# Quasiregular Mappings and Young measures

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**Abstract**

$W^{1,p}$-Gradient Young measures supported in the set $Q_2(K)$ of two dimensional K-quasiconformal matrices are studied. It is proven that these Young measures can be generated by gradients of K-quasiregular mappings. This leads e.g to the 0-1 law for quasiregular $W^{1,p}$-Gradient Young measures and other quasiregular properties such as higher integrability.

## 1 Introduction

In this note we present some applications of quasiregular mappings to the theory of $W^{1,p}$- gradient Young measures. The $W^{1,p}$- gradient Young measures, in short $W^{1,p}$-GYM's, can be understood as a generalization of a gradient, describing for a sequence of functions $f_j \in W^{1,p}(\Omega; \mathbf{R}^n)$ the limit distribution of the values attained by their gradients; for details and definitions see Section 2. They have appeared as an indispensable tool in studying variational problems where the functional is not lower semicontinuous.

In many situations GYM's supported in a given subset $E \subset \mathbf{M^{n\times m}}$, the space of $n \times m$ matrices, share the properties of genuine gradient functions having a.e. value contained in $E$. Examples of this situation are the so called two and three matrices problem and the two well problem, see [9]. However, there are also counterexamples, the canonical one being the Scheffer-Tartar square example, see the discussion on the four matrices problem in [9].

The class of quasiregular mappings is known to have strong rigidity properties, with applications also to problems related to Young measures, see e.g [3], [4], [5], [12] . Therefore it is of interest to understand the relations between these mappings and the corresponding GYM's. In this case the supporting set $E$ above is equal to

$$Q_n(K) = \{A \in \mathbf{M^{n\times n}} : \|A\|^n \leq K \det(A)\},$$

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where $1 \leq K < \infty$. Matrices contained in this set are called $K$-quasiconformal matrices. Correspondingly for $\Omega \subset \mathbf{R}^n$, a function $f \in W^{1,n}_{loc}(\Omega; \mathbf{R}^n)$ with $Df(z) \in Q_n(K)$ a.e. is called $K$-quasiregular. If $f$ is also injective then it is called $K$-quasiconformal.

In the case of Young measures we set accordingly

**Definition 1.1.** A $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ is called *$K$-quasiregular* if it is supported on $Q_n(K)$, that is $\operatorname{supp}(\nu_z) \subset Q_n(K)$ for *a.e.* $z \in \Omega$.

In this paper we shall study in planar regions gradient Young measures with support in $Q_2(K)$. It is clear that the underlying Sobolev space $W^{1,p}$ affects the qualities of the corresponding Young measure. In particular, in the theory of two dimensional quasiregular mappings, see [1] [8], regularity and rigidity results are valid only if $p > \frac{2K}{K+1}$. Hence the same regularity assumption is made also in this work. The assumed lower bound $p > \frac{2K}{K+1}$ is still weaker than the *a priori* Sobolev-regularity $p = 2$ in the definition of a quasiregular mapping.

The basic result is the following

**Theorem 1.2.** *Suppose that $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary. Assume also that $p > \frac{2K}{K+1}$. Then a $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ is supported in $Q_2(K)$ if and only if it can be generated by a sequence of (gradients of) $K$-quasiregular mappings.*

Note however, that even if $p$ is large the generating $K$-quasiregular sequence lies in general only in $W^{1,q}$ for all $q < \frac{2K}{K-1}$. Furthermore by [7], Example 5.5.5., the result fails when $p < \frac{2K}{K+1}$.

Via the theorem quasiregular properties are induced on all gradient Young measures with support in $Q_2(K)$. To illustrate this, recall [8] that the Jacobian of a non-constant quasiregular mapping can vanish only on a set of measure zero. In the case Young measures we have the following counterpart.

**Theorem 1.3.** (The 0-1 Law for Quasiregular $W^{1,p}$-GYM 's.)
*Let $\Omega \subset \mathbf{R}^2$ and let $\{\nu_z\}_{z\in\Omega}$ be a $W^{1,p}$-gradient Young measure supported in $Q_2(K)$ for some $K < \infty$. Assume that $p > \frac{2K}{K+1}$. Then either*

$\nu_z(\{0\}) = 0$ *for almost every* $z \in \Omega$, *or*

$\nu_z = \delta_0$ *for almost every* $z \in \Omega$.

As a corollary of this fact we recover the result of Šverák concerning the three matrices problem.

**Corollary 1.4.** *Let $E = \{A_1, A_2, A_3\} \subset \mathbf{M^{n\times m}}$ with $rank(A_i - A_j) > 1$ for $i \neq j$. Then every $W^{1,1} - GYM$ $\{\nu_z\}_{z\in\Omega}$ supported in $E$ satisfies $\nu_z \equiv \delta_{A_i}$ for some $i$.*

Šverák gave two deep proofs of this fact, see [12] [13] . Recently Ball and James [3] presented a different proof based on the so-called volume fraction lemma which also uses quasiregular mappings .

To obtain Corollary 1.4 from Theorem 1.3 we extend one of the methods of solving the corresponding exact problem. It is shown in [12] that by general considerations, there is no loss of generality assuming $n = m = 2$.

Then, since at least two of the determinants $\det(A_i - A_j)$ have the same sign, using affine change of variables we are reduced to the case where $A_1 = 0, \det(A_2) > 0$ and $\det(A_3) > 0$. In this case, the set $E$ is contained in $Q_2(K)$ for some $K$. Since $E$ is a compact set, we can apply the Zhang Lemma (See [14], [10]) to infer that any $W^{1,1}$-GYM $\{\nu_z\}_{z\in\Omega}$ supported in $E$ is in fact a $W^{1,\infty}$-GYM. Therefore by Theorem 1.3, the Young measure $\{\nu_z\}_{z\in\Omega}$ of the Corollary satisfies either $\nu_z \equiv \delta_{A_1}$ or $\{\nu_z\}_{z\in\Omega}$ is supported in $\{A_2, A_3\}$. In the latter case, repeating the argument, or using the fact that $A_2, A_3$ are not rank-one connected, we have either $\nu_z \equiv \delta_{A_2}$ or $\nu_z \equiv \delta_{A_3}$.

For homogeneous gradient Young measures the conclusions of Theorem 1.2 can still be considerably strengthened, to a "global" version.

**Theorem 1.5.** *Let $\nu$ be an homogeneous $W^{1,p}$-GYM with support contained in $Q_2(K)$. Assume $p > \frac{2K}{K+1}$. Then there are $K-$ quasiconformal mappings $F_j : \mathbf{R}^2 \to \mathbf{R}^2$ such that sequence $\{DF_j|_{\mathbb{D}}\}_{j\in\mathbb{N}}$ generates $\nu$.*

Here $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk of $\mathbf{R}^2$. Note also the extra feature of the theorem that the generating sequence consists of gradients of *homeomorphisms* of $\mathbf{R}^2$.

One of the basic aspects of quasiregularity is the automatically improving smoothness: We have the exponents

$$q_K < n < p_K$$

such that if $\Omega \subset \mathbf{R}^n$, $q > q_K$ and $f \in W^{1,q}_{loc}(\Omega;\mathbf{R}^n)$ satisfies $Df(x) \in Q_n(K)$ for a.e. $x$ then actually $f \in W^{1,p}_{loc}(\Omega;\mathbf{R}^n)$ for all $p < p_K$. When the dimension $n = 2$ we have $q_K = \frac{2K}{K+1}$ and $p_K = \frac{2K}{K-1}$ . In higher dimensions, however, the exact values of $q_K, p_K$ are not known.

For homogeneous gradient Young measures the above results lead to

**Corollary 1.6.** *Suppose that $p > \frac{2K}{K+1}$ and that $\nu$ is a homogeneous $W^{1,p}$-GYM supported in $Q_2(K)$. Then*

$$\int_{\mathbf{M^{2\times 2}}} |\lambda|^s d\nu(\lambda) \le C(s,K) < \infty \quad \forall s < \frac{2K}{K-1}.$$

Lastly, quasiregular gradient Young measures appear naturally also in the study of homogenization of elliptic differential operators. A new approach to this theory developed in [6] by the second author shows that the so-called $G_\theta$-closure problems are equivalent to the computation of the $\theta$-quasiconvex hulls of sets $\{K_i\}_{i=1}^n \subset Q_2(K)$. The above quasiregular results on Young measures have important applications also in this setup, for details see [6].

# 2 Quasiregular mappings and Beltrami operators

We approach Theorem 1.2 by using the Beltrami operators introduced in [2]. These are singular integral operators built on the *Beurling transform*

$$(Tf)(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dxdy.$$

As it is well known, for each $u \in W^{1,p}(\mathbb{C})$ we have $T\overline{\partial}u = \partial u$. Moreover, $T$ is a classical Calderon-Zygmund operator with Fourier multiplier $\overline{\xi}/\xi$. Therefore $T$ is bounded on $L^p(\mathbb{C})$ for each $1 < p < \infty$, $\|T\|_{L^2} = 1$.

The basic tool for this work is the following result due to Astala, Iwaniec and Saksman [2]

**Theorem 2.1.** *Let $\mu_1, \mu_2 \in L^\infty(\mathbb{C})$ be such that $k = \||\mu_1| + |\mu_2|\|_\infty < 1$. Denote $p = 1 + 1/k$ and $p' = 1 + k$. Then the Beltrami operator*

$$I - \mu_1 T - \mu_2 \overline{T} \tag{1}$$

*and its transpose $I - T\mu_1 - \overline{T}\mu_2$ are invertible on $L^q(\mathbb{C})$ for all $q \in (p', p)$.*

Note that even if $T$ is $\mathbb{C}$-linear, the Beltrami operators (1) are in general only $\mathbf{R}$-linear. Also, the above range for the exponents $q$ is the largest possible; as shown in [2], the conclusion of Theorem 2.1 fails whenever $q \leq p'$ or $q \geq p$.

In addition to Beltrami operators we need few basic results on quasiconformal mappings. The connections between the operators (1) and quasiconformality comes through the Beltrami differential equation

$$\overline{\partial} f(z) = \mu(z)\partial f(z). \tag{2}$$

In fact, a $W^{1,2}_{loc}$-homeomorphism $f$ in a planar domain is $K$-quasiconformal if and only if it satisfies (2) with $|\mu(z)| \leq \frac{K-1}{K+1} < 1$ a.e. This leads to the following result, fundamental in the theory of planar quasiconformal mappings.

**Theorem 2.2.** (Measurable Riemann mapping theorem)
*Let $\mu$ be measurable function with $\|\mu(z)\| \le k < 1$ for a.e. $z \in \mathbb{C}$. Then there exists a (unique) quasiconformal homeomorphism $F : \mathbb{C} \to \mathbb{C}$ such that $F(0) = 0$, $F(1) = 1$ and*

$$\overline{\partial} F = \mu \partial F \quad a.e.\ z \in \mathbb{C}. \tag{3}$$

For the proof of the theorem see e.g [8], Section V.1 . The ratio $\mu = \overline{\partial} F / \partial F$ is called the *complex dilatation* of the quasiconformal mapping. The same formula defines the complex dilatation $\mu = \mu_f$ for the non-injective quasiregular mappings, too. Clearly $\mu_f \equiv 0$ if and only if $\overline{\partial} f \equiv 0$; calculation of the dilatations of compositions yields then

**Proposition 2.3.** (see [8], VI.2.2 ) *Let $\Omega \subset \mathbb{C}$ be a domain and suppose $f$ is quasiregular in $\Omega$. Let $F : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping such that $\mu_f(z) = \mu_F(z)$ for a.e. $z \in \Omega$. Then*

$$f(z) = h \circ F(z), \quad z \in \Omega, \tag{4}$$

*where the function $h$ is complex-analytic in $F(\Omega)$.*

Combining the proposition with the measurable Riemann mapping theorem we see that in planar domains all quasiregular mappings are of the form (4).

For the $0 - 1$ Law of Theorem 1.3 we will need also quantitative bounds for the size of the set where the Jacobian derivative of a quasiconformal mapping is small. The following (sharp) estimate is essentially equivalent to the results [1] due to the first author.

**Theorem 2.4.** *Let $F : \mathbb{C} \to \mathbb{C}$ be $K-$quasiconformal with $F(0) = 0$ and $F(1) = 1$. Then there is a constant $C_K$, depending only on $K$, such that*

$$|\{z \in \mathbb{D} : J_F < t\}| \le C_K t^{1/(K-1)}. \tag{5}$$

*Proof.* The inverse function $F^{-1}$ is also a $K-$quasiconformal mapping fixing 0 and 1. Therefore by Theorem 1.1 in [1], cf. also the invariant formulation in Corollary 10 of [2], we have

$$|F^{-1} E| \le C_1(K) |E|^{\frac{1}{K}} \tag{6}$$

whenever $E \subset F(\mathbb{D})$. Here the constant $C_1(K)$ depends only on $K$.

In the special case $E = \{z \in F(\mathbb{D}) : J_{F^{-1}} > s\}$ we get

$$s|E| \le \int_E J_{F^{-1}} = |F^{-1} E| \le C_1(K) |E|^{\frac{1}{K}}.$$

This gives

$$|\{z \in F(\mathbb{D}) : J_{F^{-1}} > s\}| \leq \left(\frac{C_1(K)}{s}\right)^{\frac{K}{K-1}}. \tag{7}$$

We can then combine the estimates (6), (7) and obtain

$$\begin{aligned} |\{z \in \mathbb{D} : J_F < t\}| &= |F^{-1}\{w \in F(\mathbb{D}) : J_F(F^{-1}w) < t\}| \\ &\leq C_1(K)|\{w \in F(\mathbb{D}) : J_{F^{-1}} > 1/t\}|^{\frac{1}{K}} \leq C_K \, t^{\frac{1}{K-1}} \end{aligned}$$

where $C_K = C_1(K)^{\frac{2K}{K-1}}$. □

# 3 Gradient Young Measures

Oscillating sequences $\{f_j\}_{j=1}^{\infty}$ converging weakly but not strongly are inheritent in a number of questions in PDE's or Calculus of Variations. However, the weak limit itself hides much of the information on the behavior of the sequence. Therefore one is lead to embed the sequence into a bigger topological space, where the limits describe the adequate properties. Formally this procedure is done using the *Young measures.* We shall briefly recall the basis of the theory (for the case of gradients) and list a few basic facts needed later.

Let $\Omega$ be a simply connected domain in $\mathbf{R^n}$ with Lipschitz boundary. If we are given a sequence of Sobolev-functions $\{f_j\} \in W^{1,p}(\Omega; \mathbf{R^m})$ converging weakly to an element $f \in W^{1,p}(\Omega; \mathbf{R^m})$, then the associated gradient Young measure describes the local oscillations of the gradient sequence $\{Df_j\}_{j=1}^{\infty}$ at points $x \in \Omega$. More precisely, each $f_j$ defines a mapping

$$T_j : \Omega \mapsto \mathcal{M}(\mathbf{M^{n\times m}}), \quad T_j(x) = \delta_{Df_j(x)} \tag{8}$$

where $\mathcal{M}(\mathbf{M^{n\times m}})$ denotes the Radon measures on the space of $n \times m$ matrices $\mathbf{M^{n\times m}}$. Clearly $T_j$ is weakly measurable so that

$$T_j \in L^{\infty}_w(\Omega, \mathcal{M}(\mathbf{M^{n\times m}}))$$

with norm $\|T_j\| = 1$. Using the fact that $L^{\infty}_w(\Omega, \mathcal{M}((\mathbf{M^{n\times m}})) = L_1(\Omega, C_0(\mathbf{M^{n\times m}}))^*$ we may assume, retaining to a subsequence when necessary, that the sequence $\{T_j\}$ converges in the *weak*$^*$-topology towards an element $T \in L^{\infty}_w(\Omega, \mathcal{M}(\mathbf{M^{n\times m}}))$. Equivalently, for all $g \in L^1(\Omega), \varphi \in C_o(\mathbf{M^{n\times m}})$ we have

$$\int_\Omega g(z)\varphi(Df_j(z))dz \to \int_\Omega g(z) \int_{\mathbf{M^{n\times m}}} \varphi(\lambda) d\nu_z(\lambda) \; dz, \tag{9}$$

where $\{\nu_z\}_{z\in\Omega}$, $\nu_z = T(z)$, is the $W^{1,p}$*-gradient Young measure* associated to the sequence $\{f_j\}$.

Collecting these facts we have

**Definition 3.1.** A family of probability measures $\{\nu_z\}_{z\in\Omega}$ is a $W^{1,p}$-gradient Young measure if there exists a bounded sequence $\{f_j\} \subset W^{1,p}(\Omega; \mathbf{R^m})$ such that the following three properties hold:

1. $\mathrm{supp}(\nu_z) \subset \mathbf{M^{n\times m}}$ *a.e.* $z \in \Omega$.

2. The measures $\nu_z$ depend measurably on $z$. This means that for every $\varphi \in C_o(\mathbf{M^{n\times m}})$ the function

$$z \to \overline{\varphi}(z) = \int_{\mathbf{R}^m} \varphi(\lambda) d\nu_z(\lambda)$$

is measurable.

3. For each $\varphi \in C_o(\mathbf{M^{n\times m}})$,

$$\varphi(Df_j(z)) \overset{\star}{\rightharpoonup} \bar{\varphi}(z) = \int_{\mathbf{M^{n\times m}}} \varphi(\lambda) d\nu_z(\lambda) \tag{10}$$

in $L^\infty(\Omega)$.

Suppose $\nu$ is a $W^{1,p}$-GYM . Then if a sequence $\{f_j\}$ is uniformly bounded, say, in $W^{1,1}(\Omega; \mathbf{R^m})$ and if Condition 3. above holds, then we say that the gradients $\{Df_j\}$ *generates* $\nu$.

There are many different approaches and references for Young measures. We have followed the one presented in [11] and [9], c.f. also the many references there.

The cornerstone for the relations between Young measures and weak limits is the following lemma.

**Lemma 3.2.** ([11], Theorem 6.2.) *Let $\{\nu_z\}_{z\in\Omega}$ be a $W^{1,p}$-GYM generated by $\{f_j\} \subset W^{1,p}(\Omega; \mathbf{R^m})$ and let $\psi : \mathbf{M^{n\times m}} \to R$ be a continuous function such that the sequence $\{\psi(Df_j(z))\}$ is weakly convergent in $L^1(\Omega)$. Then*

$$\psi(Df_j(z)) \rightharpoonup \int_{\mathbf{M^{n\times m}}} \psi(\lambda) d\nu_z \quad \text{in} \quad L^1(\Omega)$$

In particular if $p > 1$, $Df(z)$, the gradient of the weak limit of $\{f_j\}$, equals the center of mass of $\nu_z$,

$$Df(z) = \int_{\mathbf{M^{n\times m}}} \lambda d\nu_z(\lambda) \quad a.e.\ \ z \in \Omega. \tag{11}$$

Finally, we will use the fact that in most situations the properties of Young measures depend on local phenomena. The Lebesgue differentiation theorem guarantees that this is also the case for GYM's, see [11] Theorem 8.4. This leads to the following subclass of Gradient Young measures.

**Definition 3.3.** A $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ is said to be homogeneous if there exists a probability measure $\nu$ on $\mathbf{M^{n\times m}}$ such that

$$\nu_z = \nu$$

for almost every $z \in \Omega$.

# 4 Proofs of the main theorems.

For the proof of Theorem 1.2 we need some preliminary considerations and lemmas. We assume that $n = 2$, that $\Omega$ is a planar domain with smooth boundary and that $\{\nu_z\}_{z\in\Omega}$ is a $W^{1,p}$-GYM supported in $Q_2(K_0)$, where $1 \le K_0 < \infty$ is fixed.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a bounded sequence in $W^{1,p}(\Omega; \mathbf{R}^2)$ generating $\nu$. A basic fact of the Young measures, see [11] Theorem 8.15, is that we can assume $\{|D\phi_j|^p\}_{j=1}^{\infty}$ to be equintegrable. Our task is then to find a sequence of mappings $\{f_j\}_{j=1}^{\infty}$ that are $K_0$-quasiregular and still generate the same $W^{1,p}$-GYM $\nu$.

**Lemma 4.1.** *Let $\{\nu_z\}_{z\in\Omega}$ be a $K_0$-quasiregular $W^{1,p}$-GYM and suppose $\{D\phi_j\}_{j=1}^{\infty}$ is a generating sequence such that*

$$\{|D\phi_j|^p\}_{j=1}^{\infty} \textit{ is equintegrable in } \Omega. \tag{12}$$

*Let $k_0 = \frac{K_0-1}{K_0+1}$. Then there are measurable functions $\mu_j$ such that $|\mu_j(z)| \le k_0$ a.e. $z \in \Omega$ and*

$$\lim_{j\to\infty} \|\overline{\partial}\phi_j - \mu_j \partial\phi_j\|_{L^p(\Omega)} = 0. \tag{13}$$

*Proof.* For a matrix $C \in \mathbf{M^{2\times 2}}$ consider its conformal and anticonformal parts $C_\pm$ where

$$C_\pm = \frac{1}{2}(C \pm JCJ^t), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using these let us define

$$F_{k_0}(C) = \max\{0, |C_-| - k_0|C_+|\}^p$$

where $|C|$ denotes the Hilbert-Schmidt norm of $C$. Clearly $F_{k_0}(C) = 0$ if and only if $|C_-| \le k_0|C_+|$. In terms of the operator norm $\|C\|$ this is equivalent to $\|C\|^2 \le K_0 \det(C)$. In other words, the kernel of $F_{k_0}$ is precisely $Q_2(K_0)$. Moreover, $0 \le F_{k_0}(C) \le |C|^p$.

In the complex notation $|[D\phi_i(z)]_+| = |\partial\phi_i(z)|$, $|[D\phi_i(z)]_-| = |\overline{\partial}\phi_i(z)|$. Hence for a.e. $z$ we can find complex numbers $|\mu_j(z)| \le k_0$ such that

$$F_{k_0}\Big(D\phi_i(z)\Big) = |\overline{\partial}\phi_i(z) - \mu_j(z)\partial\phi_i(z)|^p. \tag{14}$$

The condition (12) implies that also the sequence $F_{k_0}\Big(D\phi_i(z)\Big)$ is equiintegrable. By the Dunford Pettis theorem it has a weakly convergent subsequence in $L^1(\Omega)$. Thus by means of the lemma 3.2 we can conclude that

$$\lim_{j\to\infty} \int_\Omega F_{k_0}\Big(D\phi_i(z)\Big) dz = \int_\Omega \int_{\mathbf{M^{n\times m}}} F_{k_0}(\lambda) d\nu_z(\lambda) dz = 0$$

since for almost every $z$, the measure $\nu_z$ is supported in the zero set of $F_{k_0}$. With (14) this yields the claim. □

*Proof of Theorem 1.2.*
Suppose $\{\phi_j\}_1^\infty \in W^{1,p}(\Omega;\mathbb{C})$ is a sequence generating the $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ as in Lemma 4.1. Since $\nu$ is also a $W^{1,q}$-GYM for each $q < p$, we may assume that $1 + k_0 < p < 1 + 1/k_0$ where $k_0 = \frac{K_0-1}{K_0+1}$.

Let us then consider the function $h_j = \chi_\Omega(\overline{\partial}\phi_j - \mu_j\partial\phi_j)$ where $\mu_j$ is as in (13). According to the Lemma we have $\|h_j\|_{L^p(\mathbb{C})} \to 0$ as $j \to \infty$. Setting $\mu_j(z) = 0$ for $z \notin \Omega$ and using Theorem 2.1 we obtain $H_j \in L^p(\mathbb{C})$ with $H_j = (I - \mu_j T)^{-1} h_j$.

Now the functions

$$w_j(z) = -\frac{1}{\pi} \int_{\mathbf{C}} \frac{H_j(\zeta)}{\zeta - z} dm(\zeta) \tag{15}$$

satisfy $\overline{\partial} w_j = H_j$; hence $\overline{\partial} w_j - \mu_j \partial w_j = h_j$ with $w_j \in W^{1,p}_{loc}(\mathbb{C})$. In fact,

$$\|Dw_j\|_{L^p(\mathbb{C})} \le C_0 \|h_j\|_{L^p(\mathbb{C})} \to 0 \tag{16}$$

for a constant $C_0 < \infty$ depending only on $K_0$ and $p$.

Next let us define

$$f_j(z) = \phi_j(z) - w_j(z), \quad z \in \Omega.$$

Since by (16) we have $\|Df_j - D\phi_j\|_{L^p(\Omega)} \to 0$ when $j \to \infty$, the gradients $\{Df_j\}_{j\in\mathbb{N}}$ and $\{D\phi_j\}_{j\in\mathbb{N}}$ generate the same Young measure $\{\nu_z\}_{z\in\Omega}$, see [11] Lemma 6.3. Moreover, by the construction

$$\overline{\partial} f_j(z) - \mu_j \partial f_j(z) = 0, \quad z \in \Omega.$$

This means that each $f_j$ is a $W^{1,p}$-solution to the Beltrami equation (2). Since $p > 1 + k_0$ it follows, see [1] or [2], that then necessarily $f_j \in W^{1,2}_{loc}(\Omega)$. In particular, each element $f_j$ of the generating sequence is $K_0$-quasiregular. $\square$

In fact, in Theorem 1.2, specific properties of the Young measure $\nu$ naturally yield more restrictive generating gradients. We mention here two related interesting examples which can be obtained by similar methods; for details see [6] where the results are also used for further applications to elliptic PDE's.

**Remark 4.2.**

1. Given a matrix $\mu \in CO_-(2)$ with $\|\mu\| \le k < 1$, let

$$E_\mu = \{A \in \mathbf{M^{2\times 2}} : A_- = \mu A_+\}.$$

(naturally we can identify $\mu$ with the complex dilatation of each $A \in E_\mu$.) Suppose $\mu_1, \ldots, \mu_n \in CO_-(2)$ . Then any $W^{1,p}$-GYM supported in $\cup_{i=1}^n E_{\mu_i}$ is generated by quasiregular gradients such that $Df(z) \in \cup_{i=1}^n E_{\mu_i}$ for a.e. $z$.

2. The important feature of the sets $E_{\mu_i}$ is that they are unbounded sets such that $\det(A - B) > 0$ for all $A, B \in E_{\mu_i}$. More generally, we can consider the following nonlinear situation. Let $\{E_i\}_{i=1}^n$ be a collection of unbounded sets such that $\det(A - B) > 0$ for all $A, B \in E_i$. Then any $W^{1,p}$-GYM supported in $\cup_{i=1}^n E_i$ can be generated by quasiregular gradients such that $Df(z) \in \cup_{i=1}^n E_i$ for a.e. $z$. The proof uses the invertibility of the nonlinear Beltrami Operators, see [2]), and the fact that every set $E_i$ can be described as the graph of a $k$-Lipschitz function $\psi : CO_+(2) \to CO_-(2)$ where $0 \le k < 1$, see[15]).

We then turn to the study of homogeneous gradient Young measures.

*Proof of Theorem 1.5.*

We begin with some preliminary considerations. Given the homogeneous $W^{1,p}$-GYM $\nu$ with support in $Q_2(K)$, Theorem 1.2 gives us a sequence of $K$-quasiregular mappings $\{f_j\}$ whose gradients generate $\nu$. We can assume that $\frac{2K}{K+1} < p < \frac{2K}{K-1}$ and then the proof of the theorem gives

$$\sup_{j\in\mathcal{N}} \int_\Omega |Df_j(z)|^p dm(z) < \infty. \tag{17}$$

We may also assume that $f_j(0) = 0$.

Next, we saw in (11) that

$$Df_j(z) \rightharpoonup \int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) \quad \text{in } L^1(\Omega) \tag{18}$$

where by homogeneity the right hand side is a constant in $z$. This fact will be an essential feature of the argument below. For brevity let us use the notation $A = \int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda)$. Then $A \in Q_2(K)$ since the limits of $K$-quasiregular mappings remain $K$-quasiregular.

Denote then by $\mu_A$ the complex dilatation of the linear map $A(z)$, set $\mu_A = 0$ if $A = 0$. Consider now the dilatation functions $\mu_j$ where $\mu_j(z) = \mu_{f_j}(z)$ for $z \in \Omega$ and $\mu_j(z) = \mu_A$ for $z \in \mathbb{C} \setminus \Omega$. By the measurable Riemann mapping theorem we can find $K$-quasiconformal mappings $F_j : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1, for which $\mu_{F_j}(z) = \mu_j(z)$ for a.e. $z \in \mathbb{C}$. Furthermore, according to Proposition 2.3 we can factor

$$f_j = h_j \circ F_j, \quad j = 1, 2, \ldots \tag{19}$$

where the $h_j$ is analytic in $F_j(\Omega)$.

We need then the equicontinuity properties of both factors in (19). Firstly, the mappings $F_j$ are quasisymmetric, i.e. [8]

$$\frac{|F_j(y) - F_j(x)|}{|F_j(z) - F_j(x)|} \le \gamma_K\left(\frac{|y-x|}{|z-x|}\right) \quad \text{for distinct } x, y, z \in \mathbb{C}, \tag{20}$$

where the map $\gamma_K$ is an increasing homeomorphism of $[0, \infty)$ onto itself, depending only on $K$. Therefore the family $\{F_j\}$ is normal. Replacing it by a subsequence we have $F_j \to F$ uniformly on $\overline{\Omega}$, where $F : \mathbb{C} \mapsto \mathbb{C}$ is $K$-quasiconformal [8] and $F(0) = 0$, $F(1) = 1$. Moreover,

$$DF_j(z) \rightharpoonup DF(z) \quad \text{in } L^p(\Omega). \tag{21}$$

Secondly, for the factor $h_j$ note that

$$\begin{aligned}
\int_{F_j(\Omega)} |Dh_j(z)| dm(z) &= \int_\Omega |Dh_j(F_j z)| J_{F_j}(z) dm(z) \le \\
\Big(\int_\Omega |Dh_j(F_j z)|^p |DF_j(z)|^p dm(z)\Big)^{1/p} &\Big(\int_\Omega |DF_j(z)|^q dm(z)\Big)^{1/q} \\
= \Big(\int_\Omega |Df_j(z)|^p dm(z)\Big)^{1/p} &\Big(\int_\Omega |DF_j(z)|^q dm(z)\Big)^{1/q}
\end{aligned} \tag{22}$$

where $q$ is the conjugate exponent of $p$. Let $B = B(x, R)$ be a fixed disk containing $\Omega$. As $q < \frac{2K}{K-1}$ it follows from [1], see also [2] Corollary 11, that

$$\Big(\int_B |DF_j(z)|^q dm(z)\Big)^{1/q} \leq C(p, K)|F_j(B)|^{1/2} \leq c_0 \tag{23}$$

with $c_0 < \infty$ independent of $j \in \mathbb{N}$. Consequently, by (17) and (23)

$$\sup_{j\in\mathbb{N}} \int_{F_j(\Omega)} |Dh_j(z)|dm(z) < \infty. \tag{24}$$

Now the domains $F_j(\Omega)$ converge in the Hausdorff metric to the domain $F(\Omega)$ by the uniform convergence (20) . Since $h_j(0) = 0$, one may use e.g. the mean value theorem with (24) to show that the family $\{h_j : F_j(\Omega) \to \mathbb{C}\}$ is normal. Hence by choosing a further subsequence we can assume $h_j \to h$, $h'_j \to h'$ uniformly on compact subsets of $F(\Omega)$, where $h$ is analytic on $F(\Omega)$. Then $h'_j(F_j z) \to h'(Fz)$ uniformly on compact subsets of $\Omega$. It follows that

$$Df_j(z) \rightharpoonup D(h \circ F)(z) \quad \text{in } L^p(\Omega). \tag{25}$$

After these preparations we claim that $h(z) \equiv \alpha z$ for some $\alpha \in \mathbb{C}$. Here the assumption that $\nu$ is a homogeneous measure is of course needed. Indeed, (18) and (25) show that $D(h\circ F) = A$ in $\Omega$ and so, unless $h \equiv 0$, the dilatation $\mu_F(z) = \mu_A$ for $z \in \Omega$. But for $z \notin \Omega$ we have $\mu_F(z) = \mu_A$ by definition. Therefore $\mu_F$ is constant and so $F$ must be linear by the uniqueness part of Theorem 2.2. It follows that $h$, too, must be linear. Moreover, by (25), (18)

$$\alpha DF = \int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda). \tag{26}$$

To complete the proof of Theorem 1.5 choose a closed disk $\overline{B}_0 \subset \Omega$. Since now $h'_j \to \alpha$ uniformly in $\overline{B}_0$, by (19) $\|Df_j - \alpha DF_j\|_{L^p(B_0)} \to 0$. Therefore, $\{\alpha DF_j\}$ and $\{Df_j\}$ generate in $B_0$ the same $W^{1,p}$-GYM , see [11]. By homogeneity, this measure is $\nu$. Making a linear change of variables gives finally $K$-quasiconformal mappings $\tilde{F}_j : \mathbb{C} \to \mathbb{C}$ such that $D\tilde{F}_j|_{\mathbb{D}}$ generate $\nu$. This proves Theorem 1.5. $\square$

Next, we shall prove the 0-1 Law of Theorem 1.3 in the special case of homogeneous quasiregular Young measures.

**Lemma 4.3.** *Let $\nu$ be a homogeneous $W^{1,p}$-GYM with support contained $Q_2(K)$. Let $p > \frac{2K}{K+1}$. If the center of mass $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) = 0$, then $\nu = \delta_0$. On the other hand, if $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) \neq 0$, then $\nu(\{0\}) = 0$.*

*Proof:*

Let us use the sequence $\{F_j\}$ from the proof of Theorem 1.5, consisting of $K$-quasiconformal mappings fixing 0 and 1. Recall also that $F_j \to F$ uniformly on compact subsets of $\mathbb{C}$, where $F$ is linear (and $K$-quasiconformal). Similarly, let $\alpha$ be the constant as in the proof of Theorem 1.5, so that $\{\alpha DF_j|_{B_0}\}_{j=1}^{\infty}$ generates $\nu$. We can assume that $B_0 \subset \mathbb{D}$.

Now, if $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) = 0$ then by (26) $\alpha = 0$ so that $\nu = \delta_0$. If we have $\int_{\mathbf{M^{2\times 2}}} \lambda d\nu(\lambda) \neq 0$ then $\alpha \neq 0$. In this case choose a family of cut-off functions $h_\varepsilon \in C_0^\infty(\mathbf{M^{2\times 2}})$ such that $h_\varepsilon = 1$ on $B(0,\varepsilon)$, $h_\varepsilon = 0$ on $\mathbf{M^{2\times 2}} \setminus B(0,2\varepsilon)$ and $\|h_\varepsilon\|_\infty = 1$. By (9)

$$\nu(B(0,\varepsilon)) \leq \int_{\mathbf{M^{2\times 2}}} h_\varepsilon(\lambda) d\nu(\lambda) = \lim_{j\to\infty} \frac{1}{|B_0|} \int_{B_0} h_\varepsilon(\alpha DF_j(z)) dz$$

since $\nu_z = \nu$ for a.e. $z$.

Theorem 2.4 gives now

$$\begin{aligned} \nu(B(0,\varepsilon)|B_0| &\leq |\{z \in B_0 : \|DF_j\| \leq 2\varepsilon/|\alpha|\}| \\ &\leq |\{z \in \mathbb{D} : J_{F_j} \leq (2\varepsilon/|\alpha|)^2\}| \leq C_{K,\alpha}\varepsilon^{\frac{2}{K-1}}. \end{aligned} \tag{27}$$

Taking the limit $\varepsilon \to 0$ we obtain $\nu(\{0\}) = 0$. $\square$

Finally, to prove the non-homogeneous case of the 0-1 Law we only need to combine the previous results with basic properties of quasiregular mappings. It is shown e.g. in [7] that if $\{f_j\}_{j=1}^{\infty}$ is a sequence of $K$-quasiregular mappings with $f_j(0) = 0$ and

$$\sup_j \int_\Omega |Df_j(z)|^p dm(z) < \infty, \quad p > \frac{2K}{K+1} \tag{28}$$

then a subsequence $\{f_{j_n}\}$ converges uniformly on compact subsets of $\Omega$ to a $K$-quasiregular mapping $f : \Omega \to \mathbb{C}$. The second fact needed, see [8] or use Theorem 2.3, is that the Jacobian derivative of a non-constant $K$-quasiregular mapping can vanish only on a set of measure 0.

*Proof of Theorem 1.3.*

Consider a $K$-quasiregular $W^{1,p}$-GYM $\{\nu_z\}_{z\in\Omega}$ generated by the gradients of $K-$quasiregular mappings $\{f_j\}$ as in Theorem 1.2. Let $f$ be the quasiregular limit function of $\{f_j\}$; it follows from (11) that

$$Df(z) = \int_{M^{2\times 2}} \lambda d\nu_z(\lambda) \quad a.e.\ z \in \Omega. \tag{29}$$

By the localization property of GYM's, see [11] Theorem 7.2, for almost every $z$, $\nu_z$ is an homogeneous $K$-quasiregular GYM. Thus either $\nu_z(\{0\}) = 0$ or $\nu_z = \delta_0$.

Suppose that there exists a set $E \subset \Omega$ of positive measure such that for every $z \in E$ we have $\nu_z = \delta_0$. This implies that $Df(z) = 0$ for almost every $z \in E$ and hence also the Jacobian vanishes on $E$. It follows that $f$ must be constant. Therefore (29) with Lemma 4.3 shows that $\nu_z = \delta_0$ for almost every $z \in \Omega$ . □

**Remark 4.4.** The 0-1 law holds also in higher dimensions for those gradient Young measures that are generated by quasiregular mappings. Here the proof is more technical but in spirit the same; it is enough to use localizations at a point which is bounded away from the branch sets of the generating sequence. However it is not clear if Theorem 1.2 works as such in higher dimensions since then the partial differential structure associated with the quasiregular mappings is highly overdetermined.

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