ON DENSITY PROPERTIES OF THE RIESZ CAPACITIES AND THE ANALYTIC CAPACITY γ_+

Pertti Mattila and Petr V. Paramonov†

Abstract. In this paper we prove rather precise results on density properties of the Riesz capacities in $\mathbf{R}^N$ and the analytic capacity γ_+ in $\mathbf{R}^2$.

§1. Introduction and main results

Let M_h be the Hausdorff content, generated by a non-decreasing continuous function $h\colon [0,+\infty) \to [0,+\infty)$, $h(t) > 0$ for $t > 0$, $h(0) = 0$. That is, for a set E in $\mathbf{R}^N$ $(N \in \mathbf{N} = \{1, 2, \dots\})$

$$M_h(E) = \inf\left\{ \sum_{j=1}^{+\infty} h\big(d(E_j)\big) \;\middle|\; E \subset \bigcup_{j=1}^{+\infty} E_j \right\},$$

where $d(E)$ denotes the diameter of E.

The next result follows readily from [F, 2.10.17(3)].

Theorem A. *For each set $E \subset \mathbf{R}^N$ with $M_h(E) > 0$*

$$\limsup_{\delta \to 0} \frac{M_h\big(E \cap B(a,\delta)\big)}{h(\delta)} \geq 1$$

for M_h-almost all $a \in E$.

Here $B(a,\delta)$ is an open ball in $\mathbf{R}^N$, centered at $a \in \mathbf{R}^N$ and having radius $\delta > 0$ ($\overline{B}(a,\delta)$ is the corresponding closed ball).

For a reasonable h, $h(\delta)$ is comparable to $M_h\big(B(a,\delta)\big)$.

To our knowledge the failure of analogous density properties for Riesz capacities C_s in $\mathbf{R}^N$ and (positive) analytic capacity γ_+ in $\mathbf{R}^2$, which follows from our results below, was not noticed and discussed before (in [M1] such a failure was proved for the integralgeometric measure I_1^1 in $\mathbf{R}^N$). Shortly speaking, we are interested in the following more precise natural question: given a capacity C, as mentioned above, for which "density functions" h there is a compact set E such that $C\big(E \cap B(a,\delta)\big)$ is comparable to $h(\delta)$, as δ tends to zero, for all $a \in E_0$ with $C(E_0) > 0$? Theorems 1 and 2 give rather precise answers to this question.

1991 *Mathematics Subject Classification.* 30C85, 31B15.

†Supported by the grants: RFFR No.00-01-00618 and RFFR No.00-15-96008 and the Academy of Finland.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let $M_+(E)$, $E \subset \mathbf{R}^N$, be the set of all finite positive Borel measures μ with compact support, $\mathrm{supp}(\mu)$, contained in E. For $\mu \in M_+(\mathbf{R}^2)$ the Cauchy transform $\widehat{\mu}$ is defined as

$$\widehat{\mu}(z) = \int \frac{d\mu(\zeta)}{z-\zeta}, \quad z \in \mathbf{C}.$$

For $\mu \in M_+(\mathbf{R}^N)$ and $s \in (0, N)$ the s-Riesz transform of μ is

$$U^s_\mu(x) = \int \frac{d\mu(y)}{|x-y|^s}, \quad x \in \mathbf{R}^N.$$

It is easily seen (by Fubini theorem) that $\widehat{\mu}(z)$ (respectively $U^s_\mu(x)$) exists (the corresponding integrals converge absolutely) almost everywhere with respect to the Lebesgue measure $\mathcal{L}_{(2)}$ in $\mathbf{R}^2$ (respectively, to the Lebesgue measure $\mathcal{L}_{(N)}$ in $\mathbf{R}^N$), and, moreover, $\widehat{\mu} \in L_{1,\mathrm{loc}}(\mathcal{L}_{(2)})$ (respectively, $U^s_\mu \in L_{1,\mathrm{loc}}(\mathcal{L}_{(N)})$). We denote by $\|\cdot\|$ the $L_\infty(\mathcal{L}_{(N)})$-norm (on all of $\mathbf{R}^N$).

For bounded sets E in $\mathbf{R}^2$ and F in $\mathbf{R}^N$ define

$$\gamma_+(E) = \sup\{\mu(E) \mid \mu \in \mathcal{A}_+(E)\},$$
$$C_s(F) = \sup\{\mu(F) \mid \mu \in \mathcal{A}_s(F)\},$$

where $\mathcal{A}_+(E)$ (respectively, $\mathcal{A}_s(F)$) consists of all measures $\mu \in M_+(E)$ (respectively, $\mu \in M_+(F)$) such that $\|\widehat{\mu}\| \le 1$ (respectively, $\|U^s_\mu\| \le 1$).

It is easily seen that $\gamma_+\big(B(a,\delta)\big) = \delta$ and that $C_s\big(B(a,\delta)\big) = C_s\big(B(0,1)\big)\cdot\delta^s$.

Theorem 1. *Let $N \in \mathbf{N}$ and $s \in (0, N)$.*

(a) Let X be a compact set in $\mathbf{R}^N$ with $C_s(X) > 0$. If h satisfies the property

$$\int_0^1 \frac{h(t)\,dt}{t^{s+1}} < +\infty, \tag{1}$$

then for C_s-almost all $a \in X$ one has

$$\limsup_{\delta\to 0} \frac{C_s\big(B(a,\delta)\cap X\big)}{h(\delta)} = +\infty. \tag{2}$$

(b) Let h be such that

$$\int_0^1 \frac{h(t)\,dt}{t^{s+1}} = +\infty. \tag{3}$$

Suppose that for some $c > 0$ one has $ch(t) \le t^s$, $t \ge 0$, and the function $g(r) = h(r)\exp\big(-\int_r^1 \frac{ch(t)\,dt}{t^{s+1}}\big)$, $r \ge 0$, satisfies

$$\frac{g^{-1}(2^{-(j+1)N})}{g^{-1}(2^{-jN})} \in [\varepsilon, 1/2) \tag{4}$$

for some $\varepsilon \in (0, 1/2)$ and all j big enough, then there is a Cantor set X_1 with conditions

$$A_1\,h(\delta) \le C_s\big(B(a,\delta)\cap X_1\big) \le A_2\,h(\delta) \tag{5}$$

for all $a \in X_1$, $\delta \in (0,1)$ and some $0 < A_1 < A_2 < +\infty$, depending only on N, s, h, c.

Theorem 2. *(a) Let X be a compact set in $\mathbf{R}^2$ with $\gamma_+(X) > 0$. If h satisfies*

$$\int_0^1 \frac{h^2(t)\,dt}{t^3} < +\infty, \tag{6}$$

then for γ_+-almost all $a \in X$ one has

$$\limsup_{\delta\to 0} \frac{\gamma_+\big(B(a,\delta)\cap X\big)}{h(\delta)} = +\infty. \tag{7}$$

(b) Let

$$\int_0^1 \frac{h^2(t)}{t^3}\,dt = +\infty. \tag{8}$$

Suppose that for some $c > 0$ one has $ch^2(t) \le t^2$, $t \ge 0$, and the function

$$g(r) = h(r)\exp\bigg(-\int_r^1 \frac{ch^2(t)\,dt}{2t^3}\bigg), \quad r \ge 0,$$

satisfies the properties

$$\frac{g^{-1}(4^{-j-1})}{g^{-1}(4^{-j})} \in [\varepsilon, 1/2) \tag{9}$$

for some $\varepsilon \in (0, 1/2)$ and all j big enough. Then there exists a Cantor set X_1 in $\mathbf{R}^2$ with

$$A_1\,h(\delta) \le \gamma_+\big(B(a,\delta)\cap X_1\big) \le A_2\,h(\delta) \tag{10}$$

for all $a \in X_1$, $\delta \in (0, 1)$ and some $0 < A_1 < A_2 < +\infty$, depending only on h, c.

The conditions $ch(t) \le t^s$ and $ch^2(t) \le t^2$ in the parts (b) of Theorems 1 and 2 in context of (5) and (10) are quite natural. It will be clear from the proofs that conditions (4) and (9) on the function g are in some sense precise. These conditions are satisfied by all "reasonable" g (or h, cf. §3).

In §3 we give concrete examples of the functions h and discuss some other properties of densities. We also obtain several "partial" results on the densities of γ_+ and α_+ in $\mathbf{R}^N$, $N \ge 2$.

The proof of Theorem 1 can be generalized to give similar results for capacities related to general kernels K with mild regularity conditions. The integral in (1) is then replaced by $\int_0^1 h(t)\,K'(t)\,dt$. For example, $K(t)$ could be $-\log t$.

§2. Proofs

Proof of Theorem 1.

(a) Let X, s, h satisfy the conditions (a) in Theorem 1. It is enough to find at least one point $a \in X$ with conditon (2). In fact, if (2) does not hold C_s-almost everywhere in X, we can find a compact subset Y of X such that $C_s(Y) > 0$, but (2) with X replaced by Y fails for all $a \in Y$.

Take a measure $\mu \in \mathcal{A}_s(X)$ with $\|\mu\| := \int d\mu > 0$. If there exists $a \in X$ with

$$\limsup_{\delta \to 0} \frac{\mu\big(B(a,\delta)\big)}{h(\delta)} = +\infty$$

then for this a we also have (2), because for each $\delta > 0$ there is $\eta \in (0,\delta)$ with $\mu\big(\overline{B}(a,\eta)\big) \geq \mu\big(B(a,\delta)\big)/2$ and so, since $\mu\,|_{\overline{B}(a,\eta)} \in \mathcal{A}_s\big(X \cap B(a,\delta)\big)$, we get

$$C_s\big(B(a,\delta) \cap X\big) \geq \mu\big(B(a,\delta)\big)/2.$$

Suppose now that for each $a \in \operatorname{supp}(\mu)$

$$\limsup_{\delta \to 0} \frac{\mu\big(B(a,\delta)\big)}{h(\delta)} < +\infty. \tag{11}$$

We claim that $M_h(X) > 0$. In fact, suppose, by contradiction, that $M_h(X) = 0$. Then one can find balls $B_1, \dots, B_J$ such that $X \subset \bigcup_{j=1}^J B_j$ and

$$\sum_{j=1}^J \mu(B_j) \geq \|\mu\| > \sum_{j=1}^J h(2\delta_j),$$

where $2\delta_j = d(B_j)$, so that for some j (say $j = 1$) we have $\mu(B_1) > h(2\delta_1)$. Take now $X_1 = X \cap \overline{B}_1$ and $r_1 = \delta_1$, then for each $a_1 \in X_1$ we can write

$$\mu\big(X \cap B(a_1, 2r_1)\big) \geq \mu\big(X_1 \cap B(a_1, 2r_1)\big) \geq \mu(B_1) > h(2r_1).$$

Put $\mu_1 = \mu\,|_{X_1}$ and, by induction (since $M_h(X_1) = 0$), find compact sets $\{X_n\}_{n=1}^\infty$ ($X_n \subset X_{n-1}$, $n \geq 2$) and $\{r_n\}_{n=1}^\infty$ ($0 < r_n < r_{n-1}$, $n \geq 2$) such that for each $a_n \in X_n$ one has $\mu\big(X_n \cap B(a_n, 2r_n)\big) > nh(2r_n)$. For any point $a \in \bigcap_{n=1}^\infty X_n \neq \emptyset$, clearly,

$$\limsup_{\delta \to 0} \frac{\mu\big(X \cap B(a,\delta)\big)}{h(\delta)} = +\infty,$$

which contradicts (11) and proves the claim.

Now, since $M_h(X) > 0$, by Theorem A we can find $a \in X$ and a sequence $\{\delta_n\}_{n=1}^\infty$, $\delta_n \searrow 0$ as $n \nearrow +\infty$ such that for each n

$$M_h\big(B(a,\delta_n) \cap X\big) \geq h(\delta_n)/2.$$

By Frostman's lemma [C, p. 7], for each n there exists a measure $\sigma_n \in M_+\big(X \cap \overline{B}(a,\delta_n)\big)$ such that for any ball $B(x,r)$ one has $\sigma_n\big(B(x,r)\big) \leq h(r)$ and

$$\sigma_n\big(X \cap \overline{B}(a,\delta_n)\big) \geq A_1\, M_h\big(X \cap \overline{B}(a,\delta_n)\big) \geq \frac{A_1}{2}\, h(\delta_n),$$

where $A_1 > 0$ depends only on N.

We shall prove the following known fact for completeness.

Lemma 1. *Let K be a compact set in $\mathbf{R}^N$, $\sigma \in M_+(K)$ and $\sigma\big(B(x,\delta)\big) \le h(\delta)$ for any $x \in \mathbf{R}^N$ and $\delta > 0$. Then*

$$\|U^s_\sigma\| \le A \int_0^{2d} \frac{h(t)\,dt}{t^{s+1}},$$

where $d = d(K)$, $A = A(s) \in (0,+\infty)$.

Proof. Fix any $x \in \mathbf{R}^N$. Then

$$\begin{aligned} U^s_\sigma(x) &\le \int_{B(a,d)} \frac{d\sigma(y)}{|x-y|^s} + \frac{\|\sigma\|}{d^s} \le \int_0^{+\infty} \sigma\left\{ y \;\middle|\; \frac{1}{|x-y|^s} > t,\ |x-y| < d \right\} dt + \frac{h(d)}{d^s} \\ &= \int_{t_0}^{+\infty} \sigma\left(B\left(x, \left(\frac{1}{t}\right)^s \right)\right) dt + \int_0^{t_0} \sigma\big(B(x,d)\big)\,dt + \frac{h(d)}{d^s}, \end{aligned}$$

where $t_0 = d^{-s}$. Put $\tau = (1/t)^{1/s}$, then

$$U^s_\sigma(x) \le s \int_0^d \frac{h(\tau)\,d\tau}{\tau^{1+s}} + \frac{2h(d)}{d^s} \le A \int_0^{2d} \frac{h(t)\,dt}{t^{1+s}}. \quad \square$$

By Lemma 1 and (1),

$$\|U^s_{\sigma_n}\| \le A \int_0^{2\delta_n} \frac{h(t)\,dt}{t^{s+1}} =: \varepsilon_n \to 0 \quad \text{as } n \to +\infty.$$

The measure σ_n/ε_n is in $\mathcal{A}_s\big(X \cap \overline{B}(a,\delta_n)\big)$, which gives

$$C_s\big(\overline{B}(a,\delta_n) \cap X\big) \ge \frac{A_1\, h(\delta_n)}{2\varepsilon_n}$$

and, consequently, (2). In fact, from continuity of h one can find $\eta_n > 0$ such that $h(\delta_n + \eta_n) \le 2h(\delta_n)$ and therefore

$$C_s\big(B(a,\delta_n + \eta_n) \cap X\big) \ge \frac{A_1\, h(\delta_n + \eta_n)}{4\varepsilon_n}.$$

Actually, it is proved that

$$\limsup_{\delta \to 0} \frac{C_s\big(B(a,\delta) \cap X\big)}{h(\delta)/\varepsilon(\delta)} > 0,$$

where $\varepsilon(\delta) = \int_0^{2\delta} \frac{h(t)\,dt}{t^{s+1}}$.

(b) We shall construct the set X_1 with conditions (5) using estimates of C_s for Cantor sets (see [E, §1]).

Let h, c and g satisfy conditions in (b) of Theorem 1. We claim that the function g is continuous, (strictly) increasing and, moreover, $\int_0^1 \frac{g(t)\,dt}{t^{s+1}} < +\infty$.

Only the last property needs additional explanations. Since $\big(g(t)/h(t)\big)' = cg(t)/t^{s+1}$ for all $t > 0$, we have

$$\int_r^1 \frac{g(t)\,dt}{t^{s+1}} = \frac{g(t)}{ch(t)}\,\Big|_r^1.$$

But by (3), $g(t)/h(t) = \exp\big(-\int_r^1 \frac{ch(t)\,dt}{t^{s+1}}\big) \to 0$ as $r \to 0$, which gives $\int_0^1 \frac{g(t)\,dt}{t^{s+1}} = \frac{g(1)}{ch(1)} = 1/c < +\infty$ as claimed. Moreover,

$$\int_0^r \frac{g(t)\,dt}{t^{s+1}} = \frac{g(r)}{ch(r)}, \quad r > 0. \tag{12}$$

The property (4) means precisely that we can construct an N-dimensional Cantor set X_1 with parameters $\big\{l_j = g^{-1}(2^{-Nj})\big\}_{j=j_1}^{\infty}$ satisfying $l_{j+1}/l_j \in [\varepsilon, 1/2)$, $j \geq j_1 \in \mathbf{N}$.

We are not going to repeat the construction of Cantor sets (see [E, §1]), we just say that on the n-th step ($n \geq j_1$) of the construction we get 2^{Nn} congruent closed cubes $\{Q_n^m\}_{m=1}^{2^{Nn}}$ with side length l_n, so that

$$X_1 = \bigcap_{n=j_1}^{\infty} \Big(\bigcup_{m=1}^{2^{Nn}} Q_n^m\Big).$$

In particular, for each $n \geq j_1$, one has

$$X_1 = \bigcup_{m=1}^{2^{Nn}} X_n^m,$$

where X_n^m are congruent Cantor sets with parameters $\{l_j\}_{j=n}^{\infty}$ ($X_n^m \subset Q_n^m$).

One also notes that g satisfies the so-called doubling property

$$g(2r) \leq A_0\, g(r) \tag{13}$$

for all $r \in (0, l_{j_1+1})$ with constant $A_0 = 2^{2N}$, which follows from (4). Also h satisfies doubling condition, since, from the inequality $h(t) \leq t^s$ (required in (b)) one obtains

$$\frac{h(2r)}{h(r)} = \frac{g(2r)}{g(r)} \exp\left(\int_r^{2r} \frac{ch(t)\,dt}{t^{s+1}}\right) \leq 2^{2N} \exp\left(\int_r^{2r} \frac{dt}{t}\right) \leq 2^{2N+1}, \quad r \in (0, l_{j_1+1}).$$

In what follows the expression $f_1 \asymp f_2$ means that there exists a constant $A \in (1, +\infty)$, depending only on N, s, h, c (in context of γ_+ the constant A can depend only on h and c), such that $A^{-1} f_1 \leq f_2 \leq A\, f_1$ for all values of parameters in f_1 and f_2. Finally, by [E, Corollary 1.1] we get (recall that $g(l_j) = 2^{-Nj}$):

$$\begin{aligned} C_s(X_n^m) &\asymp \left(\sum_{j=0}^{\infty} 2^{-Nj} (l_{j+n})^{-s}\right)^{-1} = \left(\sum_{j=n}^{\infty} 2^{-Nj}\, 2^{Nn} (l_j)^{-s}\right)^{-1} \\ &= g(l_n) \left(\sum_{j=n}^{\infty} \frac{g(l_j)}{l_j^s}\right)^{-1} \asymp g(l_n) \left(\int_0^{l_n} \frac{g(t)\,dt}{t^{s+1}}\right)^{-1} \asymp h(l_n), \end{aligned}$$

where we used (12) and the doubling property (13). To obtain (5) it suffices to invoke the doubling property of h and the inequalities $l_{n+1} \geq \varepsilon\, l_n$, $n \geq j_1$. □

Proof of Theorem 2.

It is quite similar to that of Theorem 1, except that we need also a characterization of γ_+ in terms of the so-called curvature of a measure [T1]. Till the end of §2 $A, A_1, \dots$ are some positive constants depending only on h.

(a) Again, it is enough to check that (7) holds for at least one point $a \in X$. Take a measure $\mu \in \mathcal{A}_+(X)$, $\|\mu\| > 0$. Suppose that there is a point $a \in \operatorname{supp}(\mu) \subset X$ with

$$\limsup_{\delta \to 0} \frac{\mu\big(B(a,\delta) \cap X\big)}{h(\delta)} = +\infty.$$

We claim that also

$$\limsup_{\delta \to 0} \frac{\gamma_+\big(B(a,\delta) \cap X\big)}{h(\delta)} = +\infty.$$

For h, satisfying doubling condition (see (13)), it would be enough to check that

$$\gamma_+\big(B(a,2\delta) \cap X\big) \geq A\mu\big(B(a,\delta) \cap X\big).$$

Take $\varphi \in C_0^\infty\big(B(a,2\delta)\big)$, $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B(a,\delta)$ and $\|\overline{\partial}\varphi\| \leq \frac{A_1}{\delta}$. Put $\mu_1 = \varphi\mu$. By [V, Ch. II, §3],

$$\|\widehat{\mu}_1\| \leq A_2\, \delta\, \|\overline{\partial}\varphi\|\, \|\widehat{\mu}\| \leq A_1 A_2 = A_3^{-1}.$$

Hence, $A_3\, \mu_1 \in \mathcal{A}_+\big(B(a,2\delta) \cap X\big)$, which gives

$$\gamma_+\big(B(a,2\delta) \cap X\big) \geq A_3\, \|\mu_1\| \geq A_3\, \mu\big(B(a,\delta) \cap X\big),$$

as desired.

For a general h we apply [MPV, Lemma 3.3] (the case $N = 2$), which gives for each $B(a,\delta)$:

$$\gamma_+\big(B(a,\delta) \cap X\big) \geq A_3\, \mu\big(B(a,\delta) \cap X\big)$$

with some absolute constant $A_3 \in (0,1)$.

It now remains to consider the case when for any $a \in X$ one has

$$\limsup_{\delta \to 0} \frac{\mu\big(B(a,\delta) \cap X\big)}{h(\delta)} < +\infty.$$

As it was already shown in the proof of Theorem 1, we then get $M_h(X) > 0$. By Theorem A there exists $a \in X$ with

$$\limsup_{\delta \to 0} \frac{M_h\big(B(a,\delta) \cap X\big)}{h(\delta)} \geq 1,$$

and then we can choose $\{\delta_n\}_{n=1}^\infty \subset (0,1)$, $\delta_n \searrow 0$ as $n \nearrow +\infty$, such that

$$M_h\big(B(a,\delta_n) \cap X\big) \geq h(\delta_n)/2, \quad n \geq 1.$$

Fix $n \geq 1$ and put $h_n(t) = h(t)$ for $t \in [0,\delta_n]$ and $h_n(t) = h(\delta_n)$ for $t \geq \delta_n$. By Frostman's lemma we can find $\sigma_n \in M_+(X_n)$, $X_n = \overline{B}(a,\delta_n) \cap X$, such that

$\sigma_n\big(B(z,\delta)\big) \le h_n(\delta)$ (hence also $\sigma_n\big(\overline{B}(z,\delta)\big) \le h_n(\delta)\big)$ for all $z \in \mathbf{R}^2$, $\delta > 0$, and, moreover,

$$\|\sigma_n\| = \sigma_n(X_n) \ge A\, M_{h_n}(X_n) \ge \frac{A}{2}\, h(\delta_n).$$

Recall, that the curvature $c^2(\sigma)$ of a measure $\sigma \in M_+(\mathbf{R}^2)$ is defined by

$$c^2(\sigma) = \iiint \big(R(x,y,z)\big)^{-2}\, d\sigma(x)\, d\sigma(y)\, d\sigma(z),$$

where $R(x,y,z)$ is the radius of the circle passing through x, y and z.

Tolsa [T1, Theorem 5.1 and (59)] showed that

$$\tag{14} \begin{aligned} \gamma_+(K) \asymp \sup\big\{\|\sigma\| \mid \sigma \in M_+(K), \sigma\big(B(z,r)\big) \le r \quad &\text{for all} \\ z \in \mathbf{C} \text{ and } r > 0, \text{ and } c^2(\sigma) \le \|\sigma\|&\big\}. \end{aligned}$$

Though our definition of γ_+ is slightly different from that of [T1], we can use this result for K having zero Lebesgue measure (for which it is the same). We shall apply this result in order to obtain the corresponding lower estimate of γ_+, using the properties of measures σ_n. By Theorem 2.2 of [M2] we have:

$$\tag{15} \begin{aligned} c^2(\sigma_n) &\le 12\, \|\sigma_n\| \int_0^{+\infty} t^{-3}\, h_n^2(t)\, dt \\ &\le 12\, \|\sigma_n\| \bigg(\int_0^{\delta_n} t^{-3}\, h_n^2(t)\, dt + h_n^2(\delta_n) \int_{\delta_n}^{+\infty} t^{-3}\, dt \bigg) \\ &\le 12\, \|\sigma_n\| \bigg(\int_0^{\delta_n} t^{-3}\, h_n^2(t)\, dt + h_n^2(\delta_n)/2\delta_n^2 \bigg) \\ &\le A_1\, \|\sigma_n\| \int_0^{2\delta_n} t^{-3}\, h^2(t)\, dt =: \varepsilon_n\, \|\sigma_n\|, \end{aligned}$$

where $\varepsilon_n \to 0$ as $n \to +\infty$ by (6).

Since, evidently, $c^2(\lambda\sigma) = \lambda^3\, c^2(\sigma)$ for any $\lambda > 0$ and $\sigma \in M_+(\mathbf{R}^2)$, we have for the measure $\mu_n = \lambda_n\, \sigma_n$ with $\lambda_n = 1/\sqrt{\varepsilon_n}$ that $c^2(\mu_n) \le \|\mu_n\|$. We now have to show that

$$\tag{16} \mu_n\big(B(z,\delta)\big) \le A_2 \cdot \delta$$

for all $z \in \mathbf{R}^2$ and $\delta > 0$. To prove this consider 2 cases. The first is $\delta \ge \delta_n$. Then, since $\operatorname{supp}(\mu_n) \subset \overline{B}(a,\delta_n)$, we can write

$$\begin{aligned} \mu_n\big(B(z,\delta)\big) \le \mu_n\big(\overline{B}(a,\delta_n)\big) &\le \sigma_n\big(\overline{B}(a,\delta_n)\big)/\sqrt{\varepsilon_n} \\ &\le h(\delta_n)\bigg(A_1 \int_{\delta_n}^{2\delta_n} t^{-3}\, h^2(t)\, dt\bigg)^{-1/2} \\ &\le A_2\, h(\delta_n)\, \big(h^2(\delta_n)/\delta_n^2\big)^{-1/2} = A_2\, \delta_n \le A_2\, \delta. \end{aligned}$$

The second case is $\delta \in (0, \delta_n)$. Here we have

$$\mu_n\big(B(z,\delta)\big) \le \frac{\sigma_n\big(B(z,\delta)\big)}{\sqrt{\varepsilon_n}} \le h(\delta)\bigg(A_1 \int_\delta^{2\delta} \frac{h^2(t)}{t^3}\,dt\bigg)^{-1/2}$$
$$\le A_2\, h(\delta)\bigg(\frac{h^2(\delta)}{\delta^2}\bigg)^{-1/2} = A_2\,\delta.$$

Finally (assuming that $A_2 \ge 1$), the measure $\nu_n = \mu_n/A_2$ satisfies $c^2(\nu_n) \le \|\nu_n\|$ and $\nu_n\big(B(z,\delta)\big) \le \delta$ for all $B(z,\delta)$. Therefore, by (14), $\gamma_+(X_n) \ge A_3\,\|\nu_n\| \ge A_4\, h(\delta_n)/\sqrt{\varepsilon_n}$, which gives (7) since h is continuous. Again, it was really proved that

$$\limsup_{\delta\to 0} \frac{\gamma_+\big(B(a,\delta)\cap X\big)}{h(\delta)/\sqrt{\varepsilon(\delta)}} > 0,$$

where $\varepsilon(\delta) = \int_0^{2\delta} t^{-3}\, h^2(t)\,dt$. Recall, that here we could use (14) only whenever $\mathcal{L}_{(2)}(X) = 0$. If $\mathcal{L}_{(2)}(X) > 0$ then, since clearly $h(r)/r \to 0$ as $r \to 0$, the property (7) can be easily obtained from the following analog (in fact a corollary) of Vitushkin result on instability of analytic capacity [V, Ch. VI, §1]. Put $\mathcal{L} = \mathcal{L}_{(2)}$.

Proposition 1. *Let E be any $\mathcal{L}$-measurable set in $\mathbf{R}^2$, $\mathcal{L}(E) > 0$. Then for each $\mathcal{L}$-density point a of E (and therefore $\mathcal{L}$-a.e. on E) one has*

$$\lim_{\delta\to 0} \frac{\gamma_+\big(B(a,\delta)\cap E\big)}{\delta} = 1.$$

Proof. The corresponding proof in [V, pp. 188–192] is given for capacity α and for some more precise result (Theorem 1, p. 190). It also works for γ_+, but for the reader's convenience we choose to present here its considerably simplified version.

Let a be any $\mathcal{L}$-density point of E. Fix any $\varepsilon \in (0,1)$. There is $\delta_\varepsilon > 0$ such that for each $\delta \in (0, \delta_\varepsilon)$ one can find a compact set K_δ in $E \cap B(a,\delta)$ with $\mathcal{L}(K_\delta) \ge (1-\varepsilon^2)\,\pi\delta^2$. Put $\Omega_\delta = B(a,\delta) \setminus K_\delta$, $\mu = \mathcal{L}\big|_{B(a,\delta)}$, $\nu = \mathcal{L}\big|_{\Omega\delta}$, $\sigma = \mathcal{L}\big|_{K_\delta}$. It can be easily checked that $\widehat{\mu}(z) = \pi(\overline{z} - \overline{a})$ in $B(a,\delta)$ and $\widehat{\mu}(z) = \pi\delta^2/(z-a)$ in $\mathbf{R}^2 \setminus B(a,\delta)$, so that $\|\widehat{\mu}\| = \pi\delta$. On the other hand, $\|\widehat{\nu}\| \le 2\pi\varepsilon\delta$. In fact, fix any $z \in \mathbf{R}^2$ and let $\varrho \in (0, \varepsilon\delta]$ be such that $\mathcal{L}(\Omega_\delta) = \pi\varrho^2 = \mathcal{L}\big(B(z,\varrho)\big)$. Then

$$|\widehat{\nu}(z)| \le \int_{\Omega_\delta} \frac{d\mathcal{L}(\zeta)}{|\zeta - z|} \le \int_{B(a,\varrho)} \frac{d\mathcal{L}(\zeta)}{|\zeta - z|} = \int_0^{2\pi}\int_0^{\varrho} \frac{r\,dz\,d\varphi}{r} = 2\pi\varrho,$$

as desired. Therefore,

$$\|\widehat{\sigma}\| \le \|\widehat{\mu}\| + \|\widehat{\nu}\| \le \pi\delta(1+2\varepsilon).$$

Since $\sigma\big(\pi\delta(1+2\varepsilon)\big)^{-1} \in \mathcal{A}_+(K_\delta)$, we get

$$\gamma_+\big(E \cap B(a,\delta)\big) \ge \gamma_+(K_\delta) \ge \pi\delta^2(1-\varepsilon^2)\,\big(\pi\delta(1+2\varepsilon)\big)^{-1} = \delta\big(1 + O(\varepsilon)\big).$$

It remains to recall now that $\gamma_+\big(B(a,\delta)\cap E\big)\le\gamma_+\big(B(a,\delta)\big)=\delta$.

(b) Let h, c and g satisfy conditions in (b) of Theorem 2. The function $g(r)=h(r)\exp\big(-\int_r^1\frac{ch^2(t)}{2t^3}\,dt\big)$ satisfies $g(r)=o\big(h(r)\big)$ as $r\to 0$ (see (8)) and g is strictly increasing. Moreover,

$$\Big(\Big(\frac{g(r)}{h(r)}\Big)^2\Big)'=\exp\Big(-\int_r^1\frac{ch^2(t)}{t^3}\Big)\frac{ch^2(r)}{r^3}=\frac{cg^2(r)}{r^3}.$$

Thence, $\int_r^1 t^{-3}\,g^2(t)\,dt=g^2(t)/ch^2(t)\big|_r^1$, which gives

$$\int_0^1 t^{-3}\,g^2(t)\,dt=\frac{g^2(1)}{ch^2(1)}=\frac{1}{c}<+\infty$$

and

$$\int_0^r\frac{g^2(t)\,dt}{t^3}=\frac{g^2(r)}{ch^2(r)}. \tag{17}$$

The conditions (9) allow us to construct a 2-dimensional Cantor set X_1 with parameters $\{l_j=g^{-1}(4^{-j})\}_{j=j_1}^{\infty}$, where $\varepsilon l_j\le l_{j+1}<l_j/2$ ($j\ge j_1\in\mathbf{N}$). We use again the notation X_n^m from the proof of Theorem 1 and the same way we prove that g and h satisfy doubling properties (13) with constants $A_0=16$ and 32 respectively (we apply the condition $h(t)\le t$ and consider (13) only for $r\in(0,l_{j_1+1})$). In order to apply (14) again, we need to verify that $\mathcal{L}_{(2)}(X_1)\equiv\mathcal{L}(X_1)=0$. In fact, if, by contradiction, $\mathcal{L}(X_1)>0$ then, clearly, $\lim_{n\to\infty}l_{n+1}/l_n=1/2$. ¿From this we have (as $ch^2(t)\le t^2$):

$$\begin{aligned}\frac{h(l_{j+1})}{h(l_j)}&=\frac{g(l_{j+1})}{g(l_j)}\exp\Big(\int_{l_{j+1}}^{l_j}\frac{ch^2(t)\,dt}{2t^3}\Big)\\&\le\frac{1}{4}\Big(\exp\Big(\int_{l_{j+1}}^{2l_{j+1}}\frac{dt}{2t}+\underset{j\to\infty}{\mathrm{o}}(1)\Big)\Big)=\frac{\sqrt{2}}{4}\big(1+\underset{j\to\infty}{\mathrm{o}}(1)\big),\end{aligned}$$

which leads to a contradiction with (8).

Set $\mu=\mathcal{H}_g\big|_{X_1}$, where $\mathcal{H}_g$ is the g-Hausdorff measure. The main property of μ is that (see[G])

$$A_1\,g(r)\le\mu\big(B(z,r)\big)\le A_2\,g(r) \tag{18}$$

for each $z\in X_1$ and $r\in(0,l_{j_1})$ (all appearing A_j are positive and depend only on g).

It remains to prove that for all $n\ge j_1$ and $m=1,\dots,4^n$

$$A_3\,h(l_n)\le\gamma_+(X_n^m)\le A_4\,h(l_n). \tag{19}$$

The left inequality in (19) can be proved following part (a) of the present proof. Concretely, fix n and m, and let $\sigma_n=\mu\big|_{X_n^m}$. As in (15) and (16), using doubling conditions of g, we get

$$c^2(\sigma_n)\le A_5\,\|\sigma_n\|\int_0^{l_n}t^{-3}\,g^2(t)\,dt=:\varepsilon_n\,\|\sigma_n\|,$$

so that for $\mu_n = \sigma_n/\sqrt{\varepsilon_n}$ one has

$$c^2(\mu_n) \le \|\mu_n\| \quad \text{and} \quad \mu_n\big(B(z,\delta)\big) \le A_6\,\delta$$

($A_6 > 1$) for all z and $\delta > 0$. From (14) and (18),

$$\gamma_+(X_n^m) \ge \|\mu_n\|/A_6 \asymp \frac{g(l_n)}{\sqrt{\varepsilon_n}} \asymp h(l_n)$$

(see (17)). And that is it.

To prove the right inequality in (19) we need one more result of Tolsa. In [T2] it is proved (Theorem 1), that for each $\sigma \in M_+(\mathbf{R}^2)$ one has

$$\gamma_+\big(\{z \mid c_\sigma(z) \ge t\}\big) \le A_7 \frac{\|\sigma\|}{t}, \quad t > 0, \tag{20}$$

where

$$c_\sigma(z) = \left(\iint \frac{d\sigma(x)\,d\sigma(y)}{R^2(z,x,y)}\right)^{1/2},$$

and $A_7 > 0$ is an absolute constant.

For σ_n as just above, the same way as in [M2, Theorem 2.3], we get for all $z \in K_n^m$:

$$\big(c_{\sigma_n}(z)\big)^2 \ge A_8 \int_0^{l_n} t^{-3}\, g^2(t)\,dt =: A_9\,\varepsilon_n.$$

Applying (20) to $\sigma = \sigma_n$ and $t = \sqrt{A_9\,\varepsilon_n}$ completes the proof of Theorem 2. □

§3. Examples. Partial results for $\boldsymbol{\gamma}_+$ and $\boldsymbol{\alpha}_+$ in $\mathbf{R}^N$

For $p > 0$, $q \ge 0$ define

$$\begin{aligned} &h_{p,q}(t) = t^p/|\log t|^q, \quad t \in (0, e^{-1}), \quad h_{pq}(0) = 0, \\ &h_{p,q}(t) = e^{-p}, \quad t \ge e^{-1}. \end{aligned}$$

It can be easily checked that all $h(t) = h_{p,q}(t)$ with $p > s$, $q \ge 0$ and all $h(t) = h_{s,q}(t)$ with $q > 1$ satisfy the part (a) of Theorem 1. All functions $h(t) = h_{s,q}$ with $q \in [0,1]$ satisfy part (b) of Theorem 1 (take $c < \min\{1, N - s\}$).

The functions $h_{p,q}$ with $p > 1$, $q \ge 0$ and $h_{1,q}$, $q > 1/2$ satisfy part (a) of Theorem 2, and the functions $h_{1,q}$, $q \in [0, 1/2]$ satisfy part (b) of this theorem (take $c < 1$).

Theorems 1 and 2 can be applied even for a wider class of functions h, namely for the functions

$$h(t) = t^p\,|\log t|^{-q_1}\,|\log\,|\log t||^{-q_2}\ldots, \quad p > 0,\ q_j \ge 0$$

(finite product). From this we have

Corollary 1. *Let $N \in \mathbf{N}$, $s \in (0, N)$.*

(a) If X is a compact set in $\mathbf{R}^N$ with $C_s(X) > 0$, and $q > 1$, then for C_s-almost all $a \in X$ one has

$$\limsup_{\delta \to 0} \frac{C_s\big(B(a,\delta) \cap X\big)}{\delta^s / |\log \delta|^q} = +\infty.$$

(b) For each $q \in [0,1]$ there exists a Cantor set X_1 in $\mathbf{R}^N$ and $A_1 > 0$, $A_2 > A_1$ depending only on N, s, q, such that for each $a \in X_1$ and $\delta \in (0, e^{-1})$ one has

$$A_1\, \delta^s / |\log \delta|^q \le C_s\big(X_1 \cap B(a,\delta)\big) \le A_2\, \delta^s / |\log \delta|^q.$$

Corollary 2. *(a) For each compact set X in $\mathbf{R}^2$ with $\gamma_+(X) > 0$ and each $q > 1/2$ one has*

$$\limsup_{\delta \to 0} \frac{\gamma_+\big(X \cap B(a,\delta)\big)}{\delta / |\log \delta|^q} = +\infty$$

for γ_+-almost all $a \in X$.

(b) For each $q \in [0, 1/2]$ there exists a Cantor set X_1 in $\mathbf{R}^2$ such that for each $a \in X_1$ and $\delta \in (0, e^{-1})$ one has

$$A_1\, \delta / |\log \delta|^q \le \gamma_+\big(X \cap B(a,\delta)\big) \le A_2\, \delta / |\log \delta|^q$$

with $A_1 > 0$, $A_2 > A_1$ depending only on q.

Remark 1. ¿From Corollaries 1 and 2 (parts (b)) it follows (at least for functions h from the class $\{h_{p,q}\}_{p>0, q\ge 0}$) that for h, satisfying the requirements of the parts (b) of Theorems 1 and 2 respectively, one can find compact sets X_0 and X_∞ in $\mathbf{R}^N$ (respectively in $\mathbf{R}^2$) such that $C_s(X_0) > 0$, $C_s(X_\infty) > 0$ and

$$\lim_{\delta \to 0} \frac{C_s\big(B(a,\delta) \cap X_0\big)}{h(\delta)} = 0, \quad x \in X_0,$$
$$\lim_{\delta \to 0} \frac{C_s\big(B(a,\delta) \cap X_\infty\big)}{h(\delta)} = +\infty, \quad x \in X_\infty$$

$\Big($respectively, $\gamma_+(X_0) > 0$, $\gamma_+(X_\infty) > 0$ and

$$\lim_{\delta \to 0} \frac{\gamma_+\big(B(a,\delta) \cap X_0\big)}{h(\delta)} = 0, \quad x \in X_0,$$
$$\lim_{\delta \to 0} \frac{\gamma_+\big(B(a,\delta) \cap X_\infty\big)}{h(\delta)} = +\infty, \quad x \in X_\infty\Big).$$

In the cases, when the corresponding limits are $+\infty$ we, clearly, need to require in addition that $h(t) = o(t^s)$ ($h(t) = o(t)$ respectively) as $t \to 0$.

We propose also the following precisions to the parts (a) of Theorems 1 and 2.

Corollary 3. *(i) Suppose that h satisfies property (1) of Theorem 1 and that for some α, $0 < \alpha < N$,*

$$\lim_{t \to 0} \frac{h(t)}{t^\alpha} = +\infty. \tag{21}$$

Then there exists a Cantor-type set X in $\mathbf{R}^N$ with $C_s(X) > 0$ (and so (2) holds), but

$$\liminf_{\delta \to 0} \frac{C_s\big(B(a,\delta) \cap X\big)}{h(\delta)} = 0$$

for all $a \in X$.

(ii) Let h satisfy property (6) of Theorem 2 and for some α, $0 < \alpha < 2$,

$$\lim_{t \to 0} \frac{h(t)}{t^\alpha} = +\infty. \tag{22}$$

Then there is a Cantor-type set X in $\mathbf{R}^2$ with $\gamma_+(X) > 0$ (so (7) holds), but for all $a \in X$,

$$\liminf_{\delta \to 0} \frac{\gamma_+\big(B(a,\delta) \cap X\big)}{h(\delta)} = 0.$$

Proof. (i) We perform the following Cantor construction. Let Q_0 be a closed cube in $\mathbf{R}^n$ with side-length $1 = d_0$. Let $n_1 \in \mathbf{N}$ with $n_1 > 1$ and partition Q_0 into subcubes $P_{1,1}, \dots, P_{1,n_1^N}$ of side-length $1/n_1$. Let $0 < d_1 < 1/n_1$ and let $Q_{1,i}$ be the closed subcube of $P_{1,i}$ with the same center as $P_{1,i}$ and side-length d_1. We perform the same operation inside each $Q_{1,i}$ with parameters $n_2 \in \mathbf{N}$, $n_2 > 1$, and $0 < d_2 < d_1/n_2$. Thus we obtain altogether $(n_1\, n_2)^N$ closed cubes with side-length d_2. We continue this process and set

$$X = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{(n_1 \dots n_k)^N} Q_{k,i},$$

where $Q_{k,i}$ has side-length d_k with $0 < d_k < d_{k-1}/n_k$.

Let $s < \tau < N$ such that $\alpha < sN/\tau$ and choose n_k and d_k so that

$$n_k^N\, d_k^\tau = d_{k-1}^\tau. \tag{23}$$

Then $0 < \mathcal{H}_\varphi(X) < \infty$, $\varphi(t) = t^\tau$, and so $C_s(X) > 0$. We also choose n_k such that $n_k \to \infty$ from which one easily checks that for sufficiently large k,

$$\big(B(x, d_{k-1}/2n_k) \setminus B(x, N d_k)\big) \cap X = \emptyset$$

for all $x \in X$. Thus

$$C_s\big(B(x, d_{k-1}/2n_k) \cap X\big) = C_s\big(B(x, N d_k) \cap X\big) \le A_1\, d_k^s.$$

Hence it is enough to check that we can choose d_k so that $d_k^s/h(d_{k-1}/2n_k) \to 0$ as $k \to \infty$. But by (23),

$$\frac{d_k^s}{h(d_{k-1}/2n_k)} = d_{k-1}^{s-\alpha}\, 2^\alpha\, n_k^{\alpha - sN/\tau} \frac{(d_{k-1}/2n_k)^\alpha}{h(d_{k-1}/2n_k)}. \tag{24}$$

After fixing d_{k-1} we are free to choose n_k as large as we want. Since $\alpha - sN/\tau < 0$, we can make right hand side of (24) to tend to zero by (21).

The proof of (ii) is similar. □

Finally, we discuss the corresponding density properties of capacities γ_+ and α_+ in $\mathbf{R}^N$, $N \geq 2$.

For a bounded set E in $\mathbf{R}^N$, $N \geq 2$, define

$$\gamma_{+,N}(E) = \sup\bigl\{\mu(E) \mid \mu \in M_+(E), \||\nabla\Phi_N * \mu|\| \leq 1\bigr\},$$
$$\alpha_{+,N}(E) = \sup\bigl\{\mu(E) \mid \mu \in M_+(E), \||\nabla\Phi_N * \mu|\| \leq 1,\ \Phi_N * \mu \in C^1(\mathbf{R}^N)\bigr\},$$

where $\Phi_N(x)$ is the standard fundamental solution for the Laplace equation in $\mathbf{R}^N$ (see [MP] and [MPV]). It can be easily checked that $\gamma_{+,2} = 2\pi\gamma_+$.

¿From Egoroff's theorem it follows, that

$$C_s(F) = \sup\left\{\mu(F) \mid \mu \in \mathcal{A}_s(F) \quad \text{and} \quad \int_{B(x,r)} \frac{d\mu(y)}{|x-y|^s} \to 0 \right.$$
$$\left. \text{as } r \to 0 \text{ uniformly for } x \in \operatorname{supp}(\mu)\right\}$$

(see [C, pp. 15–16]).

¿From this (taking $s = N-1$) we have for each $N \geq 2$ that

$$\gamma_{+,N}(F) \geq \alpha_{+,N}(F) \geq A_N\, C_{N-1}(F), \tag{25}$$

$A_N > 0$ depends only on N.

Theorem 1 (a) and (25) give

Corollary 4. *Let X be a compact set in $\mathbf{R}^N$ with $\gamma_{+,N}(X) > 0$ and suppose that h (as in §1) satisfies $\int_0^1 \frac{h(t)\,dt}{t^N} < +\infty$. Then for $\gamma_{+,N}$-almost all $a \in X$ one has:*

$$\limsup_{\delta\to 0} \frac{\gamma_{+,N}\bigl(B(a,\delta)\cap X\bigr)}{h(\delta)} = +\infty. \tag{26}$$

The analogous result holds also for $\alpha_{+,N}$-capacity.

Proof. As before, it is enough to prove (26) for at least one point $a \in X$. Take $\mu \in M_+(X)$ with $\||\nabla\Phi_N * \mu|\| \leq 1$ (for $\alpha_{+,N}$ in addition we require continuity of $\nabla\Phi_n * \mu$). If there is $a \in X$ with

$$\limsup_{\delta\to 0} \frac{\mu\bigl(B(a,\delta)\cap X\bigr)}{h(\delta)} = +\infty, \tag{27}$$

then (26) also holds by [MPV, Lemma 3.3], which gives $\gamma_{+,N}\bigl(B(a,\delta)\cap X\bigr) \geq A\mu\bigl(B(a,\delta)\cap X\bigr)$ for some $A = A(N) > 0$ (for the case $\alpha_{+,N}$ we apply [MPV, Lemma 5.4]. If (27) fails for all $a \in X$ then we get $M_h(X) > 0$ and therefore (2) with $s = N-1$ for some $a \in X$. It suffices to invoke (25).

We shall need the following

Theorem 3. *Let r and δ be positive numbers, $E_N \subset B(0,r)$ in $\mathbf{R}^N$ $(N \geq 2)$ and $E_{N+1} = E_N \times [0,\delta] \subset \mathbf{R}^{N+1}$. Then*

$$\frac{A^{-1}\,\delta\,\gamma_{+,N}(E_N)}{\max\{1,(r/\delta)^2\}} \leq \gamma_{+,N+1}(E_{N+1}) \leq A\max\{\delta,r\}\,\gamma_{+,N}(E_N),$$
$$\frac{A^{-1}\,\delta\,\alpha_{+,N}(E_N)}{\max\{1,(r/\delta)^2\}} \leq \alpha_{+,N+1}(E_{N+1}) \leq A\max\{\delta,r\}\,\alpha_{+,N}(E_N),$$

with $A > 1$ depending only on N.

The proof of Theorem 3 is analogous to that of [MP, Theorem 3.1].

Corollary 5. *Let h satisfy conditions (b) of Theorem 2. Then for each $N \geq 3$ there exists a compact set X_N in $\mathbf{R}^N$ such that for each $a \in X_N$ one has*

$$A_1\,\delta^{N-2}\,h(\delta) \leq \gamma_{+,N}\big(B(a,\delta)\cap X_N\big) \leq A_2\,\delta^{N-2}\,h(\delta)$$

with $A_1 > 0$ and $A_2 > A_1$ depending only on N and h.

Proof. It is enough to take the corresponding X_1 from Theorem 2 (b) and set $X_N = X_1 \times [0,1]^{N-2}$. Theorem 3 and induction give the result. □

Therefore, here we have a gap for $\gamma_{+,N}$ and $\alpha_{+,N}$ in the sense that we do not have such complete results as in Theorems 1 and 2. Particularly, in the class of densities $\{t^{N-1}/|\log t|^q\}$, neither positive results nor counterexamples for $q \in (1/2,1]$ for $\gamma_{+,N}$ and for $q \in (0,1]$ for $\alpha_{+,N}$-capacity, $N \geq 3$.

We close this paper with the following

Proposition 2. *Let h, $c > 0$ and $g(r) = h(r)\exp\Big(-\int_r^1 \frac{ch^2(t)\,dt}{2t^3}\Big)$ satisfy conditions (9) and $ch^2(t) \leq t^2$, $t \geq 0$. Suppose that there exists some compact set X in $\mathbf{R}^2$ and $Y \subset X$ with $\gamma_+(Y) > 0$ such that $\gamma_+\big(B(y,\delta)\cap X\big) \asymp h(\delta)$ as $\delta \to 0$ for each $y \in Y$. Then there is a compact set X_1 with properties (10) for each $a \in X_1$ and $\delta \in (0,1)$.*

Proof. This h cannot satisfy condition (6), so that it satisfies (8) and then (b) of Theorem 2 applies. □

The analogous result holds for C_s-capacities.

One can construct such a compact set X, for which there exist countably many pairwise uncomparable functions h_j, satisfying Proposition 2 (they can be functions $\{t/|\log t|^{q_j}\}$, $0 \leq q_1 < q_2 \cdots < \cdots \leq 1/2$).

References

[C] L. Carleson, *Selected Problems on Exceptional Sets*, Van Nostrand, Princeton, 1966.

[E] V. Ya. Eiderman, *Estimates for potentials and δ-subharmonic functions outside exceptional sets*, Izvestiya: Mathematics **61:6** (1997), 1293–1329.

[F] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969.

[G] J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Math. 297, Springer-Verlag, 1972.

[M1] P. Mattila, *An example illustrating integralgeometric measures*, Amer. J. Math. **108** (1986), 693–702.

[M2] P. Mattila, *On the analytic capacity and curvature of some Cantor sets with non-σ-finite length*, Publ. Mat. **40** (1996), 195–204.

[MP] P. Mattila and P. V. Paramonov, *On geometric properties of harmonic* Lip_1*-capacity*, Pacific J. Math. **171:2** (1995), 469–490.

[MPV] M. S. Melnikov, P. V. Paramonov and J. Verdera, C^1*-approximation and extension of subharmonic functions*, CRM (Barcelona), Preprint 443 (2000), 25 pp. To appear in Math. Sbornik (Russia).

[T1] X. Tolsa, L^2*-boundedness of the Cauchy integral operator for continuous measures*, Duke Math. J. **98:2** (1999), 269–304.

[T2] X. Tolsa, *The analytic capacity* γ_+, Manuscript.

[V] A. G. Vitushkin, *Analytic capacity of sets in problems of approximation theory*, Uspekhi Mat. Nauk **22:6** (1967), 141–199, English transl. in Russian Math. Surveys 22:6 (1967).

P. Mattila, University of Jyväskylä, Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40351 Jyväskylä, Finland

E-mail address: pmattila@math.jyu.fi

P. V. Paramonov, Mechanics and Mathematics Faculty, Moscow State University, 119899 Moscow, Russia

E-mail address: petr@paramonov.msk.ru