Mappings of finite distortion: Sharp Orlicz-conditions

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1 Introduction

This paper is part of our program to establish the fundamentals of the theory of mappings of finite distortion [5], [1], [6], [9], [10] which form a natural generalization of the class of quasiregular mappings, also called mappings of bounded distortion. In the previous papers we considered mappings $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ of exponentially integrable distortion. Here and throughout the paper, $\Omega \subset \mathbb{R}^n$ is an open, connected set. If $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfies

$$|Df(x)|^n \leq K(x)J(x,f) \qquad \text{a.e.},$$

where $K(x) < \infty$ and if $J(\cdot, f) \in L^1_{\text{loc}}(\Omega)$, we say that f is a mapping of finite distortion. We call f a mapping of exponentially integrable distortion if furthermore $\exp(\lambda K) \in L^1_{\text{loc}}(\Omega)$ for some $\lambda > 0$. Mappings of exponentially integrable distortion in this sense were shown to have many of the nice properties of a mapping of bounded distortion. Regarding the necessity of the exponential integrability, an example from [9] shows that no topological properties like openness can be expected if we merely assume that $\exp(K/\log(e+K)^2)$ be integrable. In this paper we further examine the integrability assumptions on K. Let us replace the assumption $\exp(\lambda K) \in L^1_{\text{loc}}(\Omega)$ with $\exp(\Psi(K)) \in L^1_{\text{loc}}(\Omega)$. By the above, the critical power-like behavior of Ψ is linear. For the first theorem, we assume that Ψ is a strictly increasing, differentiable function, and we make the following two assumptions, the second of which is entirely harmless (see Remark 2.2):

(Ψ-1) $\displaystyle\int_1^\infty \frac{\Psi'(t)}{t}\,dt = \infty,$

(Ψ-2) $\displaystyle\lim_{t\to\infty} t\,\Psi'(t) = \infty.$

Then we have the following regularity result.

Theorem 1.1. *Suppose that* Ψ *satisfies* (Ψ-1) *and* (Ψ-2)*. Let* f *be a mapping of finite distortion* K *with* $\exp(\Psi(K)) \in L^1_{\mathrm{loc}}(\Omega)$ *and suppose that* $\det Df = J(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$*. Then* f *is continuous and either constant or both open and discrete. Moreover,* f *maps sets of Lebesgue measure zero to sets of measure zero.*

The continuity here means the existence of a continuous representative. The claims of Theorem 1.1 were established in [6], [9] and [10] for $\Psi(t) = \lambda t$, $\lambda > 0$. In the planar setting, Theorem 1.1 is partially covered by the results in [7].

As practical examples, Theorem 1.1 allows for

$$\Psi(t) = t\,, \ \frac{t}{\log(e+t)}\,, \ \frac{t}{\log(1+t)\,\log\log(e^e+t)}\,, \ldots$$

for any string of iterated logarithms. Regarding the sharpness, we will show, in particular, that

$$\Psi(t) = \frac{t}{t^{\epsilon}}\,, \ \frac{t}{\log^{1+\epsilon}(e+t)}\,, \ \frac{t}{\log(e+t)\,\log^{1+\epsilon}\log(e^e+t)}\,, \ldots$$

are not sufficient, for any $\epsilon > 0$. This easily follows from our next result that is a substantial improvement on the construction that we gave in [9], also see [7] regarding the part (a).

Theorem 1.2. *Suppose that* Ψ *is a strictly increasing function and*

$$\int_1^{\infty} \frac{\Psi'(s)}{s}\, ds < \infty. \tag{1.1}$$

(a) *There exists a mapping* $f : \mathbb{B} \to \mathbb{R}^n$ *of finite distortion* $K(x) = \frac{|Df(x)|^n}{J(x,f)}$*, with integrable Jacobian, with*

$$\int_{\mathbb{B}} \exp\left[\Psi(K(x))\right] dx < \infty$$

and so that f *maps* $\mathbb{B} \setminus \{0\}$ *homeomorphically onto the annulus* $\{x \in \mathbb{R}^n : 1 < |x| < b\}$*. In particular,* f *has no continuous representative.*

(b) *There exists a continuous, non-constant mapping* $f : Q_0 = [0,1]^n \to \mathbb{R}^n$ *of finite distortion* $K(x) = \frac{|Df(x)|^n}{J(x,f)}$*, with integrable Jacobian, with*

$$\int_{Q_0} \exp\left[\Psi(K(x))\right] dx < \infty$$

and so that f *is neither open, nor discrete, and it maps a set of measure zero to a set of positive measure.*

Theorem 1.1 is based on the arguments in [6], [9], [10] together with the following new observations. The integrability conditions on K, Ψ guarantee that $\Phi(|Df|)$ is locally integrable in Ω for a strictly increasing function Φ that satisfies the conditions

(Φ-1) $\displaystyle\int_1^\infty \frac{\Phi(t)}{t^{1+n}}\,dt = \infty.$

(Φ-2) There is $p \in (n-1, n)$ such that $t \mapsto t^{-p}\Phi(t)$ increases for large values of t.

Secondly, relying on recent results in [4], [11] and [3], we conclude that the point-wise Jacobian $J(x,f)$ then coincides with the so-called distributional Jacobian. This is the key fact in many of the estimates in [6], [9], [10] and we obtain the proposed topological and analytical results.

In the course of this argument we in fact establish the following result.

Theorem 1.3. *Let $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a mapping of finite distortion K. Suppose that $\Phi(|Df|) + K^q \in L^1_{\mathrm{loc}}(\Omega)$, with $q > n-1$ and Φ satisfying (Φ-1) and (Φ-2). Then f is continuous and either constant or both open and discrete. Moreover, f maps sets of Lebesgue measure zero to sets of measure zero.*

Here, the assumption (Φ-1) is critical: the examples referred to in part (b) of Theorem 1.2 satisfy

$$\Phi(|Df|) + \exp\left[\Psi(K(x))\right] \in L^1(Q_0);$$

see formulas (3.10) and (3.13). The assumption (Φ-2) is also necessary. For (a), it is enough to consider $f(x) = x(1+|x|)/|x|$, then $\Phi(Df) \in L^1(\mathbb{B})$ for any Φ violating (Φ-2). Concerning the necessity for (b), see Remark 3.1. Thus Theorem 1.3 gives a sharp extension of the celebrated results by Reshetnyak (c.f. [13], [14], [15]) on mappings of bounded distortion. It still remains unknown if the L^{n-1}-integrability of K is already sufficient under the given assumptions on $|Df|$; this is not known even when $|Df| \in L^n(\Omega)$. For this see [12], the monograph [8], and the references therein.

2 Proof of Theorem 1.1

We call a continuously differentiable and strictly increasing function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$ an Orlicz function.

In the course of this section, we associate with Ψ two other Orlicz functions (see equation (2.2)):

$$\begin{aligned} \psi(t) &= t\,\exp(\Psi(t)), \\ g(s) &= \frac{s}{\psi^{-1}(s)} - 1, \; s > 0, \text{ and } g(0) = 0. \end{aligned} \tag{2.1}$$

We notice that ψ is strictly increasing so that the inverse function ψ^{-1} makes sense. We immediately have

$$g(\psi(t)) = \exp(\Psi(t)) - 1. \tag{2.2}$$

In the first lemma we do not assume (Ψ-2).

Lemma 2.1. *Assume that Ψ is an Orlicz function satisfying* (Ψ-1). *Then*

(a) $\displaystyle\int_1^\infty \frac{g(s)}{s^2}\,ds = \infty$ *and*

(b) *given $a, b \geq 0$ we have*

$$g(ab) \leq a + \exp(\Psi(b)) - 1.$$

Proof. By the change of variables $s = \psi(t)$ and (2.2) we obtain

$$\begin{aligned}
\int_{\psi(1)}^\infty \frac{g(s)+1}{s^2}\,ds &= \int_1^\infty \frac{(g(\psi(t))+1)\,\psi'(t)}{\psi(t)^2}\,ds \\
&= \int_1^\infty \frac{\psi'(t)}{t\,\psi(t)}\,dt \\
&= \int_1^\infty \frac{(1+t\Psi'(t))\exp(\Psi(t))}{t^2\,\exp(\Psi(t))}\,dt \\
&= \int_1^\infty \left(\frac{1}{t^2} + \frac{\Psi'(t)}{t}\right)dt = \infty.
\end{aligned}$$

This proves (a). Regarding (b), we distinguish two cases; naturally we may assume that $a \neq 0 \neq b$. If $ab \leq \psi(b)$, then by (2.2)

$$g(ab) \leq g(\psi(b)) = \exp(\Psi(b)) - 1.$$

If $ab \geq \psi(b)$, then

$$g(ab) = \frac{ab}{\psi^{-1}(ab)} - 1 \leq \frac{ab}{b} - 1 = a - 1.$$

□

Remark 2.2. The condition (Ψ-1) is crucial for our considerations and shown to be necessary by our counterexamples. However, this condition alone is too weak for our purposes. To demonstrate this, let us consider a sequence $\{a_k\}$ with $a_{k+1} > k a_k$ and function Ψ which increases from $2a_{k-1}$ to a_k on $[a_k, 2a_k]$ and from a_k to $2a_k$ on $[2a_k, a_{k+1}]$. Then

$$\int_{a_k}^{2a_k} \frac{\Psi'(t)}{t}\,dt \geq \frac{a_k - 2a_{k-1}}{2a_k} \to \frac{1}{2}$$

and thus (Ψ-1) is verified. On the other hand, if

$$e^{2a_k} < a_{k+1}^{1/k}$$

then $\exp(\Psi(t))$ is not comparable with any t^q, $q > 1$. This means also that integrability of $\exp(\Psi(K))$ would not imply integrability of K^q.

This consideration shows that something should be added to the condition (Ψ-1). The condition (Ψ-1) implies that $\limsup_{t\to\infty} t\Psi'(t) = \infty$. It will not exclude important examples of Orlicz functions if we assume that this limsup turns to limit. Among power-like functions $\Psi(t) = t^\alpha$, (Ψ-1) corresponds to $\alpha < 1$, while (Ψ-2) is true for all $\alpha > 0$. This explains in what sense we regard (Ψ-2) to be "harmless".

Lemma 2.3. *Assume that Ψ is an Orlicz function satisfying (Ψ-2) and $\varepsilon \in (0,1)$. Then there exists $s_0 \in (0,\infty)$ such that the functions*

$$h : s \mapsto s^{\varepsilon-1} g(s)$$

is increasing on (s_0, ∞).

Proof. By (2.2) we rewrite

$$h(\psi(t)) = \psi(t)^{\varepsilon-1}(1 + g(\psi(t)) = t^{\varepsilon-1}\exp(\varepsilon\Psi(t)).$$

Hence

$$(h(\psi(t)))' = t^{\varepsilon-2}\exp(\varepsilon\Psi(t))\Big[\varepsilon\, t\, \Psi'(t) - (1-\varepsilon)\Big].$$

By (Ψ-2) we find a t_0 such that $h(\psi(t))$ increases for $t > t_0$. We conclude that $h(s) = s^{\varepsilon-1}(g(s)+1) - s^{\varepsilon-1}$ is increasing on (s_0,∞), where $s_0 = \psi(t_0)$. □

Now we collect results which enable us to derive regularity properties of a mapping of finite distortion from integrability of its differential. Let us consider a class $X(\Omega) \subset L^{n-1}(\Omega)$ of measurable functions on Ω satisfying the following two conditions:

(X-1) $J(\cdot, f) \in L^1_{\text{loc}}(\Omega)$ and $\det Df = \operatorname{Det} Df$ provided $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, $|Df| \in X(\Omega)$ and $J(\cdot, f) \geq 0$ a.e.

(X-2) if $g, h \geq 0$ are measurable, $g \leq ch$ for some $0 < c < \infty$ and $h \in X(\Omega)$, then $g \in X(\Omega)$.

Here the statement $\det Df = \operatorname{Det} Df$ means that

$$\int_\Omega \varphi\, J(x,f)\,dx = -\int_\Omega f_i\, J(x, f_1, \dots, f_{i-1}, \varphi, f_{i+1}, \dots, f_n)\,dx$$

for each $i = 1, \dots, n$ and for all $\varphi \in C_0^\infty(\Omega)$.

$t \mapsto t^{-p}\Phi(t)$ increases for large values of t.

The following proposition states the weak monotonicity of a mapping f, see [6, Definition 1.5], under assumptions which are adapted to our situation.

Proposition 2.4. *Let X be a space of measurable functions satisfying* (X-1) *and* (X-2). *Let $f = (f_1, \dots, f_n) \in W^{1,n-1}(\Omega)$ be a mapping of finite distortion with $|Df| \in X(\Omega)$. Then the coordinate functions of f are weakly monotone.*

Proof. We follow the standard idea as in [6, Section 4]. Let us consider a ball $B \subset\subset \Omega$. We prove e.g. that if $f_1 \le M$ on ∂B in the sense of traces, i.e. the positive part of $f_1 - M$ belongs to $W_0^{1,1}(B)$, then $f_1 \le M$ a.e. in B. We consider the truncated function $\tilde{f}_1 = \min(f_1, M)$ and the mapping $\tilde{f} = (\tilde{f}_1, f_2, \dots, f_n)$. Notice that, by (X-2), $|D\tilde{f}| \in X(\Omega)$. Let φ be a smooth test function with compact support in Ω such that $\varphi = 1$ on B. Since f_1 differs from $\tilde{f}_1$ only on B where $D\varphi = 0$, we have $f_1 D\varphi = \tilde{f}_1 D\varphi$, and thus

$$\begin{aligned}\int_\Omega \varphi\, J(x,f)\,dx &= -\int_\Omega f_1\, J(x,\varphi,f_2,\dots,f_n)\,dx \\ &= -\int_\Omega \tilde{f}_1\, J(x,\varphi,f_2,\dots,f_n)\,dx = \int_\Omega \varphi\, J(x,\tilde{f})\,dx.\end{aligned}$$

Hence, if we set $E = \{\tilde{f} \ne f\}$, we have

$$\int_E J(x,f)\,dx = \int_E J(x,\tilde{f})\,dx = 0.$$

Since $J(x,f) \ge 0$, it follows that $Jf = 0$ a.e. on E and thus, as f is a mapping of finite distortion, $Df = 0$ a.e. in E. It follows that $D(f_1 - \tilde{f}_1) = 0$ a.e. in Ω which yields that $f_1 = \tilde{f}_1 \le M$ a.e. in B. □

The following proposition summarizes the outcome of [9] and [10].

Proposition 2.5. *Let X be a space of measurable functions satisfying* (X-1) *and* (X-2). *Let $f = (f_1, \dots, f_n) \in W^{1,n-1}(\Omega)$ be a mapping of finite distortion $K \in L^q(\Omega)$, $q > n-1$, and $|Df| \in X(\Omega)$. Suppose that f is continuous. Then f is open and discrete and maps sets of measure zero to sets of measure zero.*

Proof. In [9, Theorems 2.1, 2.4, 3.1] it was shown that a mapping satisfying the hypotheses is open, discrete, and sense-preserving. By [10, Lemma 3.2], a continuous sense-preserving mapping $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, for which $\det Df = \operatorname{Det} Df$ maps sets of measure zero to sets of measure zero. We only need to check that $f \in W^{1,p}(\Omega, \mathbb{R}^n)$ for some $p > n-1$. Because of the locality of our claim, it suffices to check that $|Df| \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > n-1$, which follows by means of the Hölder inequality from the assumption $J(\cdot, f) \in L^1_{\mathrm{loc}}(\Omega)$ and from the fact that $K \in L^q_{\mathrm{loc}}(\Omega)$ with $q > n-1$. □

The assumption $\int_\Omega \Phi(|Df(x)|)\,dx < \infty$ with Φ as above has two important consequences.

Proposition 2.6. [11, Corollary 1.3] *Let Φ be an Orlicz-function that satisfies* (Φ-1) *and* (Φ-2)*. Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy $J(x, f) \geq 0$ a.e. $x \in \Omega$, and assume that $\int_\Omega \Phi(|Df(x)|)\, dx < \infty$. Then $\det Df \in L^1_{\mathrm{loc}}(\Omega)$ and $\det Df = \mathrm{Det}\, Df$.*

The following proposition is essentially [6, Theorem 1.6], but with slightly weakened assumptions on Φ.

Proposition 2.7. *Let Φ be an Orlicz-function that satisfies* (Φ-1) *and* (Φ-2)*. Let $u \in W^{1,1}(\Omega)$ be a weakly monotone function. and assume that $\int_\Omega \Phi(|Du(x)|)\, dx < \infty$. Then u has a continuous representative.*

Proof. We will follow the proof of [6, Theorem 1.6] with a small modification. By C we denote various constants which may change from line to line. Fix a point $a \in \Omega$ and $R > 0$ with $B(a, 2R) \subset \Omega$, and denote by $\omega(r)$ the essential oscillation of u on $B(a, r)$, $0 < r < R$. By [6, Lemma 7.2], for almost every radius $r \in (0, R)$ we have

$$\omega(r)^p \leq Cr^{p-n+1} \int_{\partial B(a,r)} |\nabla u|^p\, dS.$$

We consider a t_0 such that $t^{-p}\Phi(t)$ is increasing on (t_0, ∞) and a constant τ such that

$$\Phi(\tau) = \fint_{\partial B(a,r)} \Phi(\nabla u)\, dS, \tag{2.3}$$

where $\fint$ stands for the integral average. Write $\lambda = \max(\tau, t_0)$. Then we estimate

$$\begin{aligned}\int_{\partial B(a,r)} |\nabla u|^p\, dS &\leq \int_{\partial B(a,r)\cap\{|\nabla u|>\lambda\}} |\nabla u|^p\, dS + \int_{\partial B(a,r)\cap\{|\nabla u|\leq\lambda\}} |\nabla u|^p\, dS \\ &\leq \frac{\lambda^p}{\Phi(\lambda)} \int_{\partial B(a,r)} \Phi(\nabla u)\, dS + Cr^{n-1}\lambda^p \leq 2Cr^{n-1}\lambda^p.\end{aligned}$$

It follows that

$$\frac{\omega(r)}{Cr} \leq \lambda$$

and thus

$$\Phi\Big(\frac{\omega(r)}{Cr}\Big) \leq \Phi(\lambda) \leq \fint_{\partial B(a,r)} [\Phi(t_0) + \Phi(\nabla u)]\, dS.$$

Now we may continue as in the proof of [6, Theorem 1.6]. $\square$

Proof of Theorem 1.3. Proposition 2.6 shows that our space $L^\Phi(\Omega)$ qualifies for $X(\Omega)$ with (X-1) and (X-2). By Proposition 2.4 we see that the coordinate functions of f are weakly monotone which implies continuity by Proposition 2.7. Then Proposition 2.5 yields the conclusion. $\square$

Proof of Theorem 1.1. Let $\Phi(t) = g(t^n)$ where g is as in (2.1). Then by Lemma 2.1 (b)

$$\begin{aligned}\int_\Omega \Phi(|Df|)\,dx = \int_\Omega g(|Df|^n)\,dx &\le \int_\Omega g\Big(J(x,f)\,K(x)\Big)\,dx \\ &\le \int_\Omega J(x,f)\,dx + \int_\Omega \exp(K(x))\,dx < \infty.\end{aligned}$$

By Lemma 2.3 and Lemma 2.1 (a), the function Φ satisfies (Φ-1) and (Φ-2) (for all $p \in (n-1,n)$), and the inclusion $L^\Phi_{\text{loc}}(\Omega) \subset L^p_{\text{loc}}(\Omega)$ holds for all $p \in (n-1,n)$. Hence the assumptions of Theorem 1.3 are verified. $\square$

3 Proof of Theorem 1.2

We begin by giving examples of discontinuous mappings of finite distortion with the distortion function having the desired degree of regularity (also see [7]). We consider mappings $f : \mathbb{B} \to \mathbb{R}^n$ of the form

$$f(x) = \frac{x}{|x|}\,\rho(|x|). \tag{3.1}$$

The function $t \to \rho(t)$, for $0 \le t \le 1$, will continuously increase from the value 1 at $t = 0$ to $b > 1$ at $t = 1$. Thus f will map homeomorphically the punctured unit ball $\mathbb{B} \setminus \{0\}$ onto the annulus $\{x \in \mathbb{R}^n : 1 < |x| < b\}$. We may calculate the differential matrix of f and its determinant by using the familiar formulas

$$Df(x) = \frac{\rho(|x|)}{|x|}\,\mathbf{I} + \left(\rho'(|x|) - \frac{\rho(|x|)}{|x|}\right)\frac{x \otimes x}{|x|^2}, \tag{3.2}$$

where $x \otimes x$ is the $n \times n$ matrix whose i,j-entry equals $x_i x_j$, and

$$J(x,f) = \rho'(|x|)\left(\frac{\rho(|x|)}{|x|}\right)^{n-1}. \tag{3.3}$$

Our choice for ρ will satisfy

$$\rho'(t) \le \beta\,\frac{\rho(t)}{t} \tag{3.4}$$

for some $\beta \ge 1$. Consequently, the norm of differential matrix in question satisfies

$$|Df(x)| \le (\beta + 2)\,\frac{\rho(|x|)}{|x|} \tag{3.5}$$

and the dilatation function K satisfies

$$K(x) = \frac{|Df(x)|^n}{J(x,f)} \le (\beta+2)^n\,\frac{\rho(|x|)}{|x|\rho'(|x|)}. \tag{3.6}$$

We may assume that $\Psi(1)=1$. We define ρ by setting

$$\rho(t)=\exp\left(\lambda\int_{\Psi^{-1}(\log\frac{e}{t})}^{\infty}\frac{\Psi'(s)}{s}\,ds\right)$$

for $0<t<1$, where λ is a constant, whose value will be determined later. Using the change of variables

$$s=\Psi^{-1}\left(\log\frac{e}{r}\right)$$

we obtain

$$\rho(t)=\exp\left(\lambda\int_0^t\frac{dr}{r\Psi^{-1}\left(\log\frac{e}{r}\right)}\right).$$

For the Jacobian integral we compute

$$\int_{\mathbb{B}}J(x,f)\,dx=C(n)\int_0^1\rho^{n-1}(t)\rho'(t)\,dt=C(n)(\rho^n(1)-\rho^n(0))<\infty.$$

We also have

$$\frac{t\,\rho'(t)}{\rho(t)}=t\big(\log\rho(t)\big)'=\frac{\lambda}{\Psi^{-1}\left(\log\frac{e}{t}\right)}. \tag{3.7}$$

This quantity tends to zero as $t\to 0$ and thus there exists $t_0>0$ such that (3.4) follows with constant 1 for all $t\in(0,t_0)$. Fix $\lambda=3^n$. By (3.6) and (3.7) we obtain

$$\begin{aligned}\exp\Psi\left(K(x)\right) &\leq \exp\Psi\left(\frac{3^n\,\rho(|x|)}{|x|\,\rho'(|x|)}\right)\\ &\leq \exp\Psi\big(\Psi^{-1}\big(\log\frac{e}{|x|}\big)\big)=\frac{e}{|x|}\end{aligned}$$

for all $x\in B(0,t_0)\setminus\{0\}$. Hence $\exp\Psi\circ K\in L^1(\mathbb{B})$, as desired.

The construction we need for part (b) of Theorem 1.2 is a substantial improvement on the construction in [9]. For the convenience of the reader we present here also the part of the construction from [9] that need not be altered.

We begin by introducing some notation. Besides the usual euclidean norm $|x|=(x_1^2+\ldots+x_n^2)^{1/2}$ we will use the cubic norm $\|x\|=\max_i|x_i|$. Using the cubic norm, the x_0-centered closed cube with edge length $2r>0$ and sides parallel to coordinate axes can be represented in the form

$$Q(x_0,r)=\{x\in\mathbb{R}^n:\ \|x-x_0\|\leq r\}$$

We then call r the radius of Q. Let us denote $cQ(x_0,r)=Q(x_0,cr)$ if $c>0$. We will use the notation $a\lesssim b$ if there is a constant $c>0$ (not depending on (integration) variables or summation indices) such that $a \leq cb$, and we write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$. For technical reasons we will assume that $\Psi(1) = 1$

We will prove part (b) of Theorem 1.2 by giving a mapping $f : Q_0 \to \mathbb{R}^n$ so that $f = \mathrm{Id}$ on ∂Q_0, $J(x, f) < 0$ a.e. and so that the rest of the requirements hold for $|J(x, f)|$; the desired mapping is then obtained by employing an auxiliary reflection in a hyperplane.

In the following, we will construct a sequence of continuous, piecewise continuously differentiable mappings $f_k : Q_0 \to \mathbb{R}^n$. First we introduce a sequence of compact sets in Q_0 whose intersection is a Cantor set.

The unit cube Q_0 is first divided into 2^n cubes with radius $1/4$, which are each in turn divided into a subcube with radius $(1/4)/2$ and a difference of two cubes which we refer to as an annulus. The family $\mathcal{Q}_1$ consists of these 2^n subcubes. The remainder of the construction is then self-similar. The subcube is divided into 2^n cubes which are each in turn divided into a subcube with radius $4^{-2}/2$ and an annulus. The family $\mathcal{Q}_2$ consists of these 2^{2n} subcubes (see Figure 1). Continuing this way, we get the families $\mathcal{Q}_k$, $k = 1, 2, 3, \dots$, for which the radius of $Q \in \mathcal{Q}_k$ is $r(Q) = r_k = 2^{-2k-1}$ and the number of cubes in $\mathcal{Q}_k$ is $\#\mathcal{Q}_k = 2^{nk}$. It easily follows that the resulting Cantor set is of measure zero.

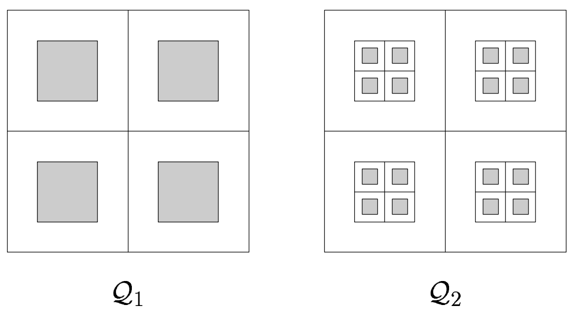

Figure 1: Families $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

We are now ready to define the mappings f_k. Define $f_0(x) = x$. We will give a mapping f_1 that leaves the boundaries $\partial(2Q)$, $Q \in \mathcal{Q}_1$ fixed, turns each annulus $2Q \setminus Q$ inside out and stretches the cube Q so that f_1 is continuous (see Figure 2). The Jacobian determinant J_{f_1} will be negative in each annulus $2Q \setminus Q$ and positive in each cube Q. Next, f_2 equals f_1 in the annulae $2Q \setminus Q$, $Q \in \mathcal{Q}_1$, turns each annulus $2Q \setminus Q$, $Q \in \mathcal{Q}_2$, inside out, stretches the cube Q and shifts the image so that f_2 is continuous. Moreover, J_{f_2} is negative a.e. in $Q_0 \setminus \bigcup_{Q \in \mathcal{Q}_2} Q$ and positive in $\bigcup_{Q \in \mathcal{Q}_2} Q$. We will then continue in this manner.

To be precise, let $f_0(x) = x$ on Q_0 and let a sequence $\{\epsilon_k\}_{k \in \mathbb{N}}$ of small

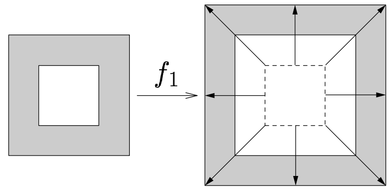

Figure 2: The mapping f_1 acting on $2Q,\ Q \in \mathcal{Q}_1$.

positive real numbers satisfy

$$\sum_{k=1}^{\infty} \epsilon_k < \infty. \tag{3.8}$$

This sequence will be fixed later. For $k = 1, 2, ...$ define

$$\varphi_k(r) = \begin{cases} 2^{-k-1}(1+\epsilon_1)\cdots(1+\epsilon_{k-1})(1+\frac{2r_k - r}{r_k}\epsilon_k), & r_k \le r \le 2r_k \\ 2^{-k-1}(1+\epsilon_1)\cdots(1+\epsilon_k)\frac{r}{r_k}, & 0 \le r \le r_k \end{cases}$$

and

$$f_k(x) = \begin{cases} f_{k-1}(x), & x \notin \bigcup_{Q\in\mathcal{Q}_k} 2Q \\ f_{k-1}(z(Q)) + \frac{x - z(Q)}{\|x-z(Q)\|}\varphi_k\big(\|x - z(Q)\|\big), & x \in 2Q,\ Q \in \mathcal{Q}_k. \end{cases}$$

Here $z(Q)$ is the center of the cube Q. Now

$$\log \prod_{j=1}^{k} \big(1+\epsilon_j\big) = \sum_{j=1}^{k} \log\big(1+\epsilon_j\big) \le \sum_{j=1}^{k} \epsilon_j,$$

and using the fact (3.8) we infer that

$$\prod_{j=1}^{\infty} \big(1+\epsilon_j\big) < \infty.$$

Thus

$$\prod_{j=1}^{k} \big(1+\epsilon_j\big) \approx 1, \qquad k = 1, 2, ... \tag{3.9}$$

Using this we obtain

$$|f_{k+1}(x) - f_k(x)| \lesssim 2^{-k}$$

and so the sum

$$\sum_{k=1}^{\infty} \big|f_{k+1}(x) - f_k(x)\big|$$

and the sequence (f_k) converge uniformly. Hence the limit $f = \lim_{k\to\infty} f_k$ is continuous. Clearly f is differentiable almost everywhere, its Jacobian determinant is strictly negative almost everywhere, and f is absolutely continuous on almost all lines parallel to coordinate axes.

We next estimate $|Df(x)|$, $|J(x,f)|$ and $K(x)$ at $x \in \operatorname{int}(2Q \setminus Q)$, $Q \in \mathcal{Q}_k$. Fix $k \in \mathbb{N}$. We see that, in the annulus $\operatorname{int}(2Q \setminus Q)$, f is a radial mapping: $f(x) = (x/\|x\|)\,\varphi_k(\|x\|)$. Hence we have

$$|Df(x)|/C_1(n) \le \max\left\{\frac{\varphi_k(\|x\|)}{\|x\|},\ |\varphi_k'(\|x\|)|\right\} \le C_1(n)|Df(x)|$$

and

$$J_f(x)/C_2(n) \le \frac{\varphi_k'(\|x\|)\varphi_k(\|x\|)^{n-1}}{\|x\|^{n-1}} \le C_2(n)J_f(x)$$

a.e. in $\operatorname{int}(2Q \setminus Q)$, see the formulas (3.2) and (3.3). By (3.8) and (3.9) we obtain

$$\big|Df(x)\big| \lesssim \frac{\varphi_k(r_k)}{r_k} \approx 2^k \tag{3.10}$$

and

$$J(x,f) \approx \left(\frac{\varphi_k(r_k)}{r_k}\right)^{n-1} \varphi_k'(r_k) \approx -\left(2^k\right)^{n-1} 2^k\,\epsilon_k = -2^{kn}\epsilon_k \tag{3.11}$$

and finally

$$K(x) := \frac{|DF(x)|}{|J(x,f)|} \lesssim \frac{2^{kn}}{2^{kn}\epsilon_k} = \frac{1}{\epsilon_k}. \tag{3.12}$$

Since

$$\left|\bigcup_{Q\in\mathcal{Q}_k} 2Q \setminus Q\right| \approx 2^{-kn}, \tag{3.13}$$

we obtain in view of (3.8)

$$\int_{Q_0} \big|J(x,f)\big|\,dx \lesssim \sum_{k=1}^{\infty} \epsilon_k < \infty.$$

By (3.12) there exists a constant β such that

$$K(x) \le \frac{\beta}{\epsilon_k},\quad x \in 2Q \setminus Q,\ Q \in \mathcal{Q}_k.$$

We now define the numbers ϵ_k explicitly by setting

$$\epsilon_k = \frac{\beta}{\Psi^{-1}(k)}. \tag{3.14}$$

Because $\int_1^\infty \frac{\Psi'(s)}{s}\,ds < \infty$, the change of variables

$$s = \Psi^{-1}(t)$$

shows that

$$\int_1^\infty \frac{dt}{\Psi^{-1}(t)} < \infty.$$

Thus the integral criterion for convergence of series establishes (3.8). By (3.12) and (3.14),

$$\exp\,\Psi\big(K(x)\big) \le \exp\,\Psi\left(\frac{\beta}{\epsilon_k}\right) = \exp\,k, \quad x \in 2Q \setminus Q,\ Q \in \mathcal{Q}_k,$$

and thus

$$\int_{Q_0} \exp\,\Psi\big(K(x)\big)\,dx \le C \sum_{k=1}^{\infty} \left(2^{-n}e\right)^k < \infty.$$

Next we will show that f maps a set of measure zero to a set of positive measure by showing that

$$Q_0 \subset f\Big(\bigcap_{k=1}^{\infty} \bigcup_{Q\in\mathcal{Q}_k} Q\Big);$$

recall that the Cantor set $\bigcap_{k=1}^{\infty}\bigcup_{Q\in\mathcal{Q}_k} Q$ has measure zero. From the construction it follows that for each $k = 1, 2, 3, \ldots$

$$f_k\Big(\bigcup_{Q\in\mathcal{Q}_k} Q\Big) \subset f_k\Big(\bigcup_{Q\in\mathcal{Q}_{k+1}} 2Q\Big) \subset f_{k+1}\Big(\bigcup_{Q\in\mathcal{Q}_{k+1}} Q\Big).$$

Since $Q_0 \subset f_1\big(\bigcup_{Q\in\mathcal{Q}_1} Q\big)$, denoting

$$H_k = \bigcup_{Q\in\mathcal{Q}_k} Q$$

we have $Q_0 \subset f_k(H_k) \subset f_l(H_k)$ for all $l \ge k \ge 1$. Now (H_k) is a decreasing sequence of compact sets, whence

$$Q_0 \subset \bigcap_{k=1}^{\infty} \bigcap_{l\ge k} f_l(H_k) \subset \bigcap_{k=1}^{\infty} f(H_k) \subset f\Big(\bigcap_{k=1}^{\infty} H_k\Big).$$

Notice that f is not open: it follows from the construction that $f(\partial Q_0) = \partial Q_0 \subset f(\operatorname{int} Q_0)$ whence $f(Q_0) = f(\operatorname{int} Q_0)$. Because $f(Q_0)$ is a nonempty compact set, $f(\operatorname{int} Q_0)$ is not open. To prove non-discreteness of f, let

$$G_k = \bigcup_{l \geq k} f\Big(\bigcup_{Q \in \mathcal{Q}_l} \operatorname{int} 2Q \setminus Q \Big).$$

Then the sets G_k are dense and open, and by the Baire category theorem their intersection is nonempty. But if $y \in \cap_k G_k$, then $f^{-1}(y)$ is an infinite compact set and thus it is not discrete.

Remark 3.1. The example can be easily modified to show sharpness of the condition (Φ-1) by setting

$$\epsilon_k = 2^{-kn}\Phi(2^k)$$

in place of (3.14).

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References

[1] Astala, K. Iwaniec, T. Koskela, P. and Martin, G.: Mappings of BMO-bounded distortion. Math. Ann. 317 (2000), no. 4, 703–726.

[2] David, G.: Solutions de l'equation de Beltrami avec $||\mu||_\infty = 1$. Ann Acad. Sci. Fenn. Ser. A I, Math. **13** No.1 (1988), 25–70.

[3] Giannetti, F., Iwaniec, T., Onninen, J. and Verde, A.: Estimates of Jacobians by subdeterminants. Manuscript.

[4] Greco, L. Sharp integrability of nonnegative Jacobians. Rend. Mat. Appl. (7) 18 (1998), no. 3, 585–600.

[5] Iwaniec, T., Koskela P. and Martin, G.: Mappings of BMO-distortion and Beltrami type operators. Manuscript.

[6] Iwaniec, T., Koskela P. and Onninen, J.: Mappings of finite distortion: Monotonicity and continuity. Invent. Math., to appear.

[7] Iwaniec, T. and Martin, G.: The Beltrami equation. Mem. Amer. Math. Soc., to appear.

[8] Iwaniec, T. and Martin, G.: Geometric function theory and nonlinear analysis. Oxford Mathematical Monographs, to appear.

[9] Kauhanen, J., Koskela, P. and Malý, J.: Mappings of finite distortion: Discreteness and openness. Arch. Rational Mech. Anal., to appear.

[10] Kauhanen, J., Koskela, P. and Malý, J.: Mappings of finite distortion: Condition N. Michigan Math. J., to appear.

[11] Koskela, P. and Zhong, X.: Minimal assumptions for the integrability of the Jacobian. Manuscript.

[12] Manfredi, J. and Villamor, E.: An extension of Reshetnyak's theorem. Indiana Univ. Math. J. **47** (1998) no. 3, 1131–1145.

[13] Reshetnyak, Yu. G.: Space Mappings with Bounded Distortion. Sibirsk. Mat. Z. **8** (1967), 629–658.

[14] Reshetnyak, Yu. G.: *Space Mappings with Bounded Distortion.* Trans. of Mathematical Monographs, Amer.Math. Soc, vol. 73, 1989.

[15] Rickman, S.: *Quasiregular mappings.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 26. Springer-Verlag, Berlin, 1993.

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