NORM ESTIMATES FOR THE KAKEYA MAXIMAL FUNCTION WITH RESPECT TO GENERAL MEASURES

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Abstract. We generalize Bourgain's theorem on the two dimensional Kakeya maximal function by proving norm estimates with respect to measures satisfying certain conditions. We use this to extend the classical result of Davies on the Hausdorff dimension of Kakeya sets in the plane.

1. Introduction

Let S^1 be the unit circle in the plane. If $\delta > 0,\ e \in S^1,\ x \in \mathbb{R}^2$ then we define

$$T_e^\delta(x) = \left\{ y \in \mathbb{R}^2 : |(y-x)\cdot e| \leq \frac{1}{2},\ |(y-x) - ((y-x)\cdot e)e| \leq \frac{\delta}{2} \right\}.$$

Thus $T_e^\delta(x)$ is a rectangle of dimensions $1 \times \delta$ centered at x such that its side with length 1 is in the e direction. The *Kakeya maximal function* $\mathcal{K}_\delta : S^1 \to \mathbb{R}$ is defined for all locally integrable functions $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathcal{K}_\delta f(e) = \sup_{x \in \mathbb{R}^2} \frac{1}{|T_e^\delta(x)|} \int_{T_e^\delta(x)} |f(y)| dy.$$

$\mathcal{K}_\delta$ was introduced by Bourgain [1]. It is one of several similar maximal functions which have been studied by many authors going back at least to Cordoba [2]. Bourgain proved, using the Fourier transform, that $\mathcal{K}_\delta$ defines a bounded $L^2(\mathbb{R}^2) \to L^2(d\sigma)$ operator, where $d\sigma$ denotes arc length measure on S^1. Namely, there exists a constant C, independent of δ, such that

$$\|\mathcal{K}_\delta f\|_{L^2(d\sigma)} \leq C \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}.$$

Interpolating with $\|\mathcal{K}_\delta f\|_\infty \leq \|f\|_\infty$ we get

$$\|\mathcal{K}_\delta f\|_{L^p(d\sigma)} \leq C_p \left(\log \frac{1}{\delta} \right)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^2)}.$$

This estimate is sharp as can be seen, for example, by the Perron tree construction due to Schoenberg [4].

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In this paper we consider the problem of obtaining non trivial $L^p(\mathbb{R}^2) \to L^p(d\mu)$ estimates for $\mathcal{K}_\delta$ where μ is a measure supported on S^1. In particular, we study the influence of the geometric properties of μ on the operator norm of $\mathcal{K}_\delta$. It is clear that little can be said if μ is completely arbitrary. So, in order to get a meaningful problem, we have to impose certain restrictions on μ. It turns out that if μ satisfies the uniform growth condition $\mu(B(e,r)) \le \phi(r),\ e \in S^1$, for some function ϕ, then $\mathcal{K}_\delta$ defines for $p \ge 2$ an $L^p(\mathbb{R}^2) \to L^p(d\mu)$ operator whose norm is bounded by a concrete function of δ. If we further assume that μ is Ahlfors regular, a notion to be defined in the next section, then our estimates are sharp. The case of measures with finite energy seems different and we only obtain a weaker estimate. Finally we use our results to give lower bounds on the generalized Hausdorff measure, also to be defined in the next section, of a wide class of Kakeya-type subsets of the plane.

2. Background

We make some definitions and introduce the terminology that we will use throughout this paper.

$B(x,r)$ is the open disc of radius r centered at x.

$|\cdot|$ denotes 2-dimensional Lebesgue measure and $\dim(\cdot)$ Hausdorff dimension.

$x \lesssim y$ means $x \le Ax$ for some absolute constant A and similarly with $x \simeq y$.

$\mathrm{spt}(\mu)$ denotes the support of the measure μ.

The *α-energy* of a Borel measure μ is defined by

$$I_\alpha(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

A Borel measure μ is said to be *s-dimensional Ahlfors regular* if there exist $C_1,\ C_2 > 0$ such that for $x \in \mathrm{spt}(\mu),\ r \le 1$

$$C_1 r^s \le \mu(B(x,r)) \le C_2 r^s.$$

A *measure function* is a non-decreasing function $h(r),\ r \ge 0$, such that $\lim_{r\to 0} h(r) = 0$.

The *generalized Hausdorff outer measure* Λ_h with respect to a measure function h is defined for $A \subset \mathbb{R}^2$ by

$$\Lambda_h(A) = \sup_{\delta>0} \inf \left\{ \sum_j h(r_j) : A \subset \bigcup_j B(x_j, r_j),\ r_j < \delta \right\}.$$

When $h(r) = r^s$, Λ_h is the usual Hausdorff outer measure denoted by $\mathcal{H}^s$.

We will make use of the following, quite standard, lemma which gives bounds on the area of intersection of two thin rectangles in terms of their angle of intersection. The proof can be easily carried out by using elementary plane geometry.

Lemma 2.1. *Let $x, y \in \mathbb{R}^2$, $e_1, e_2 \in S^1$, $\delta < 1$. Then, with notation as in the introduction, we have*

$$|T^{\delta}_{e_1}(x) \cap T^{\delta}_{e_2}(y)| \lesssim \frac{\delta^2}{\delta + \theta(e_1, e_2)},$$

where $\theta(e_1, e_2) = \cos^{-1}|e_1 \cdot e_2|$ is the acute angle between e_1 and e_2.

3. Measures satisfying the uniform growth condition

The main result of this paper is the following.

Theorem 3.1. *Let μ be a Borel measure supported on S^1 such that $\mu(B(x, r)) \leq \phi(r)$ for some non-negative function ϕ. Then for all $p \geq 2$ there exists a constant A_p such that*

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \leq A_p C(\delta)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^2)},$$

where $C(\delta)$ is given by

$$C(\delta) = \int_1^{\frac{1}{\delta}} \phi(1/r)dr.$$

Proof. We will prove the $L^2(\mathbb{R}^2) \to L^2(d\mu)$ estimate. The theorem then follows by interpolation with $\|\mathcal{K}_\delta f\|_\infty \leq \|f\|_\infty$.

Without loss of generality we may assume that $\mathrm{spt}(\mu)$ is contained in the first quadrant. We cover $\mathrm{spt}(\mu)$ with a family $\{A_j\}$ of disjoint arcs each of length δ. Pick $e_j \in A \cap \mathrm{spt}(\mu)$ and let $a_j = \mu(A_j)$. Note that if $u, v \in A_j$ then for any $x \in \mathbb{R}^2$

$$T^\delta_u(x) \subset \tilde{T}^\delta_v(x),$$

where $\tilde{T}^\delta_v(x)$ is the rectangle with dimensions $2 \times 4\delta$ and with the same center and orientation as $T^\delta_v(x)$. Therefore, for every $e \in A_j$

$$\begin{aligned} \mathcal{K}_\delta f(e) &\lesssim \sup_{x \in \mathbb{R}^2} \frac{1}{|\tilde{T}^\delta_{e_j}(x)|} \int_{\tilde{T}^\delta_{e_j}(x)} |f(y)|dy \\ &\lesssim \frac{1}{\delta} \int_{\tilde{T}^\delta_{e_j}(x_j)} |f(y)|dy, \end{aligned}$$

for some $x_j \in \mathbb{R}^2$.

We estimate $\|\mathcal{K}_\delta f\|_{L^2(d\mu)}$ by duality. Let $g \in L^2(d\mu)$ such that $\|g\|_{L^2(d\mu)} = 1$ and put

$$c_j = \left(\int_{A_j} |g(e)|^2 d\mu(e) \right)^{\frac{1}{2}}.$$

Then, letting

$$Q_g = \int \mathcal{K}_\delta f(e) g(e) d\mu(e),$$

we have

$$\begin{aligned}
Q_g &= \sum_j \int_{A_j} \mathcal{K}_\delta f(e) g(e) d\mu(e) \\
&\lesssim \frac{1}{\delta} \sum_j \int_{\tilde{T}^\delta_{e_j}(x_j)} |f(y)| dy \int_{A_j} |g(e)| d\mu(e) \\
&\leq \frac{1}{\delta} \int |f(y)| \left(\sum_j a_j^{1/2} c_j \chi_{\tilde{T}^\delta_{e_j}(x_j)}(y) \right) dy \\
&\leq \frac{1}{\delta} \|f\|_{L^2(\mathbb{R}^2)} \left(\int \left(\sum_j a_j^{1/2} c_j \chi_{\tilde{T}^\delta_{e_j}(x_j)}(y) \right)^2 dy \right)^{\frac{1}{2}} \\
&= \frac{1}{\delta} \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_{i,j} a_i^{1/2} a_j^{1/2} c_i c_j |\tilde{T}^\delta_{e_i}(x_i) \cap \tilde{T}^\delta_{e_j}(x_j)| \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_{i,j} \frac{a_i^{1/2} a_j^{1/2} c_i c_j}{\delta + |e_i - e_j|} \right)^{\frac{1}{2}}
\end{aligned}$$

where the last inequality follows from Lemma 2.1.

Now let

$$F(i,j) = \frac{a_i^{1/2} a_j^{1/2}}{\delta + |e_i - e_j|},$$

and note that for all j

$$\begin{aligned}
\sum_i a_i^{1/2} F(i,j) &= a_j^{1/2} \sum_i \frac{a_i}{\delta + |e_i - e_j|} \\
&\lesssim a_j^{1/2} \sum_i \int_{A_i} \frac{d\mu(e)}{\delta + |e - e_j|} \\
&= a_j^{1/2} \int \frac{d\mu(e)}{\delta + |e - e_j|} \\
&= a_j^{1/2} \int_0^{\frac{1}{\delta}} \mu(\{e : \delta + |e - e_j| < 1/r\}) dr \\
&\leq a_j^{1/2} \left(\int_0^1 \mu(B(e_j, 1/r)) dr + \int_1^{\frac{1}{\delta}} \phi(1/r) dr \right) \\
&\lesssim a_j^{1/2} C(\delta).
\end{aligned}$$

Similarly, for all i

$$\sum_j a_j^{1/2} F(i,j) \lesssim a_i^{1/2} C(\delta).$$

Therefore

$$\begin{aligned}
Q_g &\lesssim \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i c_i \sum_j c_j F(i,j) \right)^{\frac{1}{2}} \\
&\le \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i \left(\sum_j c_j F(i,j) \right)^2 \right)^{\frac{1}{4}} \\
&\le \|f\|_{L^2(\mathbb{R}^2)} \left(\sum_i \left(\sum_j c_j^2 a_j^{-1/2} F(i,j) \right) \left(\sum_j a_j^{1/2} F(i,j) \right) \right)^{\frac{1}{4}} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{\frac{1}{4}} \left(\sum_{i,j} c_j^2 a_j^{-1/2} F(i,j) a_i^{1/2} \right)^{\frac{1}{4}} \\
&= \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{\frac{1}{4}} \left(\sum_j c_j^2 a_j^{-1/2} \sum_i a_i^{1/2} F(i,j) \right)^{\frac{1}{4}} \\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{\frac{1}{2}} \left(\sum_j c_j^2 \right)^{\frac{1}{4}} \\
&= \|f\|_{L^2(\mathbb{R}^2)} C(\delta)^{\frac{1}{2}}.
\end{aligned}$$

We conclude that

$$\|\mathcal{K}_\delta f\|_{L^2(d\mu)} = \sup\{Q_g : \|g\|_{L^2(d\mu)} = 1\} \lesssim C(\delta)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}^2)}.$$

□

Note that when $\phi(r) = r$, that is, when μ is absolutely continuous with L^∞ density with respect to arc-length measure on S^1, we get $C(\delta) = \log \frac{1}{\delta}$ and thus recover Bourgain's result.

Taking $\phi(r) = r^s,\ 0 < s < 1$, we obtain the following estimate for measures of fractal dimension.

Corollary 3.1. *Let μ be a Borel measure on S^1 such that $\mu(B(x,r)) \le r^s$ for some $s \in (0,1)$. Then, for $p \ge 2$*

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \lesssim \left(\frac{1}{\delta}\right)^{\frac{1-s}{p}} \|f\|_{L^p(\mathbb{R}^2)}.$$

We next show that the above estimate is sharp if μ satisfies an extra regularity condition.

Proposition 3.1. *Let $s \in (0,1)$ and μ be an s-dimensional Ahlfors regular measure on S^1. Then*

$$\|\mathcal{K}_\delta\|_{L^p(\mathbb{R}^2)\to L^p(d\mu)} \simeq \left(\frac{1}{\delta}\right)^{\frac{1-s}{p}}.$$

Proof. Let $\{e_j\}_{j=1}^N$ be a maximal $\delta/2$-separated set of points in $\mathrm{spt}(\mu)$. For each j let A_j be an arc of length δ centered at e_j. Then

$$1 \leq \sum_{j=1}^N \chi_{A_j}(e) \leq 2,\ \forall e \in \mathrm{spt}(\mu).$$

Hence

$$\mu(S^1) \leq \sum_{j=1}^N \mu(A_j) \leq 2\mu(S^1). \tag{1}$$

Note that Ahlfors regularity implies that $\mu(A_j) \simeq \delta^s$. Therefore (1) yields

$$N \simeq \left(\frac{1}{\delta}\right)^s.$$

Now let

$$E_\delta = \left\{x \in \mathbb{R}^2 : 0 < |x| \leq 4,\ \frac{x}{|x|} \in \bigcup_j A_j\right\}.$$

Then

$$1 \lesssim \mathcal{K}_\delta \chi_{E_\delta}(e),\ \forall e \in \mathrm{spt}(\mu).$$

Note that

$$|E_\delta| \lesssim N\delta \simeq \delta^{1-s}.$$

Consequently

$$\begin{aligned}
\|\mathcal{K}_\delta\|_{L^p(\mathbb{R}^2)\to L^p(d\mu)} &\geq \frac{\|\mathcal{K}_\delta \chi_{E_\delta}\|_{L^p(d\mu)}}{\|\chi_{E_\delta}\|_{L^p(\mathbb{R}^2)}} \\
&\gtrsim \left(\frac{1}{|E_\delta|}\right)^{\frac{1}{p}} \\
&\gtrsim \left(\frac{1}{\delta}\right)^{\frac{1-s}{p}}.
\end{aligned}$$

$\square$

4. Measures with finite energy

The next result shows that we can weaken the condition $\mu(B(x,r)) \leq r^s$ in Corollary 3.1 at the expense of obtaining a weaker estimate.

Theorem 4.1. *Let μ be a Borel measure on S^1 such that $I_s(\mu) < \infty$ for some $s \in (0,1)$. Then for every $p \geq 1$ and $\epsilon > 0$ there exists $C_{p,\epsilon} > 0$ such that*

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \leq C_{p,\epsilon} \left(\frac{1}{\delta}\right)^{\frac{1-s}{2p}+\epsilon} \|f\|_{L^{2p}(\mathbb{R}^2)}.$$

Proof. Fix $\delta > 0$ and let

$$A_i = \left\{ x \in S^1 : 2^{i-1} < \sup_{r \geq \delta} \frac{\mu(B(x,r))}{r^s} \leq 2^i \right\}.$$

Then there are $\lesssim \log \frac{1}{\delta}$ values of i such that $A_i \neq \emptyset$. Using the Besicovitch covering theorem we can find, for each i, a family of discs $\{B(r_{ij})\}_j$ such that

$$2^{i-1} < \frac{\mu(B(r_{ij}))}{r_{ij}^s} \tag{2}$$

and

$$\chi_{A_i} \leq \sum_j \chi_{B(r_{ij})} \leq C, \tag{3}$$

for some constant C. Note that for every $x \in B(r_{ij})$

$$\frac{\mu(B(r_{ij}))}{r_{ij}^s} \lesssim \int_{B(r_{ij})} \frac{d\mu(y)}{|x-y|^s}.$$

Therefore, by (2)

$$2^{i-1}\mu(B(r_{ij})) \lesssim \int_{B(r_{ij})} \int_{B(r_{ij})} \frac{d\mu(x)d\mu(y)}{|x-y|^s}.$$

Consequently

$$\begin{aligned} \mu(A_i) &\leq \sum_j \mu(B(r_{ij})) \\ &\lesssim \frac{1}{2^i} \sum_j \int_{B(r_{ij})} \int_{B(r_{ij})} \frac{d\mu(x)d\mu(y)}{|x-y|^s} \\ &\lesssim \frac{1}{2^i} I_s(\mu), \end{aligned}$$

where the last inequality follows from (3). Now let $\mu_i = \mu(A_i)\mu|A_i$. Then

$$\mu_i(B(x,r)) \lesssim r^s \; \forall r \geq \delta.$$

Hence, by Theorem 3.1

$$\|\mathcal{K}_\delta f\|_{L^{2p}(d\mu_i)} \lesssim \left(\frac{1}{\delta}\right)^{\frac{1-s}{2p}} \|f\|_{L^{2p}(\mathbb{R}^2)}.$$

It follows that

$$\begin{aligned} \int (\mathcal{K}_\delta f(e))^p d(\mu|A_i)(e) &= \frac{1}{\mu(A_i)} \|\mathcal{K}_\delta f\|^p_{L^p(d\mu_i)} \\ &\leq \frac{(\mu_i(S^1))^{1/2}}{\mu(A_i)} \|\mathcal{K}_\delta f\|^p_{L^{2p}(d\mu_i)} \\ &= \|\mathcal{K}_\delta f\|^p_{L^{2p}(d\mu_i)} \\ &\lesssim \left(\frac{1}{\delta}\right)^{\frac{1-s}{2}} \|f\|^p_{L^{2p}(\mathbb{R}^2)}. \end{aligned}$$

Summing over i we get

$$\begin{aligned} \|\mathcal{K}_\delta f\|^p_{L^p(d\mu)} &\lesssim \log\frac{1}{\delta}\left(\frac{1}{\delta}\right)^{\frac{1-s}{2}} \|f\|^p_{L^{2p}(\mathbb{R}^2)} \\ &\lesssim \left(\frac{1}{\delta}\right)^{\frac{1-s}{2}+\epsilon p} \|f\|^p_{L^{2p}(\mathbb{R}^2)}. \end{aligned}$$

□

Note that if μ satisfies $\mu(B(x,r)) \leq r^s$ then $I_t(\mu) < \infty$ for all $t < s$. So in view of Corollary 3.1, one would expect that an $L^p(\mathbb{R}^2) \to L^p(d\mu)$ estimate like

$$\|\mathcal{K}_\delta f\|_{L^p(d\mu)} \leq C_{p,\epsilon}\left(\frac{1}{\delta}\right)^{\frac{1-s}{p}+\epsilon} \|f\|_{L^p(\mathbb{R}^2)},\ p \geq 2$$

should hold. We don't, however, know how to prove this.

5. An application to Kakeya-type sets

If $e \in S^1$ and $x \in \mathbb{R}^2$ then we define

$$l_e(x) = \left\{x + te : t \in \left[-\frac{1}{2}, \frac{1}{2}\right]\right\}.$$

Let A be a subset of S^1. A set $E \subset \mathbb{R}^2$ is said to be an *A-Kakeya set* if for every $e \in A$ there exists $x_e \in \mathbb{R}^2$ such that $l_e(x_e) \subset E$. In other words, an A-Kakeya set contains a unit line segment in the e-direction for every $e \in A$. We will use Theorem 3.1 to obtain lower bounds on the size of an A-Kakeya set in terms of the size of A, generalizing the classical result of Davies [3] on the Hausdorff dimension of usual Kakeya sets in the plane. This type of argument originates in [1].

Theorem 5.1. *Let A be a subset of S^1 such that $\Lambda_h(A) > 0$ for some measure function h. If E is an A-Kakeya set in the plane then for all $\epsilon > 0$, $\Lambda_{r^{2-\epsilon}\psi(r)}(E) > 0$, where ψ is given by*

$$\psi(r) = \int_1^{\frac{2}{r}} h(1/s)ds.$$

Proof. Using Frostman's lemma we can find a measure μ supported on A such that $\mu(B(x,r)) \leq h(r)$. Let $\{B(a_j,r_j)\}_j$ be a countable covering of E by discs. Without loss of generality we may assume that $r_j \leq 1$. Define

$$J_k = \left\{ j : 2^{-k} < r_j \leq 2^{-(k-1)} \right\},$$

$$E_k = \bigcup_{j \in J_k} B(a_j, r_j),$$

$$\tilde{E}_k = \bigcup_{j \in J_k} B(a_j, 2r_j).$$

By the definition of E, for each $e \in A$ there is $x_e \in \mathbb{R}^2$ such that $l_e(x_e) \subset E$. Using the pigeonhole principle and the fact that $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ one can show that for each $e \in A$ there exists k_e such that

$$\mathcal{L}^1(E_{k_e} \cap l_e(x_e)) \geq \frac{6}{\pi^2 k_e^2},$$

where $\mathcal{L}^1$ is linear Lebesgue measure. By the pigeonhole principle again, there exists k such that

$$\mu^* \left(\left\{ e \in A : \mathcal{L}^1(E_k \cap l_e(x_e)) \geq \frac{6}{\pi^2 k^2} \right\} \right) \geq \frac{6\mu(S^1)}{\pi^2 k^2},$$

where μ^* is the corresponding outer measure. Using measure theory, we can find a Borel set $\tilde{A} \subset S^1$ with

$$\mu(\tilde{A}) \geq \frac{\mu(S^1)}{2k^2}$$

such that for all $e \in \tilde{A}$ there is $x'_e \in \mathbb{R}^2$ with

$$\mathcal{L}^1(E_k \cap l_e(x'_e)) \geq \frac{1}{2k^2}.$$

With this k, note that if $e \in \tilde{A}$ then

$$|T_e^{2^{-k}}(x'_e) \cap \tilde{E}_k| \gtrsim \frac{1}{k^2} |T_e^{2^{-k}}(x'_e)|$$

and consequently

$$\mathcal{K}_{2^{-k}} \chi_{\tilde{E}_k}(e) \gtrsim \frac{1}{k^2}.$$

Therefore, by Theorem 3.1

$$\begin{aligned}\frac{1}{k^6} &\lesssim \int_{\tilde{A}} \left(\mathcal{K}_{2^{-k}}\chi_{\tilde{E}_k}(e)\right)^2 d\mu(e)\\ &\lesssim C(2^{-k})|\tilde{E}_k|\\ &\lesssim C(2^{-k})|J_k|2^{-2k}.\end{aligned}$$

It follows that

$$|J_k| \gtrsim \frac{2^{2k}}{k^6 C(2^{-k})}.$$

Hence

$$\begin{aligned}\sum_j r_j^{2-\epsilon}\psi(r_j) &\gtrsim |J_k|\left(\frac{1}{2^k}\right)^{2-\epsilon}\psi(2^{-k+1})\\ &\gtrsim \frac{2^{k\epsilon}\psi(2^{-k+1})}{k^6 C(2^{-k})}\\ &\gtrsim 1.\end{aligned}$$

Therefore

$$\Lambda_{r^{2-\epsilon}\psi(r)}(E) > 0.$$

$\square$

If we specialize to the case of the usual Hausdorff measure by taking $h(r) = r^s$ then we obtain the following.

Corollary 5.1. *Let E be an A-Kakeya set in the plane. Then* $\dim(E) \geq \dim(A) + 1$.

Note that the trivial example

$$E = \left\{x \in \mathbb{R}^2 : |x| > 0,\ \frac{x}{|x|} \in A\right\}$$

shows that the above estimate is sharp.

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