \operatorname{length}_\rho(\alpha). \tag{4.1}$$

On the other hand, our assumptions imply $|u - v| \gtrsim c_2(\varepsilon)(1 - |x|)$ and therefore it is easily seen that the hyperbolic geodesic α contains a subcurve

α' with $\text{length}(\alpha') \geq c_3(\varepsilon)(1-|x|)$ and $\sup_{y\in\alpha'} d_h(x,y) \leq c_4(\varepsilon)$. It follows from HI($A$) that

$$\text{length}_\rho(\alpha) \geq \text{length}_\rho(\alpha') \geq c_5(A,\varepsilon)(1-|x|)\rho(x) = c_5(A,\varepsilon) r_x. \tag{4.2}$$

Inequalities (4.1) and (4.2) imply the claim. □

Proposition 4.8. *Suppose $\mathcal{E} \subseteq \mathcal{C}$, $z \in \mathbb{B}^n$, $\varepsilon > 0$, and γ is a compact curve in $\partial\mathbb{B}^n$. If*

(a) $S_x \subseteq S_z$ *for* $x \in \mathcal{E}$,
(b) $\gamma \cap \partial_\rho \mathbb{B}^n \subseteq \bigcup_{x\in\mathcal{E}} 5S_x$,
(c) $\text{diam}(\gamma) \geq \varepsilon(1-|z|)$,

then $\sum_{x\in\mathcal{E}} r_x \geq c r_z$, *where* $c = c(A,B,n,\varepsilon) > 0$.

Proof. We may assume $\sum_{x\in\mathcal{E}} r_x < \infty$, for otherwise there is nothing to prove.

By Lem. 4.6 there exists $\lambda = \lambda(A,B,n) > 0$ such that $B_\rho(x, \lambda r_x)$ separates a tail of $[0,\zeta)$ for all $\zeta \in 5S_x$, $x \in \mathbb{B}^n$. Then by Lem. 4.1 the connected components $\Omega_1, \Omega_2, \dots$ of the open set $U = \bigcup_{x\in\mathcal{E}} B_\rho(x,\lambda r_x) \subseteq \mathbb{B}^n$ form a family of regions whose boundary points in $\mathbb{B}^n$ are accessible. Moreover, the family is locally finite in $\mathbb{B}^n$.

Our choice of λ implies $R(B_\rho(x,\lambda r_x)) \supseteq 5S_x$ for $x \in \mathcal{E}$. Our assumption (b) hence shows that each point $\zeta \in \gamma \cap \partial_\rho\mathbb{B}^n$ lies in the interior (w.r.t. $\partial\mathbb{B}^n$) of some $R(\Omega_\nu)$.

By Lem. 4.4 there is at least one of the regions Ω_ν, say Ω, for which the set $R(\Omega)$ is maximal among the sets $R(\Omega_\nu)$ and contains a point of $\gamma\cap\partial_\rho\mathbb{B}^n$ in its interior.

We may assume that γ has a parametrization $\gamma\colon [0,1] \to \partial\mathbb{B}^n$. Let $I \subseteq [0,1]$ be the set of $t \in [0,1]$ for which $\gamma(t)$ lies in the interior (w.r.t. $\partial\mathbb{B}^n$) of $R(\Omega)$. Then $I \neq \emptyset$ is a relative open set in $[0,1]$. We claim that $I = [0,1]$.

Otherwise, there exists a point $t_0 \in [0,1]$ in the relative boundary of I in $[0,1]$. Since I is open in $[0,1]$, $t_0 \notin I$, but t_0 is a limit point of points in I. This implies that $\zeta_0 = \gamma(t_0)$ does not lie in the interior of $R(\Omega)$, but is a limit point of interior points of $R(\Omega)$. In particular, ζ_0 lies in the boundary of $R(\Omega)$.

Assume $\zeta_0 \in \partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n$. Then $\text{diam}_\rho(\Omega) = \infty$ by Lem. 4.5. This is impossible, since the ρ-diameter of every component Ω_ν of U is bounded by $2\lambda\sum_{x\in\mathcal{E}} r_x < \infty$. To see this note that two points $u, v \in U$ lie in the same component of U if and only if there exist distinct points $x_1, \dots, x_k \in \mathcal{E}$ such that $u \in B_\rho(x_1, \lambda r_{x_1})$, $v \in B_\rho(x_k, \lambda r_{x_k})$ and

$$B_\rho(x_\nu, \lambda r_{x_\nu}) \cap B_\rho(x_{\nu+1}, \lambda r_{x_{\nu+1}}) \neq \emptyset$$

for $\nu \in \{1, \dots, k-1\}$. The points u and v can then be joined by a curve in the component which has ρ-length at most

$$2\lambda \sum_{\nu=1}^{k} r_{x_\nu} \leq 2\lambda \sum_{x\in\mathcal{E}} r_x.$$

Thus $\zeta_0 = \gamma(t_0) \in \gamma \cap \partial_\rho \mathbb{B}^n$. Then $\zeta_0 \subseteq 5S_x$ for some $x \in \mathcal{E}$, and so ζ_0 lies in the interior of some $R(\Omega_l)$. We have seen above that ζ_0 is a limit point of points in $R(\Omega)$ distinct from ζ_0. Therefore, $R(\Omega_l) \cap R(\Omega) \neq \emptyset$. By Lem. 4.2 and the maximality of $R(\Omega)$ this implies $R(\Omega_l) \subseteq R(\Omega)$. In particular, ζ_0 would lie in the interior of $R(\Omega)$. Since we know that this is not true, we get a contradiction. This shows that $I = [0,1]$ as claimed. Hence γ lies in $R(\Omega)$.

Choose $\zeta_1, \zeta_2 \in \gamma$ such that $|\zeta_1 - \zeta_2| = \operatorname{diam} \gamma$. Since $\zeta_1, \zeta_2 \in R(\Omega)$, we have $[0, \zeta_1) \cap \Omega \neq \emptyset$ and $[0, \zeta_2) \cap \Omega \neq \emptyset$. From the consideration above it follows that there are pairwise distinct points $x_1, \dots, x_k \in \mathcal{E}$ such that $[0, \zeta_1) \cap B_\rho(x_1, \lambda r_{x_1}) \neq \emptyset$, $[0, \zeta_2) \cap B_\rho(x_k, \lambda r_{x_k}) \neq \emptyset$ and

$$B_\rho(x_\nu, \lambda r_{x_\nu}) \cap B_\rho(x_{\nu+1}, \lambda r_{x_{\nu+1}}) \neq \emptyset$$

for $\nu \in \{1, \dots, k-1\}$. In particular, there exists a curve α in $\mathbb{B}$ connecting $[0, \zeta_1)$ and $[0, \zeta_2)$ with

$$\operatorname{length}_\rho(\alpha) \leq 2\lambda \sum_{x \in \mathcal{E}} r_x. \tag{4.3}$$

Moreover, we may assume that α passes through at least one of the points $x_1, \dots, x_k$, i.e., there exists $x_0 \in \mathcal{E}$ with $x_0 \in \alpha$. The point x_0 lies on some ray $[0, \zeta_1')$, $\zeta_1' \in \partial \mathbb{B}^n$. Since $|\zeta_1 - \zeta_1'| + |\zeta_2 - \zeta_1'| \geq |\zeta_1 - \zeta_2| = \operatorname{diam}(\gamma)$, there is a point $\zeta_2' \in \{\zeta_1, \zeta_2\}$ for which

$$|\zeta_1' - \zeta_2'| \geq \tfrac{1}{2}|\zeta_1 - \zeta_2| = \tfrac{1}{2} \operatorname{diam}(\gamma) \geq \tfrac{1}{2}\varepsilon(1 - |z|).$$

Since $S_{x_0} \subseteq S_z$ by assumption (a), there exists a point $y \in B_z \cap [0, x_0]$ by Lem. 2.1. In other words, y lies in on the ray $[0, \zeta_1')$ "above" x_0. The curve α connects $[x_0, \zeta_1') \subseteq [y, \zeta_1')$ to $[0, \zeta_2')$ If $x_0 \in B_y$ then $r_{x_0} \approx r_y \approx r_z$, and the statement of the proposition is then trivially true.

If $x_0 \notin B_y$, then α connects $[y, \zeta_1') \setminus B_y$ to $[0, \zeta_2')$. Moreover, $|\zeta_1' - \zeta_2'| \geq \frac{1}{2}\varepsilon(1 - |z|) \geq \frac{3}{4}\varepsilon(1 - |y|)$.

By Lem. 4.7 there is a constant $c = c(A, B, n, \varepsilon) > 0$ such that

$$\operatorname{length}_\rho(\alpha) \geq c r_y \approx c r_z. \tag{4.4}$$

The claim now follows follows from (4.3) and (4.4). □

5. Modulus estimates

Suppose Γ is a curve family in $\mathbb{R}^n$ and $\rho\colon \mathbb{B}^n \to [0, \infty]$ is Borel measurable. The density ρ is called admissible for Γ if

$$\int_\gamma \rho(z)\,|dz| \geq 1 \tag{5.1}$$

for all locally rectifiable curves $\gamma \in \Gamma$.

The modulus of Γ is defined as

$$\operatorname{mod}_n \Gamma = \inf \int_{\mathbb{R}^n} \rho^n \, dm_n, \tag{5.2}$$

where the infimum is taken over all densities admissible for Γ.

If Γ is a curve family in $\partial\mathbb{B}^n$, $n \geq 2$, the spherical modulus is defined by

$$\operatorname{mod}^\sigma_{n-1} \Gamma = \inf \int_{\partial\mathbb{B}^n} \rho^{n-1}\, d\sigma_{n-1}, \tag{5.3}$$

where the infimum ranges over all Borel densities that are admissible for Γ. Obviously, we can in addition assume that ρ is supported on $\partial\mathbb{B}^n$. We emphasize the difference between theses modulus concepts by the additional superscript σ (for "spherical") in (5.3).

Proposition 5.1. *Suppose $0 < \alpha \leq n-1$. Let $x \in \mathbb{B}^n$, let $E \subseteq S_x$ be a Borel set, and Γ be the family of curves in $\mathbb{B}^n$ connecting $\overline{B}_x$ and E. Then*

$$\mathcal{H}^{\alpha,\infty}(E) \leq c(1-|x|)^\alpha \operatorname{mod}_n \Gamma,$$

where $c = c(\alpha, n) > 0$.

Proof. For $x = 0$ this is [6, Prop. 4.3]. The general case follows from a modification of the argument to our present situation (cf. [7, Form. (3.3)]). Alternatively, we can reduce to the case $x = 0$ by using a Möbius transformation T fixing $\mathbb{B}^n$ and mapping x to 0. Note that T preserves the modulus of curve families and expands distances in S_x by a factor comparable to $(1-|x|)^{-1}$. □

Lemma 5.2. *Let $n \geq 3$. Then there exist a numerical constant $c_1 > 0$ and constants $\varepsilon = \varepsilon(n) > 0$, $c_2 = c_2(n) > 0$ with the following property. Let $x \in \mathbb{B}^n$, $M \subseteq S_x$, and $\mathcal{H}^{n-2,\infty}(M) \leq \varepsilon(1-|x|)^{n-2}$. Denote by Γ the family of all compact curves $\gamma \subseteq S_x \setminus M$ such that* $\operatorname{diam}(\gamma) \geq c_1(1-|x|)$. *Then $\operatorname{mod}^\sigma_{n-1} \Gamma \geq c_2$.*

Proof. Fix a center for the spherical ball S_x and shrink its radius $r \leq \pi$ which is comparable to $1-|x|$ by a factor 2 so that the smaller ball fits into a hemisphere. Consider only curves γ in this new spherical ball. It can be mapped onto the ball $B = B(0, R) \subseteq \mathbb{R}^{n-1}$ of radius $R := 1-|x|$ by a bilipschitz map whose bilipschitz constant is bounded by a fixed number.

By using this auxiliary map, we are reduced to a Euclidean situation and it is enough to show that if $\varepsilon = \varepsilon(n) > 0$ is small, and $M \subseteq B$ is a set with $\mathcal{H}^{n-2,\infty}(M) < \varepsilon R^{n-2}$, then for the family of all compact curves $\gamma \subseteq B \setminus M$ such that $\operatorname{diam}(\gamma) \geq \frac{1}{3}R$ we have $\operatorname{mod}_{n-1}\Gamma \geq c(n) > 0$.

Consider the annulus $A := \bar{B}(0, R_2) \setminus B(0, R_1) \subseteq B$, where $R_1 = \frac{1}{3}R$ and $R_2 = \frac{2}{3}R$. Let $M' = A \cap M$ and Γ' be the family of all closed segments in $A \setminus M'$ which are subsets of rays starting from 0 and have their end points on the different boundary components of A. Then $\Gamma' \subseteq \Gamma$, and so

$$\operatorname{mod}_{n-1}\Gamma \geq \operatorname{mod}_{n-1}\Gamma'. \tag{5.4}$$

Since $M' \subseteq M$, we can find balls $B(x_k, r_k) \subseteq \mathbb{R}^{n-1}$ such that $x_k \in M'$ for $k \in \mathbb{N}$, $M \subseteq \bigcup_{k\in\mathbb{N}} B(x_k, r_k)$ and $\sum_{k=1}^\infty r_k^{n-2} < \varepsilon(2R)^{n-2}$. Note that the factor 2 in this last inequality is caused by the requirement $x_k \in M'$ for $k \in \mathbb{N}$. If $\varepsilon = \varepsilon(n) > 0$ is small enough, then this implies $r_k < \frac{1}{10}R$. In particular, $B(0, \frac{1}{10}R) \cap B(x_k, r_k) = \emptyset$ and $B(x_k, r_k) \subseteq B(0, R)$ for $k \in \mathbb{N}$. Then r_k^{n-2} is comparable to the solid angle under which $B(x_k, r_k)$ is seen from the origin in $\mathbb{R}^{n-1}$. Assuming $\varepsilon = \varepsilon(n) > 0$ is small enough, we have that

$$\sigma_{n-2}(N) \leq \tfrac{1}{2}\sigma_{n-2}(\partial\mathbb{B}^{n-2}),$$

where

$$N = \{\zeta \in \partial\mathbb{B}^{n-2} : \exists t \geq 0 : t\zeta \in \bigcup_{k\in\mathbb{N}} B(x_k, r_k)\}.$$

If $\zeta \in \partial\mathbb{B}^n \setminus N$, then the line segment $\{t\zeta : R_1 \leq t \leq R_2\}$ belongs to Γ'. If ρ is an admissible density for Γ', then it follows from Hölder's inequality that

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \rho^{n-1}\, dm_{n-1} &\geq \int_{\partial\mathbb{B}^{n-2}\setminus N} \left[\int_{R_1}^{R_2} \rho(r\zeta)^{n-1} r^{n-2}\, dr\right] d\sigma_{n-2}(\zeta)\\
&\geq \frac{1}{[\log(R_2/R_1)]^{n-1}} \int_{\partial\mathbb{B}^{n-2}\setminus N} \left[\int_{R_1}^{R_2} \rho(r\zeta)\, dr\right]^{n-1} d\sigma_{n-2}(\zeta)\\
&\geq \frac{1}{2[\log 2]^{n-1}}\sigma_{n-2}(\partial\mathbb{B}^{n-2}) = c(n) > 0.
\end{aligned}$$

The claim follows from this inequality and (5.4). □

The following Lemma is Lem. 3.2 from [3].

Lemma 5.3. *Suppose $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density satisfying* HI(A) *and* VG(B). *There exists a constant $C = C(B,n) > 0$ with the following property.*

Let E be a nonempty subset of $\mathbb{B}^n$ and suppose $L \geq \delta > 0$. Assume $\operatorname{diam}_\rho(E) \leq \delta$ *and that Γ is a family of curves in $\mathbb{B}^n$ so that γ has one end point in E and* $\operatorname{length}_\rho(\gamma) \geq L$ *for every $\gamma \in \Gamma$. Then*

$$\operatorname{mod}_n \Gamma \leq \frac{C}{[\log(1 + L/\delta)]^{n-1}}.$$

In order to prove the next proposition we need the following result. We denote by $||f||_{L^p} = (\int_{\partial\mathbb{B}^n} |f|^p\, d\sigma_{n-1})^{1/p}$ the $L^p(\sigma_{n-1})$-norm of a measurable function $f\colon \partial\mathbb{B}^n \to \mathbb{R}$.

Lemma 5.4. *Suppose $n \geq 2$, $1 \leq p < \infty$ and $\lambda \geq 1$. Let $\{\Sigma_i\}_{i\in\mathbb{N}}$ be a collection of spherical balls in $\partial\mathbb{B}^n$ and let $a_i \geq 0$ for $i \in \mathbb{N}$. Then*

$$\|\sum_{i\in\mathbb{N}} a_i\chi_{\lambda\Sigma_i}\|_{L^p} \leq C(n,p,\lambda)\|\sum_{i\in\mathbb{N}} a_i\chi_{\Sigma_i}\|_{L^p}$$

For a proof of a similar result see e.g. [1]. The case $p = 1$ is easy and the proof for the case $p > 1$ is based on L^p-duality and on the boundedness of the maximal function in L^q, where q is the conjugate exponent of p. These considerations apply to the setting of Lem. 5.4, since σ_{n-1} is a doubling measure with respect to the spherical metric.

Proposition 5.5. (Main Modulus Estimate) *Suppose $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density satisfying* HI(A) *and* VG(B). *If $\varepsilon \in (0, \frac{1}{2}]$, then there exists a constant $C(n, A, B, \varepsilon) > 0$ with the following property.*

Let $z \in \mathbb{B}^n$, $\mathcal{E} \subseteq \mathcal{C}$, and Γ be a family of compact curves in $\partial\mathbb{B}^n$. Assume that

(a) $S_x \subseteq S_z$ *for* $x \in \mathcal{E}$,

(b) $\gamma \cap \partial\mathbb{B}^n \subseteq \bigcup_{x\in\mathcal{E}} S_x$ *for* $\gamma \in \Gamma$,

(c) $\operatorname{diam}(\gamma) \geq \varepsilon(1-|z|)$ *for* $\gamma \in \Gamma$.

Then

$$\operatorname{mod}^{\sigma}_{n-1} \Gamma \leq \frac{C}{r_z^{n-1}} \sum_{x\in\mathcal{E}} r_x^{n-1}. \tag{5.5}$$

Proof. In the proof the constants of comparability will only depend on A, B, n, and ε.

A standard covering argument shows that there exists a set $\mathcal{E}' \subseteq \mathcal{E}$ such that

$$S_x \cap S_y = \emptyset \quad \text{for} \quad x, y \in \mathcal{E}',\ x \neq y, \tag{5.6}$$

and

$$\bigcup_{x\in\mathcal{E}} S_x \subseteq \bigcup_{x\in\mathcal{E}'} 5S_x. \tag{5.7}$$

By (a)–(c) the density $\frac{1}{\varepsilon(1-|z|)}\chi_{S_z}$ is admissible for Γ. Hence

$$\operatorname{mod}^{\sigma}_{n-1} \Gamma \lesssim 1. \tag{5.8}$$

Suppose for some $x \in \mathcal{E}'$ the diameter of B_x is comparable to the diameter of B_z. Then (a) and Lem. 2.1 imply that the hyperbolic distance of B_x and B_z is controlled. So r_x and r_z are comparable by property HI(A) of ρ. In this case (5.5) follows from (5.8). Therefore, we may assume that $\operatorname{diam}(B_x)$ and hence $\operatorname{diam}(S_x)$ is small compared to $\operatorname{diam}(B_z) \approx 1-|z|$ for all $x \in \mathcal{E}'$. More precisely we assume that

$$\operatorname{diam}(S_x) \approx \operatorname{diam}(6S_x) < \varepsilon(1-|z|) \leq 1/2 \quad \text{for} \quad x \in \mathcal{E}'. \tag{5.9}$$

Now let

$$\tilde{\rho}(\zeta) = \frac{1}{r_z} \sum_{x\in\mathcal{E}'} \rho(x)\chi_{6S_x}(\zeta), \quad \text{for} \quad \zeta \in \partial\mathbb{B}^n.$$

To see that $\tilde{\rho}$ multiplied with a constant only depending on A, B, n, and ε is admissible for Γ, we have to show $\int_\gamma \tilde{\rho}(\zeta)\,|d\zeta| \gtrsim 1$ for $\gamma \in \Gamma$. For this note that γ cannot be contained in any $6S_x$, $x \in \mathcal{E}'$, for otherwise by (5.9)

$$\operatorname{diam}(\gamma) \leq \operatorname{diam}(6S_x) < \varepsilon(1-|z|),$$

which contradicts (c).

This implies that if $\gamma \cap 5S_x \neq \emptyset$ for some $x \in \mathcal{E}'$, then γ connects $5S_x$ and $\partial\mathbb{B}^n \setminus 6S_x$, and so

$$\operatorname{length}(\gamma \cap 6S_x) \geq \operatorname{dist}(\partial\mathbb{B}^n \setminus 6S_x, 5S_x) \gtrsim 1-|x|.$$

Here we used (5.9) and fact that $\varepsilon \in (0, \frac{1}{2}]$ which implies that $6S_x$ is contained in a hemisphere.

Now if $\gamma \in \Gamma$ let $\mathcal{C}'_\gamma := \{x \in \mathcal{E}' : \gamma \cap 5S_x \neq \emptyset\}$. If $\gamma \in \Gamma$, then by (b) and (5.7) we can apply Proposition 4.8 to $\mathcal{C}'_\gamma$ and we obtain

$$\begin{aligned}\int_\gamma \tilde{\rho}(x)\,|dx| &= \frac{1}{r_z}\sum_{x\in\mathcal{E}'}\rho(x)\,\mathrm{length}(\gamma\cap 6S_x)\\ &\gtrsim \frac{1}{r_z}\sum_{x\in\mathcal{C}'_\gamma}\rho(x)(1-|x|) = \frac{1}{r_z}\sum_{x\in\mathcal{C}'_\gamma} r_x \gtrsim 1.\end{aligned} \tag{5.10}$$

Hence inequality (5.5) follows, since by Lem. 5.4 and (5.6) we have

$$\begin{aligned}\operatorname{mod}^\sigma_{n-1}\Gamma &\lesssim \int_{\partial\mathbb{B}^n}\tilde{\rho}^{n-1}\,d\sigma_{n-1} = \frac{1}{r_z^{n-1}}\Big\|\sum_{x\in\mathcal{E}'}\rho(x)\chi_{6S_x}\Big\|^{n-1}_{L^{n-1}}\\ &\lesssim \frac{1}{r_z^{n-1}}\Big\|\sum_{x\in\mathcal{E}'}\rho(x)\chi_{S_x}\Big\|^{n-1}_{L^{n-1}} = \frac{1}{r_z^{n-1}}\sum_{x\in\mathcal{E}'}\rho(x)^{n-1}\sigma_{n-1}(S_x)\\ &\lesssim \frac{1}{r_z^{n-1}}\sum_{x\in\mathcal{E}'}\rho(x)^{n-1}(1-|x|)^{n-1} \le \frac{1}{r_z^{n-1}}\sum_{x\in\mathcal{E}} r_x^{n-1}.\end{aligned}$$

The proof is complete. □

Now we can prove Thm. 1.1.

Proof of Thm. 1.1. Suppose $n = 2$ and assume that for some $\mathcal{E} \subseteq \mathcal{C}$ we have $\bigcup_{x\in\mathcal{E}} S_x \supseteq \partial_\rho\mathbb{B}^n$. Prop 4.8 with $z = 0$ and $\gamma = \partial\mathbb{B}^2$ shows that

$$\sum_{x\in\mathcal{E}} r_x \ge c_1 r_0 = c_1\rho(0),$$

where $c_1 = c_1(A,B) > 0$. Hence $\Phi^{1,\infty}_\rho(\partial_\rho\mathbb{B}^2) \ge c_1\rho(0)$.

Prop. 3.1 then implies

$$\mathcal{H}^1_\rho(\partial_\rho\mathbb{B}^2) \ge \mathcal{H}^{1,\infty}_\rho(\partial_\rho\mathbb{B}^2) \ge c_2\Phi^{1,\infty}_\rho(\partial_\rho\mathbb{B}^2) \ge c_1c_2\rho(0) > 0,$$

where $c_2 = c_2(A,B) > 0$. Thus $\dim_\rho(\partial_\rho\mathbb{B}^2) \ge 1$.

If $n \ge 3$ and then Lem. 5.2 for $x = 0$ shows that for the family Γ of all compact curves γ in $\partial\mathbb{B}^n$ with $\operatorname{diam}(\gamma) \ge c_3$, we have

$$\operatorname{mod}^\sigma_{n-1}\Gamma \ge c_4 > 0, \tag{5.11}$$

where $c_3 > 0$ and $c_4 = c_4(n) > 0$. On the other hand, the Main Modulus Estimate shows that if $\bigcup_{x\in\mathcal{E}} S_x \supseteq \partial_\rho\mathbb{B}^n$ for some set $\mathcal{E} \subseteq \mathcal{C}$, then

$$\sum_{x\in\mathcal{E}} r_x^{n-1} \ge c_5\rho(0)^{n-1}\operatorname{mod}^\sigma_{n-1}\Gamma \ge c_6\rho(0)^{n-1}, \tag{5.12}$$

where $c_5 = c_5(A,B,n) > 0$ and $c_6 = c_6(A,B,n) > 0$. Hence $\Phi^{n-1,\infty}_\rho(\partial_\rho\mathbb{B}^n) \ge c_6\rho(0)^{n-1}$.

Therefore, by Prop. 3.1 we have that

$$\mathcal{H}^{n-1}_\rho(\partial_\rho\mathbb{B}^n) \ge \mathcal{H}^{n-1,\infty}_\rho(\partial_\rho\mathbb{B}^n) \ge c_7\Phi^{n-1,\infty}_\rho(\partial_\rho\mathbb{B}^n) \ge c_6c_7\rho(0)^{n-1} > 0,$$

where $c_7 = c_7(A,B,n) > 0$. This shows $\dim_\rho(\partial_\rho\mathbb{B}^n) \ge n-1$. □

The proof actually shows the stronger result $\mathcal{H}^{n-1}_\rho(\partial_\rho\mathbb{B}^n) \ge C(A,B,n)\rho(0)^{n-1}$.

6. The Hanson Conjecture

From now on we will deal with quasiconformal maps $f\colon \mathbb{B}^n \to \mathbb{R}^n$.

A homeomorphism f of $\mathbb{B}^n$ onto a region $\Omega \subseteq \mathbb{R}^n$ is called K-quasiconformal for $K \geq 1$ if the following condition is satisfied: If we define

$$H_f(x) := \limsup_{r\to 0^+} \frac{\max\{|f(y)-f(x)| : |y-x| = r\}}{\min\{|f(y)-f(x)| : |y-x| = r\}},$$

then $H_f(x) \leq K$ for $x \in \mathbb{B}^n$.

A quasiconformal map lies in the Sobolev space $W^{1,n}_{\mathrm{loc}}$ and is differentiable a.e. in $\mathbb{B}^n$. If the Jacobian determinant of f is denoted by J_f, we define

$$a_f(x) = \left(\frac{1}{m_n(B_x)} \int_{B_x} J_f\, dm_n\right)^{1/n} \quad \text{for} \quad x \in \mathbb{B}^n.$$

It can be shown that for a K-quasiconformal map f, the density $\rho = a_f$ is positive and continuous and satisfies HI(A) and VG(B) with constants only depending on n and K (cf. [3, 2.4]). In the following, $\rho = a_f$ unless otherwise stated. Moreover, in inequalities like $a \lesssim b$ the constants of comparability are understood to depend only on n and K.

It is convenient to have special notation for the distance of a point to the boundary of the image region Ω of f. So for $x \in \Omega$ we let

$$\delta_\Omega(x) = \mathrm{dist}(x, \partial\Omega).$$

It follows from the local quasisymmetry of a quasiconformal map (cf. [8, Sec. 18]) that

$$\begin{aligned} r_x = \rho(x)(1-|x|) &= a_f(x)(1-|x|) \\ &\approx \mathrm{diam}(f(B_x)) \approx \delta_\Omega(f(x)) \quad \text{for} \quad x \in \mathbb{B}^n. \end{aligned} \tag{6.1}$$

The internal metric l_Ω and the quasihyperbolic metric k_Ω on Ω are for $x, y \in \Omega$ defined by

$$l_\Omega(x,y) = \inf \,\mathrm{length}(\gamma) \quad \text{and} \quad k_\Omega(x,y) = \inf \int_\gamma \frac{|dz|}{\delta_\Omega(z)},$$

respectively, where the infimum is taken over all curves γ in Ω connecting x and y. It follows from standard distortion estimates for quasiconformal mappings that

$$l_\Omega(f(x), f(y)) \approx d_\rho(x,y) \text{ whenever } x, y \in \mathbb{B}^n \text{ and } d_h(x,y) \geq 1. \tag{6.2}$$

For $\zeta \in \partial\mathbb{B}^n$ we let $f(\zeta)$ be the radial limit $\lim_{r\to 1} f(r\zeta)$ if it exists. In this case, $f(\zeta) \in \partial\Omega$. Since

$$\begin{aligned} \mathrm{length}_\rho([r\zeta,\zeta)) &\approx \sum\{r_x : x \in \mathcal{C},\, B_x \cap [r\zeta,\zeta) \neq \emptyset\} \\ &\approx \sum\{\mathrm{diam}(f(B_x)) : x \in \mathcal{C},\, B_x \cap [r\zeta,\zeta) \neq \emptyset\} \\ &\geq \mathrm{diam}\big(f([r\zeta,\zeta))\big) \quad \text{for} \quad r \in [0,1),\, \zeta \in \partial\mathbb{B}^n, \end{aligned}$$

the limit $\lim_{r\to 1} f(r\zeta)$ exists whenever $\mathrm{length}_\rho([r\zeta,\zeta)) \to 0$ for $r \to 1$. In particular, $f(\zeta)$ is defined for $\zeta \in \partial_\rho\mathbb{B}^n$. So there exists an exceptional set

$E_0 \subseteq \partial\mathbb{B}^n$ of vanishing n-capacity such that $\partial_\rho\mathbb{B}^n \subseteq \partial\mathbb{B}^n \setminus E_0$ and $f(\zeta)$ is defined for $\zeta \in \partial\mathbb{B}^n \setminus E_0$.

For $M \subseteq \partial\mathbb{B}^n$ we let $f(M) \subseteq \partial\Omega$ denote the set $\{f(\zeta) : \zeta \in (\partial\mathbb{B}^n \setminus E_0) \cap M\}$.

If $x = f(\zeta)$ with $\zeta \in \partial_\rho\mathbb{B}^n$ for a boundary point $x \in \partial\Omega$, then x is a rectifiably accessible boundary point of $\partial\Omega$, i.e., there exists a rectifiable curve $\gamma\colon [0,1] \to \mathbb{R}^n$ such that $\gamma([0,1)) \subseteq \Omega$ and $\gamma(1) = x$.

Lemma 6.1. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is K-quasiconformal, $a \in \partial\Omega$, $R > 0$, and $\mathcal{E} = \{x \in \mathcal{C} : f(B_x) \cap \partial B(a,R) \neq \emptyset\}$. Then*

$$\sum_{x\in\mathcal{E}} r_x^{n-1} \leq CR^{n-1},$$

where $C = C(n,K) > 0$.

Proof. For each $x \in \mathcal{E}$ there exists $y_x \in B_x$ such that $|f(y_x) - a| = R$. Since $\frac{1}{2}(1-|x|) \leq 1 - |y_x| \leq \frac{3}{2}(1-|x|)$, we have $\frac{1}{4}B_{y_x} \subseteq B(x, (\frac{1}{2}+\frac{3}{16})(1-|x|)) \subseteq \frac{3}{2}B_x$. The local quasisymmetry of f implies that there exists $c = c(n,K) > 0$ such that $f(\frac{1}{4}B_{y_x}) \supseteq B\big(f(y_x), c\delta_\Omega(f(y_x))\big)$. Note that $\delta_\Omega(f(y_x)) \approx r_{y_x} \approx r_x$.

On the other hand, $\delta_\Omega(f(y_x)) \leq |f(y_x) - a| = R$. Therefore, $\partial B(a,R) \cap B\big(f(y_x), c\delta_\Omega(f(y_x))\big)$ is a spherical ball on $\partial B(a,R)$ with spherical measure comparable to $c^{n-1}\delta_\Omega(f(y_x))^{n-1}$, i.e.,

$$\sigma_{n-1}\Big(\partial B(a,R) \cap B\big(f(y_x), c\delta_\Omega(f(y_x))\big)\Big) \approx \delta_\Omega(f(y_x))^{n-1} \approx r_x^{n-1}.$$

Note that the balls $B\big(f(y_x), c\delta_\Omega(f(y_x))\big)$, $x \in \mathcal{E}$, have bounded overlap, because by (2.7)

$$\sum_{x\in\mathcal{E}} \chi_{B\left(f(y_x), c\delta_\Omega(f(y_x))\right)} \circ f \leq \sum_{x\in\mathcal{E}} \chi_{\frac{1}{4}B_{y_x}} \leq \sum_{x\in\mathcal{E}} \chi_{\frac{3}{2}B_x} \leq \sum_{x\in\mathcal{C}} \chi_{\frac{3}{2}B_x} \lesssim 1.$$

Hence

$$\begin{aligned}\sum_{x\in\mathcal{E}} r_x^{n-1} &\approx \sum_{x\in\mathcal{E}} \delta_\Omega(f(y_x))^{n-1} \approx \sum_{x\in\mathcal{E}} \sigma_{n-1}\Big(\partial B(a,R) \cap B\big(f(y_x), c\delta_\Omega(f(y_x))\big)\Big) \\ &\lesssim \sigma_{n-1}(\partial B(a,R)) \approx R^{n-1}.\end{aligned}$$

□

Lemma 6.2. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is K-quasiconformal and $M \subseteq \partial_\rho\mathbb{B}^n$. Then there exist positive constants c_1, c_2, c_3 only depending on n and K such that*

$$\mathcal{H}^{n-1,\infty}(f(M)) \geq c_1\tilde{\Phi}_\rho^{n-1,\delta}(M) \quad \textit{for} \quad \delta > c_2 \operatorname{diam}(M),$$

and

$$\mathcal{H}^{n-1}(f(M)) \geq c_3\Phi_\rho^{n-1,\delta}(M) \quad \textit{for} \quad \delta > 0.$$

Proof. We may assume $\operatorname{diam}(M) > 0$ for otherwise the right hand sides of the inequalities to be proved are 0 by an argument similar to the one given in the beginning of the proof of Lem. 3.2. Then we can find a point $z \in \mathbb{B}^n$ such that $M \subseteq S_z$ and $1 - |z| \approx \operatorname{diam}(M)$.

Assume the balls $B(u_k, R_k)$, $k \in \mathbb{N}$, cover $f(M) \subseteq \partial\Omega$. Discarding balls from the collection if necessary we may assume $B(u_k, R_k) \cap f(M) \neq \emptyset$. In this case we may also assume that $u_k \in f(M)$, since we can shift the center of each ball to a point in $f(M)$ and increase its radius by a factor 2 if necessary. In this way the sum $\sum_{k\in\mathbb{N}} {R_k}^{n-1}$ that we have to bound from below increases by a factor 2^{n-1} at most. We now consider two cases.

I. $f(B_z) \cap \overline{B}(u_l, R_l) \neq \emptyset$ for some $l \in \mathbb{N}$.

Then $f(y) \in \overline{B}(u_l, R_l)$ for some $y \in B_z$. It follows that

$$r_y \approx \delta(f(y)) \leq |f(y) - u_l| \leq R_l. \tag{6.3}$$

If $\mathcal{E}$ is the set of all $x \in \mathcal{C}$ with $B_x \cap B_z \neq \emptyset$, then $\#\mathcal{E} \lesssim 1$ and $r_x \approx r_z \approx r_y \lesssim R_l$ for $x \in \mathcal{E}$. Moreover, $1 - |x| \approx 1 - |z| \approx \operatorname{diam}(M)$ for $x \in \mathcal{E}$, and $\bigcup_{x\in\mathcal{E}} B_x \supseteq B_z$ implies $\bigcup_{x\in\mathcal{E}} S_x \supseteq S_z \supseteq M$. This shows that for $\delta \gtrsim \operatorname{diam}(M)$

$$\sum_{k\in\mathbb{N}} R_k^{n-1} \geq R_l^{n-1} \gtrsim \sum_{x\in\mathcal{E}} r_x^{n-1} \geq \tilde{\Phi}_\rho^{n-1,\delta}(M). \tag{6.4}$$

II. $f(B_z) \cap \overline{B}(u_k, R_k) = \emptyset$ for all $k \in \mathbb{N}$.

Since $S_z \supseteq M$, for every $\zeta \in M$ we can pick a point $y_\zeta \in [0, \zeta) \cap B_z$. Let $\mathcal{E}$ be the set of all $x \in \mathcal{C}$ such that $B_x \cap [y_\zeta, \zeta) \neq \emptyset$ for some $\zeta \in M$ and $f(B_x) \cap \partial B(u_k, R_k) \neq \emptyset$ for some $k \in \mathbb{N}$. Note that

$$1 - |x| \lesssim 1 - |z| \approx \operatorname{diam}(M) \quad \text{for} \quad x \in \mathcal{E}. \tag{6.5}$$

Moreover, if $\zeta \in M$ is arbitrary, then $f(\zeta) \in B(u_k, R_k)$ for some $k \in \mathbb{N}$. Since $f(y_\zeta) \in f(B_z)$, the curve $f([y_\zeta, \zeta])$ has its initial point outside $\overline{B}(u_k, R_k)$ and has its end point in $B(u_k, R_k)$. Therefore, there exists a point $v \in [y_\zeta, \zeta)$ with $|f(v) - u_k| = R_k$. For some $x \in \mathcal{C}$ we have $v \in B_x$. Then $x \in \mathcal{E}$ and $\zeta \in S_x$. This shows $\bigcup_{x\in\mathcal{E}} S_x \supseteq M$.

Lem. 6.1 and (6.5) now imply that for $\delta \gtrsim \operatorname{diam}(M)$

$$\begin{aligned} \sum_{k\in\mathbb{N}} R_k^{n-1} &\gtrsim \sum_{k\in\mathbb{N}} \sum \{r_x^{n-1} : x \in \mathcal{C},\ f(B_x) \cap \partial B(u_k, R_k) \neq \emptyset\} \\ &\geq \sum_{x\in\mathcal{E}} r_x^{n-1} \geq \tilde{\Phi}_\rho^{n-1,\delta}(M). \end{aligned} \tag{6.6}$$

The first statement of the lemma follows from (6.4) and (6.6).

To prove the second statement we start as in the first part of the proof, but may in addition assume that the radii R_k of the covering balls are smaller than a given arbitrary positive constant $\delta > 0$. In both cases considered above we have that

$$\sup_{x\in\mathcal{E}} r_x \lesssim \sup_{l\in\mathbb{N}} R_l < \delta.$$

Hence we get the inequalities (6.4) and (6.6) where $\tilde{\Phi}^{n-1,\delta}$ is replaced by $\Phi^{n-1,\delta}$. By considering appropriate limits based on these inequalities the second statement of the lemma follows. $\square$

Theorem 6.3. *Suppose $n \geq 3$ and $f \colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then there exist constants $\varepsilon = \varepsilon(n) > 0$ and $c = c(n, K) > 0$ such that* *for every* $x \in \mathbb{B}^n$ *and for every set* $M \subseteq S_x$ *with* $\mathcal{H}^{n-2,\infty}(M) \leq \varepsilon(1-|x|)^{n-2}$ *we have*

$$\mathcal{H}^{n-1,\infty}(f(S_x \setminus M)) \geq c\delta_\Omega(f(x))^{n-1}.$$

Corollary 6.4. (Hanson conjecture) *Suppose* $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ *is a* K*-quasiconformal map. Then*

$$\mathcal{H}^{n-1,\infty}(f(S_x)) \geq c\delta_\Omega(f(x))^{n-1} \quad \textit{for} \quad x \in \mathbb{B}^n,$$

where $c = c(n, K) > 0$.

Note that in the Corollary we allow the case $n = 2$.

Proof. Assume first that $n \geq 3$. Let $N = M \cup (\partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n)$. Since $\dim(\partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n) = 0$, we have $\mathcal{H}^{n-2,\infty}(N) = \mathcal{H}^{n-2,\infty}(M)$. By Lem. 5.2 we can find $\varepsilon_1 = \varepsilon_1(n) > 0$, $c_1 > 0$, and $c_2 = c_2(n) > 0$ with the following property. If $\mathcal{H}^{n-2,\infty}(M) \leq \varepsilon(1-|x|)^{n-1}$ and Γ is the family of all compact curves $\gamma \subseteq S_x \setminus N$ with $\operatorname{diam}(\gamma) \geq c_1(1-|x|)$, then $\operatorname{mod}_{n-1}^{\sigma} \Gamma \geq c_2$. By Lem. 6.2, for some number $\delta \approx (1-|x|) \gtrsim \operatorname{diam}(S_x \setminus N)$ we have that

$$\mathcal{H}^{n-1,\infty}(f(S_x \setminus M)) \geq \mathcal{H}^{n-1,\infty}(f(S_x \setminus N)) \gtrsim \tilde{\Phi}_\rho^{n-1,\delta}(S_x \setminus N). \tag{6.7}$$

To estimate the last expression in (6.7), assume $\mathcal{E} \subseteq \mathcal{C}$ is a set such that $\bigcup_{y\in\mathcal{E}} S_y \supseteq S_x \setminus N$ and $1 - |y| \leq \delta$ for $y \in \mathcal{E}$. In order to give a lower bound for $\sum_{y\in\mathcal{E}} r_y^{n-1}$, we can make the further assumption $S_y \cap S_x \neq \emptyset$ for $y \in \mathcal{E}$, because we can discard all $y \in \mathcal{E}$ from $\mathcal{E}$ for which $S_y \cap S_x = \emptyset$, without affecting the inclusion $\bigcup_{y\in\mathcal{E}} S_y \supseteq S_x \setminus N$.

Since $1 - |y| \leq \delta \approx 1 - |x|$ for $y \in \mathcal{E}$, the union $\bigcup_{y\in\mathcal{E}} S_y$ is then contained in a spherical ball with the same center ζ as S_x and a spherical radius not much larger than the radius of S_x. Therefore, we can find a point $z \in [0, x]$ such that $|z - x| \lesssim 1 - |x|$ and $S_z \supseteq \bigcup_{y\in\mathcal{E}} S_y$. Note that $d_h(z, x) \lesssim 1$ and so $r_x \approx r_z$.

The Main Modulus Estimate now shows

$$\sum_{y\in\mathcal{E}} r_y^{n-1} \gtrsim r_z^{n-1} \approx r_x^{n-1} \approx \delta_\Omega(f(x))^{n-1}.$$

Thus for δ as above we have

$$\tilde{\Phi}_\rho^{n-1,\delta}(S_x \setminus N) \gtrsim \delta_\Omega(f(x))^{n-1}. \tag{6.8}$$

Inequalities (6.7) and (6.8) imply the statement of the theorem for $n \geq 3$. The corollary follows from the theorem for $n \geq 3$ by taking $M = \emptyset$.

Finally, the corollary in the case $n = 2$ follows from Prop. 4.8 by the above reasoning, if for the curve γ in Prop. 4.8 we take a closed arc with $\gamma \subseteq S_x$ and $\operatorname{diam}(\gamma) \approx 1 - |x|$. □

In the statement of the next theorem we use the following notation. If $\Omega \subseteq \mathbb{R}^n$ is a region, then $B_{l_\Omega}(y, r)$ is the ball centered at y with radius $r > 0$ in the internal metric of Ω, i.e., the set all points $z \in \bar{\Omega}$ for which there exists a curve $\alpha\colon [0, 1] \to \mathbb{R}^n$ with $\alpha(0) = 0$, $\alpha(1) = z$, $\alpha([0, 1)) \subseteq \Omega$, and $\operatorname{length}(\alpha) < r$.

Theorem 6.5. ("Wall" Theorem) *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then*

$$\mathcal{H}^{n-1,\infty}\big(\partial\Omega \cap B_{l_\Omega}(y, 2\delta_\Omega(y))\big) \geq c\delta_\Omega(y)^{n-1} \quad \textit{for} \quad y \in \Omega,$$

where $c = c(n,K) > 0$.

Note that this theorem implies Thm. 1.3 stated in the introduction, because $B_{l_\Omega}(y,r) \subseteq B(y,r)$, whenever $y \in \Omega$ and $r > 0$.

Proof. First note that it is enough to show that there exists $\lambda = \lambda(n,K) > 1$ such that

$$\mathcal{H}^{n-1,\infty}\big(\partial\Omega \cap B_{l_\Omega}(y, \lambda\delta_\Omega(y))\big) \gtrsim \delta_\Omega(y)^{n-1} \quad \text{for} \quad y \in \Omega. \tag{6.9}$$

For if (6.9) is true, let $y \in \Omega$ and $z \in \partial\Omega$ with $|y - z| = \delta_\Omega(y)$. Define $y' = ty + (1-t)z$, where $t = 1/\lambda \in (0,1)$. Note that $y' \in \Omega$, $\delta_\Omega(y') = |y' - z| = t\delta_\Omega(y)$ and $B_{l_\Omega}(y', \lambda\delta_\Omega(y')) \subseteq B_{l_\Omega}(y, 2\delta_\Omega(y))$. Inequality (6.9) applied to y' then gives

$$\begin{aligned}\mathcal{H}^{n-1,\infty}(\partial\Omega \cap B_{l_\Omega}\big(y, 2\delta_\Omega(y))\big) &\geq \mathcal{H}^{n-1,\infty}(\partial\Omega \cap B_{l_\Omega}\big(y', \lambda\delta_\Omega(y'))\big) \\ &\gtrsim \delta_\Omega(y')^{n-1} \approx \delta_\Omega(y)^{n-1}.\end{aligned}$$

Now assume $n \geq 3$. In order to prove (6.9), let $x \in \mathbb{B}^n$ be the unique point with $y = f(x)$. For arbitrary $s \geq 1$ define M_s to be the set of all $\zeta \in S_x$ for which the ρ-length of the hyperbolic geodesic γ_ζ joining x and ζ is bigger than sr_x.

We show that for $s = s(n,K)$ sufficiently large, we have

$$\mathcal{H}^{n-2,\infty}(M_s) \leq \varepsilon(1 - |x|)^{n-2}, \tag{6.10}$$

where $\varepsilon = \varepsilon(n)$ is the constant of Thm. 6.3. To see this consider the family Γ_s of all curves γ in $\mathbb{B}^n$ connecting $\overline{B}_x$ and M_s. If $\delta = \operatorname{diam}_\rho(\overline{B}_x)$, then $\delta \leq c_1 r_x$, where $c_1 = c_1(n,K) > 0$. By the Gehring-Hayman theorem, there exists $c_2 = c_2(n,K)$ such that

$$\operatorname{length}_\rho(\gamma_\zeta) \leq c_2 \operatorname{length}_\rho(\alpha)$$

for any curve α in $\mathbb{B}^n$ connecting $\zeta \in M_s$ and x. It follows that

$$\operatorname{length}_\rho(\gamma) \geq sr_x/c_2 - \delta \geq r_x(s/c_2 - c_1) =: L_s \quad \text{for} \quad \gamma \in \Gamma_s.$$

In particular, $L_s \geq \delta$ for $s \geq 2c_1c_2$. For these s, Lem. 5.3 then implies

$$\operatorname{mod}_n \Gamma_s \leq \frac{c_3}{[\log(1 + L_s/\delta)]^{n-1}}, \tag{6.11}$$

where $c_3 = c_3(n,K) > 0$. On the other hand, by Prop. 5.1

$$\mathcal{H}^{n-2,\infty}(M_s) \leq c_4(1 - |x|)^{n-2} \operatorname{mod}_n \Gamma_s,$$

where $c_4 = c_4(n,K) > 0$. By (6.11) we can choose $s_0 = s_0(n,K) > 0$ large enough so that $c_4 \operatorname{mod}_n \Gamma_s \leq \varepsilon$ for $s \geq s_0$. For these s, inequality (6.10) is true.

Thm. 6.3 and (6.10) now imply that

$$\mathcal{H}^{n-1,\infty}(f(S_x \setminus M_s)) \geq c_6\delta_\Omega(y)^{n-1}, \tag{6.12}$$

where $c_6 = c_6(n,K) > 0$.

Whenever $\zeta \in S_x \setminus M_s$, the hyperbolic geodesic γ_ζ has ρ-length less or equal to $sr_x \approx s\delta_\Omega(y)$. This implies that there exists a curve $\alpha\colon [0,1] \to \mathbb{R}^n$ such that $\alpha(0) = y$, $\alpha([0,1)) \subseteq \Omega$, $\alpha(1) = f(\zeta)$ and length$(\alpha) < c_7 s\delta_\Omega(y)$, where $c_7 = c_7(n,K) > 0$. To see this note that the internal Euclidean metric in Ω is roughly comparable to the ρ-metric.

In particular, if $\lambda = c_7 s_0$, then λ only depends on n and K and we have $B_{l_\Omega}(y, \lambda\delta_\Omega(y)) \cap \partial\Omega \supseteq f(S_x \setminus M_{s_0})$. Inequality (6.9) then follows from (6.12), and the proof is complete for $n \geq 3$.

For $n = 2$ the proof is elementary and much easier. Let $y \in \Omega$, and $z \in \partial\Omega$ be a point with $|y - z| = \delta_\Omega(y)$. Consider the circles $C_t = \partial B(z, t\delta_\Omega(y))$ for $0 < t \leq 1$. Starting at the unique intersection point of the line segment (z, y) with C_t and moving along C_t in positive orientation, let z_t be the first point in the boundary of Ω that we hit. Such a point exists, because otherwise $C_t \subseteq \Omega$ would separate the point at infinity from $z \in \Omega$, which is impossible, since the region Ω is simply connected. On the one hand, for the set $M = \{z_t : 0 < t \leq 1\}$ we have $M \subseteq \partial\Omega \cap B_{l_\Omega}(y, (1 + 2\pi)\delta_\Omega(y))$. On the other hand, it is easy to see $\mathcal{H}^{1,\infty}(M) \geq \mathcal{H}^{1,\infty}((z, y]) = \frac{1}{2}\delta_\Omega(y)$. The statement follows. $\square$

Remark 6.6. If appropriately formulated, the main results of this section (Thm. 6.3, Cor. 6.4, and Thm. 6.5) remain true for general continuous densities $\rho\colon \mathbb{B}^n \to (0, \infty)$ satisfying HI(A) and VG(B).

7. The Riesz-Privalov Theorem in Higher Dimensions

If $f\colon \mathbb{B}^2 \to \Omega$ is a conformal map of the unit disc in the complex plane onto a Jordan region Ω, then

$$\Lambda(\partial\Omega) = \sup_{r<1} \int_{\partial\mathbb{B}^2} |f'(r\zeta)|\,|d\zeta| =: ||f'||_1.$$

Here $\Lambda(\partial\Omega)$ is the length of the Jordan curve $\partial\Omega$. This statement is the classical Riesz-Privalov Theorem. In particular, $\Lambda(\partial\Omega) < \infty$ if and only if $||f'||_1 < \infty$. This statement can easily be generalized for quasiconformal maps $f\colon \mathbb{B}^2 \to \Omega$, where Ω is not necessarily a Jordan region (see below). The generalization of this statement to higher dimensions is not so obvious. This is the main topic of this section.

If $\rho\colon \mathbb{B}^n \to (0, \infty)$ is a continuous density, and $p \geq 0$, we define the integral mean

$$I_p(\rho, t) := \frac{1}{\sigma_{n-1}(\partial\mathbb{B}^n)} \int_{\partial\mathbb{B}^n} \rho(t\zeta)^p \, d\sigma_{n-1}(\zeta) \quad \text{for} \quad 0 \leq t < 1.$$

Moreover, set

$$||\rho||_p := \sup_{t<1} I_p(\rho, t)^{1/p}.$$

Lemma 7.1. *Suppose $\rho\colon \mathbb{B}^n \to (0, \infty)$ is a continuous density satisfying* HI(A) *and* VG(B). *Then there exist constants $c_1 = c_1(A, n) > 0$ and $c_2 = c_2(A, B, n) > 0$ such that*

(a) *for* $p \geq 0$ *and* $0 \leq t < 1$ *we have*

$$\frac{1}{c_1^p} I_p(\rho, t) \leq (1-t)^{n-1-p} \sum \{r_x^p : x \in \mathcal{C},\, B_x \cap \partial B(0,t) \neq \emptyset\} \leq c_1^p I_p(\rho, t),$$

(b) $I_{n-1}(\rho, t) \leq c_2 I_{n-1}(\rho, s)$ *for* $p \geq 0$, $0 \leq t \leq s < 1$.

Statement (b) says that the integral $(n-1)$-mean of ρ is essentially non-decreasing as a function of t.

Proof. (a) This statement follows from

$$\begin{aligned}\Big[I_p(\rho,t)(1-t)\Big]^{1/p} &\approx \Big[\int_{||z|-t| \leq \frac{1}{2}(1-t)} \rho(z)^p \, dm_n(z)\Big]^{1/p} \\ &\approx \Big[(1-t)^{n-1-p} \sum_{x \in \mathcal{C},\, B_x \cap \partial B(0,t) \neq \emptyset} r_x^p\Big]^{1/p}.\end{aligned}$$

To see this use Harnack's inequality for ρ and note that the set $\{z \in \mathbb{B}^n : ||z| - t| \leq \frac{1}{2}(1-t)\}$ is a spherical shell around the sphere $\partial B(0,t)$. The volume of this shell is comparable to $(1-t)$. The constants of comparability depend only on A and n.

(b) Here the constants of comparability will depend on A, B, and n. Let $\mathcal{C}_t = \{x \in \mathcal{C} : B_x \cap \partial B(0,t) \neq \emptyset\}$ for $0 \leq t < 1$. For $x \in \mathcal{C}_t$ let $\mathcal{C}_s(x) = \{y \in \mathcal{C}_s : S_y \cap S_x \neq \emptyset\}$. Since $\bigcup_{x \in \mathcal{C}_t} S_x \supseteq \partial\mathbb{B}^n$ for all $0 \leq t < 1$, we have $\bigcup_{y \in \mathcal{C}_s(x)} S_y \supseteq S_x$ for all $0 \leq t \leq s < 1$, $x \in \mathcal{C}_t$. Note that $\operatorname{diam}(S_y) \lesssim 1 - s \leq 1 - t$ for $y \in \mathcal{C}_s(x)$. This shows that

$$\operatorname{diam}\Big(\bigcup_{y \in \mathcal{C}_s(x)} S_y\Big) \lesssim 1 - t \approx 1 - |x| \quad \text{for} \quad x \in \mathcal{C}_t.$$

So whenever $x \in \mathcal{C}_t$, by traveling a distance comparable to $1-t$ along the ray $[0,x]$ from x, we can find a point $x' \in \mathbb{B}^n$ such that

$$r_{x'} \approx r_x \text{ and } S_{x'} \supseteq \bigcup_{y \in \mathcal{C}_s(x)} S_y \supseteq S_x. \tag{7.1}$$

By Lem. 5.2, for each $x \in \mathbb{B}^n$ there exists a family Γ_x of compact curves with $\gamma \subseteq S_x$ and $\operatorname{diam}(\gamma) \approx 1 - |x|$ for $\gamma \in \Gamma_x$ such that

$$\operatorname{mod}_{n-1}^{\sigma} \Gamma_x \gtrsim 1. \tag{7.2}$$

The Main Modulus Estimate, (7.1), and (7.2) then imply

$$\sum_{y \in \mathcal{C}_s(x)} r_y^{n-1} \gtrsim r_x^{n-1} \quad \text{for} \quad x \in \mathcal{C}_t. \tag{7.3}$$

If $x_1, x_2 \in \mathcal{C}_t$ and $\mathcal{C}_s(x_1) \cap \mathcal{C}_s(x_2) \neq \emptyset$, then $|x_1 - x_2| \lesssim 1 - t$. Since $1 - |x_1| \approx 1 - |x_2| \approx 1 - t$, we get $d_h(x_1, x_2) \lesssim 1$. This shows that $\#\{x \in \mathcal{C}_t : y \in \mathcal{C}_s(x)\} \lesssim 1$ for all $y \in \mathcal{C}_s$.

Hence by (7.3)

$$\sum_{x \in \mathcal{C}_t} r_x^{n-1} \lesssim \sum_{x \in \mathcal{C}_t} \sum_{y \in \mathcal{C}_s(x)} r_y^{n-1} \approx \sum_{y \in \mathcal{C}_s} r_y^{n-1}.$$

Now this and (a) imply (b). □

From now on we will again consider K-quasiconformal maps $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$. Unless otherwise stated, $\rho = a_f$ and all constants of comparability will depend only on n and K.

Proposition 7.2. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map.*
(a) *If $n = 2$, then there exists $c_1 = c_1(K) > 0$ such that*

$$||a_f||_1 \leq c_1 \mathcal{H}^1(\partial\Omega).$$

(b) *If $n \geq 3$, then there exist $c_2 = c_2(n, K) > 0$ and $\varepsilon = \varepsilon(n) > 0$ such that*

$$||a_f||_{n-1}^{n-1} \leq c_2 \mathcal{H}^{n-1}(f(\partial\mathbb{B}^n \setminus M)),$$

whenever $M \subseteq \partial\mathbb{B}^n$ is a Borel set with $\mathcal{H}^{n-2,\infty}(M \cap \Sigma(a,r)) \leq \varepsilon r^{n-2}$ for all $a \in \partial\mathbb{B}^n$, $0 < r \leq \pi$. In particular, $||a_f||_{n-1}^{n-1} \leq c_2 \mathcal{H}^{n-1}(\partial\Omega)$.

Proof. We first prove (b).

Since $\dim(\partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n) = 0$, we may assume $\partial\mathbb{B}^n \setminus \partial_\rho\mathbb{B}^n \subseteq M$. Lem. 5.2 shows that if $M \subseteq \partial\mathbb{B}^n$ has the property that $\mathcal{H}^{n-2,\infty}(M \cap \Sigma(a,r)) \leq \varepsilon r^{n-2}$ for all $a \in \partial\mathbb{B}^n$, $0 < r \leq \pi$, where $\varepsilon = \varepsilon(n) > 0$ is sufficiently small, then the following statement is true. If $x \in \mathbb{B}^n$ and Γ_x is the family of all curves $\gamma \subseteq S_x \setminus M$ with $\operatorname{diam}(\gamma) \geq c(1 - |x|)$, where $c > 0$, then

$$\operatorname{mod}_{n-1}^{\sigma} \Gamma_x \gtrsim 1. \tag{7.4}$$

Let $0 \leq t < 1$ be arbitrary. As in the proof of Lem. 7.1 let $\mathcal{C}_t = \{x \in \mathcal{C} : B_x \cap \partial B(0,t) \neq \emptyset\}$. To estimate $\tilde{\Phi}_\rho^{n-1,\delta}(\partial\mathbb{B}^n \setminus M)$ where $\rho = a_f$ and $\delta \leq 1 - t$ suppose $\mathcal{E} \subseteq \mathcal{C}$ is a set such that $\bigcup_{y \in \mathcal{E}} S_y \supseteq \partial\mathbb{B}^n \setminus M$ and $1 - |y| \leq \delta$ for $y \in \mathcal{E}$.

Define $\mathcal{E}(x) = \{y \in \mathcal{E} : S_y \cap S_x \neq \emptyset\}$ for $x \in \mathcal{C}_t$. Then $\operatorname{diam}(S_y) \approx 1 - |y| \leq \delta \leq 1 - t \approx 1 - |x|$ whenever $x \in \mathcal{C}_t$ and $y \in \mathcal{E}(x)$. As in the proof of Lem. 7.1 for every $x \in \mathcal{C}_t$ we can find a point $x' \in [0, x]$ such that

$$r_{x'} \approx r_x \text{ and } S_{x'} \supseteq \bigcup_{y \in \mathcal{E}(x)} S_y \supseteq S_x \setminus M. \tag{7.5}$$

In particular, $\gamma \subseteq \bigcup_{y \in \mathcal{E}(x)} S_y$ for every $\gamma \in \Gamma_x$, $x \in \mathcal{C}_t$. The Main Modulus Estimate, (7.4), and (7.5) then imply

$$r_x^{n-1} \lesssim \sum_{y \in \mathcal{E}(x)} r_y^{n-1} \quad \text{for} \quad x \in \mathcal{C}_t. \tag{7.6}$$

As in the proof of Lem. 7.1 we have that

$$\#\{x \in \mathcal{C}_t : y \in \mathcal{E}(x)\} \lesssim 1 \quad \text{for} \quad y \in \mathcal{E}.$$

Hence by Lem. 7.1 (a)

$$I_{n-1}(a_f, t) \approx \sum_{x \in \mathcal{C}_t} r_x^{n-1} \lesssim \sum_{x \in \mathcal{C}_t} \sum_{y \in \mathcal{E}(x)} r_y^{n-1} \lesssim \sum_{y \in \mathcal{E}} r_y^{n-1}.$$

The constants involved in this inequality only depend on n and K and not on t and $\mathcal{E}$. Therefore,

$$I_{n-1}(a_f, t) \lesssim \tilde{\Phi}_\rho^{n-1,\delta}(\partial\mathbb{B}^n \setminus M) \quad \text{if} \quad \delta \leq 1 - t. \tag{7.7}$$

Hence by Lem. 3.3 and Lem. 6.2 we have that

$$I_{n-1}(a_f, t) \lesssim \mathcal{H}^{n-1}(f(\partial\mathbb{B}^n \setminus M)) \quad \text{for} \quad 0 \le t < 1.$$

From this (b) follows.

The proof of (a) runs along similar lines. In this case the crucial inequality (7.6) follows from Prop. 4.8 taking for γ some compact curve with $\gamma \subseteq S_x$ and $\mathrm{diam}(\gamma) \approx 1 - |x|$. □

We can now prove the following version of the Riesz-Privalov Theorem for planar quasiconformal mappings in the unit disc.

Theorem 7.3. *Suppose $f\colon \mathbb{B}^2 \to \Omega \subseteq \mathbb{R}^2$ is a K-quasiconformal map. Then*

$$(1/c)||a_f||_1 \le \mathcal{H}^1(\partial\Omega) \le c||a_f||_1,$$

where $c = c(K) \ge 1$.

Proof. We have $||a_f||_1 \lesssim \mathcal{H}^1(\partial\Omega)$ by Prop. 7.2. Hence if suffices to show $\mathcal{H}^1(\partial\Omega) \lesssim ||a_f||_1$. For this purpose let $C < \mathcal{H}^1(\partial\Omega)$ be arbitrary. Since $\lim_{\delta\to 0}\mathcal{H}^{1,\delta}(\partial\Omega) = \mathcal{H}^1(\partial\Omega)$, there exists $\delta > 0$ such that $\mathcal{H}^{1,\delta}(\partial\Omega) > C$ and $2\delta < \mathrm{diam}(\partial\Omega)$.

The set $M = \{f(\zeta) : \zeta \in \partial\mathbb{B}^2,\ \lim_{r\to 1} f(r\zeta) \text{ ex.}\}$ is dense in $\partial\Omega$. To see this note first that the set of rectifiably accessible boundary points of Ω is dense in $\partial\Omega$. For if $y \in \partial\Omega$ is arbitrary, then there exist points $z \in \Omega$ arbitrarily close to y. If $y' \in \partial\Omega$ is a boundary point with $|z - y'| = \mathrm{dist}(z, \partial\Omega)$, then $[z, y') \subseteq \Omega$, so y' is rectifiably accessible. Moreover, $|y' - y| \le 2|z - y|$, and so there are rectifiably accessible boundary points arbitrarily close to y.

Secondly, note that if $\gamma\colon [0,1] \to \mathbb{R}^2$ is a rectifiable curve with $\gamma([0,1)) \subseteq \Omega$ and $\gamma(1) \in \partial\Omega$, then $\lim_{t\to 1} f^{-1}\circ\gamma(t) \in \partial\mathbb{B}^2$ exists. Calling this limit ζ, it follows that the radial limit of f at ζ exists, and we have $f(\zeta) = \gamma(1)$.

Therefore, the image set of the points of $\partial\mathbb{B}^2$ where the radial limit of f exists contains the rectifiably accessible boundary points of Ω. Hence the set is dense.

It follows that $\bigcup_{y\in M} B(y, \delta/5) \supseteq \partial\Omega$. By a standard covering argument we can choose a subset $N \subseteq M$ such that the balls $B(y, \delta/5)$, $y \in N$, are pairwise disjoint and $\bigcup_{y\in N} B(y,\delta) \supseteq \partial\Omega$. The set N is countable, and since $2\delta < \mathrm{diam}(\partial\Omega)$ we have $\#N \ge 2$. The definition of $\mathcal{H}^{1,\delta}(\partial\Omega)$ shows that

$$C < \mathcal{H}^{1,\delta}(\partial\Omega) \le \sum_{y\in N} \delta.$$

Therefore, we can find $l \in \mathbb{N}$ with $2 \le l \le \#N$ such that $\delta l > C$, and find points $\zeta_\nu = e^{i\vartheta_\nu}$, $1 \le \nu \le l$, where $0 \le \vartheta_1 < \ldots < \vartheta_l < 2\pi$, such that the radial limit $f(\zeta_\nu) = \lim_{r\to 1} f(r\zeta_\nu)$ exists and $|f(\zeta_\nu) - f(\zeta_{\nu+1})| \ge \frac{2}{5}\delta$ for $\nu \in \{1, \ldots, l\}$ (we set $f(\zeta_{l+1}) = f(\zeta_1)$).

Now choose $t < 1$ close enough to 1 such that $|f(t\zeta_\nu) - f(t\zeta_{\nu+1})| > \frac{1}{5}\delta$ and $d_h(t\zeta_\nu, t\zeta_{\nu+1}) \ge 1$ for $1 \le \nu \le l$. If l_Ω is the internal Euclidean metric in Ω and $\rho = a_f$, then $d_\rho(x,y) \approx l_\Omega(f(x), f(y))$, whenever $d_h(x,y) \ge 1$. Thus we get

$$\begin{aligned}C < \delta l &\le 5\sum_{\nu=1}^{l}|f(t\zeta_\nu)-f(t\zeta_{\nu+1})| \le 5\sum_{\nu=1}^{l} l_\Omega(f(t\zeta_\nu), f(t\zeta_{\nu+1}))\\ &\lesssim \sum_{\nu=1}^{l} d_\rho(t\zeta_\nu, t\zeta_{\nu+1}) \le \sum_{\nu=1}^{l}\int_{\vartheta_\nu}^{\vartheta_{\nu+1}} \rho(te^{i\vartheta})\,d\vartheta\\ &= \int_{\partial\mathbb{B}^2} \rho(t\zeta)\,d\sigma_1(\zeta) = 2\pi I_1(a_f,t) \le 2\pi||a_f||_1.\end{aligned}$$

Since this is true for arbitrary $C < \mathcal{H}^1(\partial\Omega)$ and the constants involved in this inequality only depend on K, we get $\mathcal{H}^1(\partial\Omega) \lesssim ||a_f||_1$ as desired. □

Lemma 7.4. *Suppose $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map, $\zeta \in \partial_\rho\mathbb{B}^n$, and $x \in [0,\zeta)$. Then there exists a constant $c = c(n,K) > 0$ such that*

$$|f(x) - f(\zeta)| \le c\,\mathrm{length}_\rho([x,\zeta)).$$

Proof. Note that $\mathrm{length}_\rho([x,\zeta)) < \infty$ and $\lim_{r\to1} f(r\zeta) = f(\zeta)$ exists. Let $x_0 = x$ and define $x_k \in [x,\zeta)$ such that $x_k \in [x_{k-1},\zeta)$ and $d_h(x_{k-1},x_k) = 1$ for $k \in \mathbb{N}$. Let $a_k = \mathrm{length}_\rho([x_k,x_{k+1}])$, $k \in \mathbb{N}_0$. Then

$$l_\Omega(f(x_k), f(x_{k+1})) \approx d_\rho(x_k,x_{k+1}) \lesssim a_k \quad \text{for} \quad k \in \mathbb{N}_0.$$

Therefore we can choose a curve γ_k in Ω joining $f(x_k)$ and $f(x_{k+1})$ such that $\mathrm{length}(\gamma_k) \lesssim a_k$. Then $\gamma = \gamma_0 \cup \gamma_1 \cup \dots$ is a curve in Ω with initial point $f(x)$. Moreover $\mathrm{length}(\gamma) \lesssim \sum_{k=0}^{\infty} a_k = \mathrm{length}_\rho([x,\zeta)) < \infty$. Since $a_k \to 0$, we have $\mathrm{diam}(\gamma_k) \to 0$ for $k \to \infty$. Since $f(x_k) \to f(\zeta)$ for $k \to \infty$, the curve γ has the limit $f(\zeta)$ as we travel along the curve starting at the initial point. Hence $|f(\zeta) - f(x)| \le \mathrm{length}(\gamma) \lesssim \mathrm{length}_\rho([x,\zeta))$. □

Definition 7.5. *If $\Omega \subseteq \mathbb{R}^n$ is a region and $\lambda \ge 1$, then the λ-porous part of the boundary of Ω is*

$$\begin{aligned}\partial_\lambda\Omega := \{y \in \partial\Omega :\exists \text{ sequence } (y_l) \text{ in } \Omega \text{ s.t.}\\ \lim_{l\to\infty} y_l = y \text{ and } |y_l - y| \le \lambda\,\mathrm{dist}(y_l,\partial\Omega) \text{ for } l \in \mathbb{N}\}.\end{aligned}$$

Note that $\partial_{\lambda'}\Omega \subseteq \partial_\lambda\Omega$ if $\lambda' \le \lambda$.

Theorem 7.6. *Suppose $n \ge 3$ and $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ is a K-quasiconformal map. Then there exist $\lambda = \lambda(n,K) \ge 1$ and $c = c(n,K) > 0$ such that*

$$||a_f||_{n-1}^{n-1} \le c\mathcal{H}^{n-1}(\partial_\lambda\Omega).$$

Proof. By Prop. 7.2 (b) it suffices to find $\lambda = \lambda(n,K) \ge 1$ and a set $M \subseteq \partial\mathbb{B}^n$ such that $\mathcal{H}^{n-2}(M) = 0$ and $f(\partial\mathbb{B}^n \setminus M) \subseteq \partial_\lambda\Omega$.

By Thm. 5.2 in [3] we can choose $M \subseteq \partial\mathbb{B}^n$ such that $\mathcal{H}^{n-2}(M) = 0$ and $a_f(t\zeta) = \rho(t\zeta) = o((1-t)^{-2/n})$ as $t \to 1$ for all $\zeta \in \partial\mathbb{B}^n \setminus M$.

The proof of Lem. 7.5 in [3] shows that there exists $\lambda_1 = \lambda_1(n) \ge 1$ such that for every $\zeta \in \partial\mathbb{B}^n\setminus M$ there exists a sequence (x_l) on $[0,\zeta)$ with $(x_l) \to \zeta$ and $\mathrm{length}_\rho([x_l,\zeta)) < \lambda_1 r_{x_l}$ for $l \in \mathbb{N}$. Since $r_{x_l} \approx \delta_\Omega(f(x_l))$, it follows from Lem. 7.4 that there exists $\lambda_2 = \lambda_2(n,K) > 0$ such that

$$|f(x_l) - f(\zeta)| \le \lambda_2\delta(f(x_l)) \quad \text{for} \quad l \in \mathbb{N}. \tag{7.8}$$

Note that $\delta_\Omega(f(x_l)) \approx r_{x_l} = \rho(x_l)(1-|x_l|) = o((1-|x_l|)^{1-2/n})$ as $l \to \infty$. Therefore, $\delta_\Omega(f(x_l)) \to 0$ for $l \to \infty$. Hence (7.8) implies $f(\zeta) \in \partial_{\lambda_2}\Omega$ for all $\zeta \in \partial\mathbb{B}^n \setminus M$. □

Lemma 7.7. *Suppose $n \geq 3$ and $\rho\colon \mathbb{B}^n \to (0,\infty)$ is a continuous density satisfying* HI(A) *and* VG(B)*. Then there exist constants $L = L(A,B,n) > 0$ and $C = C(A,B,n) > 0$ with the following property.*

If $\mathcal{F} \subseteq \mathcal{C}$ and $d_\rho(x,y) > L(r_x + r_y)$ for all $x, y \in \mathcal{F}$, $x \neq y$, then

$$\sum_{x \in \mathcal{F}} r_x^{n-1} \leq C||\rho||_{n-1}^{n-1}.$$

Proof. In this proof the constants of comparability will depend only on A, B, and n.

We may assume that the set $\mathcal{F}$ is finite and so $\mathcal{F} = \{y_1, \dots, y_l\}$. We have to show that if $L = L(A,B,n)$ is sufficiently large and the ρ-balls $\Omega_\nu = B_\rho(y_\nu, Lr_{y_\nu})$ are pairwise disjoint, then $\sum_{y \in \mathcal{F}} r_y^{n-1} \lesssim ||\rho||_{n-1}^{n-1}$. We proceed in several steps.

I. If $L \geq L_1(A,B,n)$ is large enough, then $B_{y_\nu} \subseteq \Omega_\nu$, $S_{y_\nu} \subseteq R(\Omega_\nu)$, and there exist $\zeta_\nu \in S_{y_\nu}$ such that Ω_ν contains a tail of $[0,\zeta_\nu)$ for all $\nu \in \{1,\dots,l\}$.

The first statement is clear and the second follows from Lem. 4.6. For the third statement note that $\operatorname{mod}_n \Gamma_\nu \gtrsim 1$, where Γ_ν is the family of curves γ in $\mathbb{B}^n$ connecting B_x and S_x. From Lem. 5.3 it follows that $\operatorname{length}_\rho(\alpha_\nu) \lesssim r_{y_\nu}$ for some $\alpha_\nu \in \Gamma_\nu$. If ζ_ν is the end point of α_ν on S_{y_ν}, then for some $y'_\nu \in B_{y_\nu} \cap [0,\zeta_\nu)$ we have $\operatorname{length}_\rho([y'_\nu,\zeta_\nu)) \lesssim r_{y_\nu}$. Since $\operatorname{diam}_\rho(B_{y_\nu}) \lesssim r_{y_\nu}$ it follows $d_\rho(y_\nu, x) \lesssim r_{y_\nu}$ for all x on the tail $[y'_\nu, \zeta_\nu)$ of $[0,\zeta_\nu)$.

II. Suppose $L \geq L_1$. For $\nu, \mu \in \{1,\dots,l\}$, $\nu \neq \mu$, we have $R(\Omega_\nu) \subsetneqq R(\Omega_\mu)$ or $R(\Omega_\mu) \subsetneqq R(\Omega_\nu)$ or $R(\Omega_\nu) \cap R(\Omega_\mu) = \emptyset$.

In view of Lem. 4.1 and Lem. 4.2 it suffices to show that $\nu \neq \mu$ implies $R(\Omega_\nu) \neq R(\Omega_\mu)$; recall that $\Omega_\nu \cap \Omega_\mu = \emptyset$ provided $\nu \neq \mu$. If we travel on the ray $[0, y_\nu]$ from 0 until we hit one of the sets $\bar{\Omega}_\nu$ or $\bar{\Omega}_\mu$, we see from the fact that each boundary point of Ω_ν or Ω_μ in $\mathbb{B}^n$ is accessible that there exists a curve α in $\mathbb{B}^n$ connecting one of the sets Ω_ν or Ω_μ with the origin without hitting the other one. Assume α connects Ω_ν to 0 and $\Omega_\mu \cap \alpha = \emptyset$, say. By I. there exists a point $\zeta_0 \in \partial\mathbb{B}^n$ such that Ω_ν contains a tail of $[0,\zeta_0)$. Since Ω_ν is connected, any tail of $[0,\zeta_0)$ can be connected in Ω_ν with the initial point of α. Since $\Omega_\nu \cap \Omega_\mu = \emptyset$, we see that any tail of $[0,\zeta_0)$ can be connected to 0 by a curve that avoids Ω_μ. Therefore, $\zeta_0 \notin R(\Omega_\mu)$. On the other hand, $\zeta_0 \in R(\Omega_\nu)$, since Ω_ν contains a tail of $[0,\zeta_0)$. Hence $R(\Omega_\nu) \neq R(\Omega_\mu)$.

III. For each $\nu \in \{1,\dots,l\}$ we can find points $y'_\nu, y''_\nu \in \mathbb{B}^n$ such that $\operatorname{dist}_h(y_\nu, y'_\nu) \lesssim 1$, $\operatorname{dist}_h(y_\nu, y''_\nu) \lesssim 1$, and

$$S_{y'_\nu} \supseteq 2S_{y_\nu} \supseteq S_{y_\nu} \supseteq 2S_{y''_\nu}. \tag{7.9}$$

Moreover, we can require

$$\operatorname{dist}(\partial\mathbb{B}^n \setminus S_{y_\nu}, S_{y''_\nu}) \gtrsim 1 - |y_\nu| \quad \text{for} \quad \nu \in \{1,\dots,l\}. \tag{7.10}$$

Then

$$r_{y'_\nu} \approx r_{y_\nu} \approx r_{y''_\nu} \text{ and } 1-|y'_\nu| \approx 1-|y_\nu| \approx 1-|y''_\nu| \tag{7.11}$$

for $\nu \in \{1,\dots,l\}$.

For $t \in [0,1)$, let $\mathcal{C}_t = \{x \in \mathcal{C} : B_x \cap \partial B(0,t) \neq \emptyset\}$. Then by Lem. 7.1 we have that $\sum_{x\in\mathcal{C}_t} r_x^{n-1} \approx I_{n-1}(\rho,t) \leq ||\rho||_{n-1}^{n-1}$. Note that $\bigcup_{x\in\mathcal{C}_t} S_x \supseteq \partial\mathbb{B}^n$ and $\operatorname{diam}(S_x) \approx 1-t$ for $x \in \mathcal{C}_t$. So by (7.9) and (7.10), we can fix $s \in [0,1)$ sufficiently close to 1 such that $S_x \cap S_{y_\nu} \neq \emptyset$ implies $S_x \subseteq S_{y'_\nu}$ and $S_x \cap S_{y''_\nu} \neq \emptyset$ implies $S_x \subseteq S_{y_\nu}$ for $\nu \in \{1,\dots,l\}$, $x \in \mathcal{C}_s$.

We then have

$$\begin{aligned} S_{y'_\nu} &\supseteq \bigcup\{S_x : x \in \mathcal{C}_s,\, S_x \cap S_{y_\nu} \neq \emptyset\} \\ &\supseteq \bigcup\{S_x : x \in \mathcal{C}_s,\, S_x \subseteq S_{y_\nu}\} \supseteq S_{y''_\nu} \quad \text{for} \quad \nu \in \{1,\dots,l\}. \end{aligned} \tag{7.12}$$

IV. For $\nu \in \{1,\dots,l\}$ let $\mathcal{E}_\nu \subseteq \mathcal{C}_s$ be the set of all $x \in \mathcal{C}_s$ such that $S_x \subseteq S_{y_\nu}$ and the set $R(\Omega_\nu)$ is minimal among the sets $R(\Omega_\mu)$ for which $S_x \subseteq S_{y_\mu}$. Then $\mathcal{E}_\nu \cap \mathcal{E}_\mu = \emptyset$ for $\nu \neq \mu$. For if $x \in \mathcal{E}_\nu \cap \mathcal{E}_\mu$, then $S_x \subseteq S_{y_\nu} \cap S_{y_\mu}$. By the minimality of $R(\Omega_\nu)$ and $R(\Omega_\mu)$, we have $R(\Omega_\nu) \cap R(\Omega_\mu) = \emptyset$ by II. But this is impossible since $\emptyset \neq S_x \subseteq S_{y_\nu} \cap S_{y_\mu} \subseteq R(\Omega_\nu) \cap R(\Omega_\mu)$ by I.

This shows that

$$\sum_{\nu=1}^{l} \sum_{x\in\mathcal{E}_\nu} r_x^{n-1} \leq \sum_{x\in\mathcal{C}_s} r_x^{n-1} \lesssim ||\rho||_{n-1}^{n-1}. \tag{7.13}$$

V. We claim that $r_{y_\nu}^{n-1} \lesssim \sum_{x\in\mathcal{E}_\nu} r_x^{n-1}$ for $\nu \in \{1,\dots,l\}$ if $L = L(A,B,n)$ is large enough. By (7.13) this will prove the Lemma.

In order to see that the claim is true, let $E_\nu = \bigcup\{S_x : x \in \mathcal{C}_s \setminus \mathcal{E}_\nu,\, S_x \subseteq S_{y_\nu}\}$ and Γ'_ν be the family of all curves in $\mathbb{B}^n$ connecting B_{y_ν} and E_ν. By (7.12) we have

$$S_{y'_\nu} \supseteq \bigcup_{x\in\mathcal{E}_\nu} S_x \supseteq S_{y''_\nu} \setminus E_\nu. \tag{7.14}$$

Suppose $\zeta \in E_\nu$. Then there exists $x \in \mathcal{C}_s \setminus \mathcal{E}_\nu$ such that $\zeta \in S_x \subseteq S_{y_\nu}$. This is only possible if there exists $\mu \neq \nu$ such that $S_x \subseteq S_{y_\mu}$ and $R(\Omega_\mu) \subsetneqq R(\Omega_\nu)$. Then $\zeta \in S_x \subseteq S_{y_\mu} \subseteq R(\Omega_\mu)$ by I. Since $\zeta \in S_{y_\nu}$, there exists $z_1 \in [0,\zeta) \cap B_{y_\nu} \subseteq [0,\zeta) \cap \Omega_\nu$. We must have $[z_1,\zeta) \cap \Omega_\mu \neq \emptyset$. To see this note that there exists a curve α connecting Ω_ν and 0 without hitting Ω_μ. The curve α is constructed as in II. by traveling from 0 to a point in Ω_ν until we first hit Ω_ν or Ω_μ. We must first hit Ω_ν by Lem. 4.3. If $[z_1,\zeta) \cap \Omega_\mu = \emptyset$, then we could connect z_1 to the initial point of α in Ω_ν without hitting Ω_μ. Then no tail of $[0,\zeta)$ would be separated by Ω_μ contradicting $\zeta \in R(\Omega_\mu)$. Hence there exists $z_2 \in \Omega_\mu \cap [z_1,\zeta)$.

Then

$$\begin{aligned} \operatorname{length}_\rho([z_1,\zeta)) &\geq \operatorname{length}_\rho([z_1,z_2]) \\ &\geq d_\rho(y_\nu,y_\mu) - d_\rho(z_1,y_\nu) - d_\rho(z_2,y_\mu) \\ &\geq (L-c_1)r_{y_\nu}, \end{aligned}$$

where $c_1 = c_1(A, B, n) > 0$ is a constant such that

$$\operatorname{diam}_\rho(B_{y_\nu}) \le c_1 r_{y_\nu} \quad \text{for} \quad \nu \in \{1, \dots, l\}.$$

If γ is any curve connecting B_{y_ν} to ζ we obtain from the Gehring-Hayman Theorem

$$\operatorname{length}_\rho(\gamma) + \operatorname{diam}_\rho(B_{y_\nu}) \ge c_2 \operatorname{length}_\rho([z_1, \zeta)),$$

where $c_2 = c_2(A, B, n) > 0$. This shows

$$\operatorname{length}_\rho(\gamma) \ge (c_2 L - c_3) r_{y_\nu} \quad \text{for all} \quad \gamma \in \Gamma'_\nu,\ \nu \in \{1, \dots, l\},$$

where $c_3 = c_3(A, B, n) > 0$.

Lem. 5.3 shows that $\operatorname{mod}_n \Gamma'_\nu$ is arbitrarily small, when L is large enough. In particular, we can choose $L_2 = L_2(A, B, n) \ge L_1$ such that for $L \ge L_2$ we have (cf. Lem. 5.2 and Prop. 5.1)

$$\mathcal{H}^{n-2,\infty}(E_\nu) \le c(1 - |y''_\nu|)^{n-2} \operatorname{mod}_n \Gamma'_\nu \le \varepsilon(1 - |y''_\nu|)^{n-2},$$

for $\nu \in \{1, \dots, l\}$, where $\varepsilon = \varepsilon(n) > 0$ is as in Lem. 5.2 and c as in Prop. 5.1. Lem. 5.2 then shows that if $L \ge L_2$, we have $\operatorname{mod}^\sigma_{n-1} \tilde\Gamma_\nu \gtrsim 1$, where $\tilde\Gamma_\nu$ is the family of all curves γ in $S_{y''_\nu} \setminus E_\nu$ such that $\operatorname{diam}(\gamma) \gtrsim (1 - |y''_\nu|)$. The Main Modulus Estimate and (7.11) then imply

$$r_{y_\nu}^{n-1} \lesssim \sum_{x \in \mathcal{E}_\nu} r_x^{n-1} \quad \text{for} \quad \nu \in \{1, \dots, l\}.$$

The proof is complete. □

Theorem 7.8. *Suppose $n \ge 3$ and let $f : \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ be a K-quasiconformal map. Then for all $\lambda \ge 1$ there exists $c = c(n, K, \lambda) > 0$ such that*

$$\mathcal{H}^{n-1}(\partial_\lambda \Omega) \le c||a_f||^{n-1}_{n-1}. \tag{7.15}$$

Proof. Let $C < \mathcal{H}^{n-1}(\partial_\lambda \Omega)$ be arbitrary. By definition of $\mathcal{H}^{n-1}(\partial_\lambda \Omega)$ there exists $\delta > 0$ such that $C < \mathcal{H}^{n-1,\delta}(\partial_\lambda \Omega)$. By definition of $\partial_\lambda \Omega$, for each $z \in \partial_\lambda \Omega$ we can find a point $y_z \in \Omega$ arbitrarily close to z such that $z \in \bar B(y_z, \lambda \delta_\Omega(y_z))$. In particular, we can require that $5\lambda\delta_\Omega(y_z) \le \delta$ for all $z \in \partial_\lambda\Omega$. A standard covering argument shows that there exists a set $Z \subseteq \partial_\lambda \Omega$ such that the balls $B(y_z, \lambda\delta_\Omega(y_z))$ are pairwise disjoint for $z \in Z$, and $\bigcup_{z \in Z} B(y_z, 5\lambda\delta_\Omega(y_z)) \supseteq \partial_\lambda\Omega$. Note that the set Z is countable. By the definition of $\mathcal{H}^{n-1,\delta}(\partial_\lambda\Omega)$ we have

$$C < (5\lambda)^{n-1} \sum_{z \in Z} \delta_\Omega(y_z)^{n-1}.$$

Choosing a finite subset of Z, we can find points $y_1, \dots, y_l \in \Omega$ such that the balls $B(y_\nu, \lambda\delta_\Omega(y_\nu))$ are pairwise disjoint and

$$C < (5\lambda)^{n-1} \sum_{\nu=1}^{l} \delta_\Omega(y_\nu)^{n-1}. \tag{7.16}$$

Let $x'_\nu \in \mathbb{B}^n$ be the preimage of y_ν under f; i.e., $y_\nu = f(x'_\nu)$ for $\nu \in \{1, \dots, l\}$.

If $\lambda \geq \lambda_1(n,K)$, then $d_h(x'_\nu, x'_\mu) \geq 1$ for $\nu \neq \mu$. For if $d_h(x'_\nu, x'_\mu) < 1$, $\nu \neq \mu$, then the quasihyperbolic distance $k_\Omega(y_\nu, y_\mu)$ of y_ν and y_μ is bounded by a constant $c_1 = c_1(n,K) > 0$. Hence

$$\log(\lambda) \leq \log\left(\frac{|y_\mu - y_\nu|}{\delta_\Omega(y_\nu)}\right) \leq k(y_\nu, y_\mu) \leq c_1,$$

and so $\lambda < \lambda_1 := \exp(c_1)$.

There are points $x_\nu \in \mathcal{C}$ such that $d_h(x_\nu, x'_\nu) \lesssim 1$. Then $r_{x_\nu} \approx r_{x'_\nu} \approx \delta_\Omega(y_\nu)$ and $d_\rho(x_\nu, x'_\nu) \lesssim r_{x_\nu}$. In particular, there exists a constant $c_2 = c_2(n,K) > 0$ such that $d_\rho(x_\nu, x'_\nu) \leq c_2 r_{x_\nu}$ and $(1/c_2) r_{x_\nu} \leq \delta_\Omega(y_\nu) \leq c_2 r_{x_\nu}$ for $\nu \in \{1, \dots, l\}$.

If $\lambda \geq \lambda_1$, then $d_h(x'_\nu, x'_\mu) \geq 1$ for $\nu \neq \mu$ and so $l_\Omega(y_\nu, y_\mu) \approx d_\rho(x'_\nu, x'_\mu)$. In particular, for some $c_3 = c_3(n,K) > 0$ we have $l_\Omega(y_\nu, y_\mu) \leq c_3 d_\rho(x'_\nu, x'_\mu)$, $\nu \neq \mu$. These inequalities and the estimate $\delta_\Omega(y_\nu) \geq (1/c_2) r_{x_\nu}$ for all ν show that for $\nu \neq \mu$

$$\begin{aligned} d_\rho(x_\nu, x_\mu) &\geq d_\rho(x'_\nu, x'_\mu) - c_2 r_{x_\nu} - c_2 r_{x_\mu} \\ &\geq \frac{1}{c_3} l_\Omega(y_\nu, y_\mu) - c_2 r_{x_\nu} - c_2 r_{x_\mu} \\ &\geq \frac{1}{c_3} \lambda(\delta_\Omega(y_\nu) + \delta_\Omega(y_\mu)) - c_2 r_{x_\nu} - c_2 r_{x_\mu} \\ &\geq (\frac{1}{c_2 c_3}\lambda - c_2)(r_{x_\nu} + r_{x_\mu}). \end{aligned}$$

This inequality shows that if $\lambda \geq \lambda_2(n,K) \geq \lambda_1$, then the points $x_1, \dots, x_l$ satisfy the hypothesis of Lem. 7.7. For some constant $c_4 = c_4(n,K) > 0$ we then have

$$\sum_{\nu=1}^{l} r_{x_\nu}^{n-1} \leq c_4 ||a_f||_{n-1}^{n-1}.$$

This implies

$$C < (5\lambda)^{n-1} \sum_{\nu=1}^{l} \delta_\Omega(y_\nu)^{n-1} \leq (5c_2\lambda)^{n-1} \sum_{\nu=1}^{l} r_{x_\nu}^{n-1} \leq c_4 (5c_2\lambda)^{n-1} ||a_f||_{n-1}^{n-1},$$

provided $\lambda \geq \lambda_2$.

Since $C < \mathcal{H}^{n-1}(\partial_\lambda \Omega)$ was arbitrary we conclude

$$\mathcal{H}^{n-1}(\partial_\lambda \Omega) \leq c_4 (5c_2\lambda)^{n-1} ||a_f||_{n-1}^{n-1}, \quad \text{if} \quad \lambda \geq \lambda_2.$$

If $1 \leq \lambda < \lambda_2$, then $\partial_\lambda \Omega \subseteq \partial_{\lambda_2} \Omega$ and so

$$\mathcal{H}^{n-1}(\partial_\lambda \Omega) \leq c_4 (5c_2\lambda_2)^{n-1} ||a_f||_{n-1}^{n-1}.$$

The proof is complete. □

Note that Thm. 1.4 follows from Thm. 7.6 and Thm. 7.8.

Remark 7.9. Again, appropriate versions of the main results of this section (Prop. 7.2, Thm. 7.3, Thm. 7.6, and Thm. 7.8) are true for continuous densities $\rho\colon \mathbb{B}^n \to (0,\infty)$ satisfying HI(A) and VG(B).

Example 7.10. The question arises whether Thm. 1.4 may be improved to a statement similar to Thm. 7.3 valid in higher dimensions.

The answer is in the negative: we will construct an example of a K-quasiconformal map $f\colon \mathbb{B}^n \to \Omega \subseteq \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(\partial\Omega) = \infty$, but $||a_f||_{n-1} < \infty$. Not all the details of the construction will be given.

The idea is to find a region $\Omega \subseteq \mathbb{R}^n$ such that $\mathcal{H}^{n-1}(\partial_\lambda \Omega) < \infty$ for all $\lambda \geq 1$, but $\mathcal{H}^{n-1}(\partial\Omega) = \infty$, and then show that Ω is the image of $\mathbb{B}^n$ under a quasiconformal map. The region Ω is constructed by putting thin tubes around a tree T consisting of line segments in $\mathbb{R}^n$.

To outline this construction, let W be the set of all finite words $w = a_1 \dots a_l$ with letters in the alphabet consisting of -1 and 1. The empty word e is considered a member of W. If $w = a_1 \dots a_l \in W \setminus \{e\}$, we let $w' = a_1 \dots a_{l-1} \in W$.

Denote by $e_1, \dots, e_n$ the unit vectors in $\mathbb{R}^n$. Suppose we are given a sequence of numbers (λ_ν) with $1 = \lambda_{-1} = \lambda_0 > \lambda_1 > \dots > 0$.

Now for $w \in W \setminus \{e\}$ define the point $P(w) \in \mathbb{R}^n$ by

$$P(w) = \sum_{\nu=1}^{l(w)} \lambda_{[\frac{\nu-1}{n}]} \frac{a_\nu}{2^{[\frac{\nu-1}{n}]+1}} e_{\nu - n[\frac{\nu-1}{n}]},$$

where $l(w)$ is the length of w and $w = a_1 \dots a_l$. Set $P(e) = 0$.

Moreover, for each $w \in W$ with $n|l(w)$ let $\tilde{Q}(w)$ and $Q(w)$ be the closed cubes with center $P(w)$, faces parallel to the coordinate axes, and side-length equal to $2\lambda_{l(w)/n-1} \cdot \frac{1}{2^{l(w)/n-1}}$ and $2\lambda_{l(w)/n} \cdot \frac{1}{2^{l(w)/n}}$, respectively. Then $\tilde{Q}(e) = Q(e) = [-1,1]^n$ and it can be shown by induction that for $w \in W$, $n|l(w)$, the points $P(wb_1 \dots b_n)$, $b_1, \dots, b_n \in \{-1, 1\}$ are the centers of the cubes which arise by decomposing the cube $Q(w)$ into 2^n subcubes by using the n hyperplanes perpendicular to the coordinate axes which pass through the center $P(w)$ of $Q(w)$. These subcubes are the cubes $\tilde{Q}(wb_1 \dots b_n)$, $b_1, \dots, b_n \in \{-1, 1\}$. The cubes $Q(wb_1 \dots b_n)$ are then obtained from $\tilde{Q}(wb_1 \dots b_n)$ by shrinking with the factor $\lambda_{l(w)/n+1}/\lambda_{l(w)/n}$ which is less than 1 by assumption.

If

$$M_l = \bigcup \{Q(w) : w \in W,\, l(w) = ln\} \quad \text{for} \quad l \in \mathbb{N}_0,$$

then $M_0 = [-1,1]^n \supseteq M_1 \supseteq \dots$. It is obvious that the set F of all limit points of $\{P(w) : w \in W\}$ is $F = \bigcap_{l \in \mathbb{N}_0} M_l$. Moreover, $m_n(M_l) = 2^n \lambda_l^n$, and thus $m_n(F) = 2^n \lim_{l\to\infty} \lambda_l^n = (2\lambda)^n$, where $\lambda = \lim_{l\to\infty} \lambda_l$. For $w \in W$ let $S(w)$ be the closed line segment $[P(w1), P(w-1)]$. Then $T = \bigcup_{w\in W} S(w)$ is a regular 3-tree with vertices $P(w)$, $w \in W \setminus \{e\}$.

We have $S(w_1) \cap S(w_2) \neq \emptyset$ if and only if $w_1 = w_2$ or $w_1' = w_2$ or $w_2' = w_1$. If none of these equalities holds, then

$$\operatorname{dist}(S(w_1), S(w_2)) \geq \left[\lambda_{[\frac{k-1}{n}]} - \lambda_{[\frac{k-1}{n}]+1}\right] \cdot \frac{1}{2^{[\frac{k-1}{n}]+1}},$$

where $k \in \mathbb{N}_0$ is the largest integer such that for the words $w_1 = a_1 \dots a_{l_1}$ and $w_2 = b_1 \dots b_{l_2}$ we have $a_\nu = b_\nu$ for $\nu \in \{1, \dots, k\}$.

For $w \in W$ define thin tubes by

$$C(w) = \bigcup_{x \in S(w)} B\left(x, \lambda_{l(w)/n} \cdot \frac{\delta_{l(w)}}{2^{l(w)/n}}\right),$$

where (δ_ν) is a sequence of numbers in $(0,1)$. Put $\Omega = \bigcup_{w \in W} C(w)$.

It can be shown that if (δ_ν) decreases to 0 sufficiently fast, then there exists a K-quasiconformal map $f\colon \mathbb{B}^n \to \Omega$. The dilatation K will be independent of the choice of (λ_ν) and (δ_ν).

The map $f\colon \mathbb{B}^n \to \Omega$ can be constructed inductively as follows. There exists a map $f_0\colon \mathbb{B}^n \to C(e)$, which is K-quasiconformal with K independent of δ_0. The map f_0 is then modified near the preimage of the points where $S(-1)$ and $S(1)$ penetrate $C(e)$. In this manner, we obtain a K-quasiconformal map $f_1\colon \mathbb{B}^n \to C(e) \cup C(-1) \cup C(1)$ with fixed K. Continuing in this manner and passing to the limit we obtain a K-quasiconformal map $f\colon \mathbb{B}^n \to \Omega$ (see [5, Sec. 4] and [10, Sec. 10] for the necessary details). Obviously,

$$\partial\Omega \subseteq F \cup \bigcup_{w \in W} \partial C(w).$$

This implies

$$\partial\Omega \setminus F \subseteq \partial_\lambda \Omega \quad \text{for} \quad \lambda \geq 1.$$

On the other hand, $F \subseteq \partial\Omega$, but $F \cap \partial_\lambda\Omega = \emptyset$ for $\lambda \geq 1$ if $(\delta_\nu) \to 0$.

By Thm. 1.4 we have for some $\lambda \geq 1$

$$||a_f||_{n-1}^{n-1} \approx \mathcal{H}^{n-1}(\partial_\lambda\Omega) \leq \mathcal{H}^{n-1}(\bigcup_{w \in W} \partial C(w))$$

$$\leq \sum_{w \in W} \mathcal{H}^{n-1}(\partial C(w)) \lesssim \sum_{l=0}^{\infty} \delta_l^{n-1} 2^n < \infty,$$

if (δ_ν) decreases to 0 fast enough.

On the other hand, if we choose (λ_ν) such that $\lambda = \lim_{\nu\to\infty} \lambda_\nu > 0$, then $m_n(F) > 0$ and so $\mathcal{H}^{n-1}(\partial\Omega) \geq \mathcal{H}^{n-1}(F) = \infty$.

References

[1] B. Bojarski, "Remarks on Sobolev imbedding inequalities". In: Proc. of the Conference on Complex Analysis (Joensuu 1987), Lecture Notes in Math. 1351, Springer, Berlin, 1988, pp. 52–68.

[2] M. Bonk, J. Heinonen and S. Rohde, "Doubling conformal densities", to appear in: J. Reine Angew. Math.

[3] M. Bonk, P. Koskela and S. Rohde, "Conformal metrics on the unit ball in Euclidean space", Proc. London Math. Soc. (3) 77 (1998), 635–664.

[4] B. Hanson, "Quasiconformal analogues of a theorem of Smirnov", Math. Scand. 75 (1994), 133–149.

[5] J. Heinonen, "The boundary absolute continuity of quasiconformal mappings II", Rev. Mat. Iberoamericana 12 (1996), 697–725.

[6] P. Koskela and S. Rohde, "Hausdorff dimension and mean porosity", Math. Ann. 309 (1997), 593–609.

[7] T. Tossavainen, "On the connectivity properties of the ρ-boundary of the unit ball", Ann. Acad. Sci. Fenn. Math. Diss. 123 (2000), 1–38.

[8] J. Väisälä, "Lectures on n-dimensional quasiconformal mappings", Lecture Notes in Mathematics 229, Springer, Berlin, 1971.

[9] J. Väisälä, "The wall conjecture on domains in Euclidean spaces", Manuscripta Math. 93 (1997), 515–534.

[10] J. Väisälä, "The free quasiworld. Freely quasiconformal and related maps in Banach spaces." In: Quasiconformal geometry and dynamics (Lublin 1996), Banach Center Publ. 48, Polish Acad. Sci., Warsaw, 1999, pp. 55–118.

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