Composites and quasiconformal mappings: new optimal bounds

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1 Introduction

We study a two dimensional G-closure problem. We consider multiphase composites made of an arbitrary number of linearly conducting phases. We will establish new optimal bounds for the homogenized tensor of a large class of composites.

To explain our results in detail, let us introduce some notations. Let $Q = (0,1) \times (0,1)$ be the unit square in $\mathbf{R}^2$. Let $\sigma(x)$ be a symmetric, uniformly elliptic and bounded matrix function, having measurable entries and being defined for all $x \in \mathbf{R}^2$. Assume also that σ is Q-periodic. Then the effective conductivity σ_{hom} is the unique, constant and symmetric matrix satisfying

$$\forall A \in \mathcal{M} \ , \ \mathrm{Tr}[A\sigma_{\text{hom}}A^t] = \inf_{U \in W^{1,2}_\sharp(Q;\mathbf{R}^2)} \int_Q \mathrm{Tr}[(DU(x)+A)\sigma(x)(DU(x)+A)^t]dx \tag{1.1}$$

where the space $W^{1,2}_\sharp(Q;\mathbf{R}^2)$ denotes the completion of the space of Q-periodic vector fields with respect to the $W^{1,2}$ norm, $\mathcal{M}$ denotes the set of two by two real matrices and Tr denotes the trace.

We now describe the class of conductivity matrices considered in our study. Fix an arbitrary integer $N \geq 1$ and N real numbers

$$0 < \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_N \ . \tag{1.2}$$

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Given N real numbers $p_i > 0$, (called the the volume fractions) with $\sum_{i=1}^N p_i = 1$, and a real number $K \geq 1$, we consider a Q-periodic conductivity tensor of the form

$$\sigma(x) = R^t(x) \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix} R(x)\chi_1(x)\sigma_1 + \sum_{i=2}^N \chi_i(x)\sigma_i I, \qquad x \in Q, \tag{1.3}$$

where the χ_i are Q-periodic characteristic functions of measurable sets satisfying $\frac{1}{|Q|}\int_Q \chi_i(x)dx = p_i$, $x \mapsto R(x)$ is a measurable field of rotation matrices (i.e. $R(x) \in SO(2)$ for each x) and I denotes the identity matrix. The corresponding G-closure problem consists of characterizing the (closure of the) set of constant matrices which arise as a homogenized (or effective) conductivity when one allows the "microgeometry" i.e. both the field of rotations R and the characteristic functions χ_i to vary over their respective ranges. We emphasize that the set of values of K, σ_i and p_i are given parameters which we call for brevity, the G-closure parameters.

For clarity of exposition we will consider only bounds that are optimal when the composite is isotropic, i.e. $\sigma_{\text{hom}} = hI$ for some real number h. However, as in [5], [26] similar methods do yield bounds also for non isotropic composites; we leave the details for the reader.

With these notations, we begin by recalling the following generalization of the celebrated Hashin-Shtrikman bounds [14].

Theorem 1.1 *(Kohn-Milton, [25], Section 4, formula (4.22)) Let σ be of the form (1.3), (1.2). If $K = 1$, then σ_{hom} satisfies the following inequality*

$$\frac{2}{\mathrm{Tr}(\sigma_{\text{hom}}) + 2\sigma_1} \leq \sum_{i=1}^N \frac{p_i}{\sigma_i + \sigma_1} \quad . \tag{1.4}$$

It will be convenient to use the notation

$$c_i = \frac{\sigma_i - \sigma_1}{\sigma_i + \sigma_1} \quad , \quad i = 2, 3, \ldots, N \quad . \tag{1.5}$$

Then we write (1.4) in the equivalent form

$$\frac{2}{\mathrm{Tr}(\sigma_{\text{hom}}) + 2\sigma_1} \leq \frac{1}{2\sigma_1}\left[1 - \sum_{i=2}^N c_i p_i\right] \quad . \tag{1.6}$$

Milton [22] was the first to prove the following sufficient condition for the optimality of the bounds (1.6).

Theorem 1.2 *(See [25], Section 4, formula (4.29)) Under the same assumptions of Theorem 1.1, if* σ_{hom} *is isotropic, then the bounds (1.6) are optimal provided the following inequality holds*

$$c_2 \geq \sum_{i=2}^{N} c_i p_i \quad . \tag{1.7}$$

In fact a slightly better result can be proved.

Theorem 1.3 *(Gibiansky-Sigmund, [13], Section 5.3, formula (80)) The inequality (1.4) is optimal if the following condition holds*

$$p_1 \geq \frac{2\sigma_1(\sigma_3 - \sigma_2)(\sqrt{p_2} - p_2)}{(\sigma_2 + \sigma_1)(\sigma_3 - \sigma_1)} \quad . \tag{1.8}$$

For more details see Section 5 of [13].

From the composites viewpoint, our main result is a generalization of Theorems 1.1 and 1.2 to the class of polycrystalline composites described at the beginning of the section. Indeed, we prove the following results.

Theorem 1.4 *Let* σ *be of the form (1.3), (1.2) and let* $K \geq 1$ *be arbitrary. Then,* σ_{hom} *satisfies the following inequality*

$$\frac{2}{\text{Tr}(\sigma_{\text{hom}}) + 2\sigma_1} \leq \frac{1}{2\sigma_1}\left[1 - \left(\sum_{i=2}^{N} c_i^{\frac{1}{K}} p_i\right)^K\right] \quad , \tag{1.9}$$

where the c_i *are defined in (1.5).*

The proof will be presented in Section 3.

Theorem 1.5 *Under the same assumptions of Theorem 1.4, , if* σ_{hom} *is isotropic, then the bounds (1.9) are optimal provided*

$$c_2 \geq \left(\sum_{i=2}^{N} c_i^{\frac{1}{K}} p_i\right)^K \quad . \tag{1.10}$$

Section 4 is devoted to the proof of the optimality. Note that our condition (1.10) reduces to (1.7) as K tends to one. It would be desirable to have a condition reducing to (1.8) but we were unable to achieve this. In fact among the class of microgeometries which are known to be optimal for the case $K = 1$, only some appear to admit a natural generalization to the polycrystalline case.

Let us briefly put our results into context by considering some special cases. In the following, we restrict attention to the case of prescribed volume fractions and isotropic composites.

Case 1: $N = 2$, $K = 1$. The bounds (1.9) reduce to those of Tartar and Murat [34] and Lurie and Cherkaev [19]. If in addition one assumes that σ_{hom} is isotropic, the bounds reduce to those of Hashin-Shtrikman [14]. They are optimal for any choice of the G-closure parameters.

Case 2: $N = 2$, $K > 1$. The bounds (1.9) reduce to those of Nesi [29]. When $\sigma_1 < \sigma_2$, they are optimal for any choice of the remaining G-closure parameters.

Case 3: $N > 2$, $K = 1$. The bounds reduce to those of Kohn and Milton [25], (see also Zhikov [37]). For isotropic composites, the bounds are due to Hashin and Shtrikman [14]. They are optimal for any choice of the G-closure parameters satisfying (1.8).

Case 4: $N > 2$, $K > 1$. The bounds are new. When $\sigma_1 < \sigma_2$, they are optimal for any choice of the remaining G-closure parameters satisfying (1.10).

We wish to emphasize that from the point of view of the bounds, the new results for the cases $N > 2$ and $K > 1$ are obtained using new sharp distortion estimates for planar quasiconformal mappings. These inequalities are explained in detail later in this section and proved in Section 2. In contrast to some recent work on the subject, c.f. [29], [5], [26], the already known results on quasiconformal mappings due to Astala [3] do not suffice as such in the present G-closure problem. The new results which are needed were in fact suggested exactly by the G-closure problem under study.

From the point of view of optimal microstructures, we generalize previous work by combining ideas due to Cherkaev and Lurie [21], Kohn and Milton [25] and Schulgasser [32].

Although already restricted to two dimensional problems in linear conductivity, our discussion of bounds and their optimality is far from being complete. We quote here some of the relevant literature on the subject. When the volume fraction of the phases is not prescribed and $N = 2$, the so-called two phase polycrystalline problem has been fully solved by Francfort and Murat [12], completing previous work by Lurie and Cherkaev [20]. In [11] there are results concerning the case of unconstrained volume fractions with an arbitrary number of phases. A few more recent papers attempt the task of finding good candidates for optimal microgeometries (both isotropic and anisotropic). They include [13], [33], [36]. We mention in particular the work by Cherkaev and Gibiansky, [9], where new optimal anisotropic microgeometries are exhibited for $N = 3$ and $K = 1$. It is important to recall that, even for $K = 1$, if $N \geq 3$, it is known that the bounds (1.4) cannot be optimal for all the values of the G-closure parameters because, in certain regimes, better bounds are available [28].

It is worth mentioning that Cherkaev, [8], found a new type of isotropic composite for the case $N = 3$ and $K = 1$. This microgeometry is likely to be optimal under an additional constraint on the G-closure parameters when (1.8) is not satisfied. The reason to conjecture optimality is that its effective conductivity approaches the value predicted by the bounds in [28] as σ_3 diverges.

Another issue which deserves a comment is the exact definition of an optimal bound. Let us emphasize that in our Theorem 1.4, we understand optimality in its most stringent sense. Namely we show that for any value of the G-closure parameters satisfying (1.10), one can find a Q-periodic σ satisfying (1.3) and (1.2), such that the corresponding σ_{hom} is isotropic and it satisfies (1.9) as an equality. Very often, optimality is understood in a weaker sense. For instance in Theorem 1.3 optimality was considered in the sense of an approximating procedure.

Let us then turn to the quasiconformal aspects of our work. Recall that a mapping $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ is K-*quasiregular*, $K \geq 1$, if $f \in W^{1,2}_{\mathrm{loc}}$ and

$$\|Df(x)\|^2 \leq K J_f(x) \quad \text{a.e. } x \in \mathbf{R}^2. \tag{1.11}$$

Here $\| \; \|$ denotes the operator norm. It follows that quasiregular mappings are open, discrete and continuous. If, in addition, f is a homeomorphism, then the map f is called K-*quasiconformal.* The are a number of other (nontrivially) equivalent ways of defining these mappings, c.f. Section 2.

Basically, the connection of quasiconformality to composites arises in this paper in a manner similar to the fundamental work [29] where these relations were first established. However, in the case of multiphase problems one needs instead of [3] results on the distortion of *weighted* area under a $K-$quasiconformal mapping f. To obtain sharp bounds we shall use for f the so called hydrodynamical normalization $f \in \Sigma_K$; for details see Section 2, formula (2.6). Under this assumption we prove

Theorem 1.6 *Suppose $f \in \Sigma_K$ is $K-$quasiconformal in $\mathbf{R}^2$. Assume that we have a measurable set $E \subset \mathbf{D} = \{x \in \mathbf{R}^2 : |x| < 1\}$ and a (measurable) weight $w(x) \geq 0$, $x \in E$.*

If $f|_E$ is conformal, meaning that $\overline{\partial} f(x) = 0$ a.e. $x \in E$, then

$$\Big(\int_E w(x)^{1/K} dm(x)\Big)^K \leq \int_E w(x) J_f(x) dm(x) \leq \Big(\int_E w(x)^K dm(x)\Big)^{1/K}. \tag{1.12}$$

As we shall see Theorem 1.6 is sharp; there is a large class of weights and $K-$quasiconformal mappings f for which one of the bounds (1.12) holds as an equality, see Section 2.

From the point of view of composites, Theorem 1.6 is the result needed for the $G-$closure bounds and a reader looking for a minimal approach to composites may proceed with (1.12) directly to Section 3. However, from the point of view of quasiconformal and quasiregular mappings (1.12) has interesting consequences to their regularity theory. In particular, recall [3] that in dimension $n = 2$ any $K-$quasiconformal mapping f is contained in the Sobolev class $f \in W^{1,p}_{\text{loc}}$ for each $p < p_K \equiv \frac{2K}{K-1}$ while there are simple examples showing that in general $f \notin W^{1,p_K}_{\text{loc}}$. Somewhat surprisingly, when $f|_E$ is conformal as in Theorem 1.6, then the derivatives of f do integrate at the borderline case $p = p_K$, even if the set E can be of quite irregular character.

Theorem 1.7 *Suppose $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a $K-$quasiregular mapping and $E \subset \mathbf{R}^2$ is measurable and bounded. If $\overline{\partial} f(x) = 0$ a.e. $x \in E$, then, for $K > 1$,*

$$\int_E |Df(x)|^p dx < \infty \quad \text{for } p = \frac{2K}{K-1}. \tag{1.13}$$

Again we show by an example that for some f satisfying the assumptions, (1.13) fails for all exponents $p > \frac{2K}{K-1}$. If in addition, $f \in \Sigma_K$ we prove the sharp bound

$$\int_E J_f(x)^p dm(x) \leq 1, \quad p = \frac{K}{K-1}.$$

The higher integrability has similar consequences for solutions to elliptic PDE's, see Section 5.

The organization of the paper is as follows: Section 2 is devoted to the proof of Theorem 1.6. In Section 3 we prove the composite bounds of Theorem 1.4 and in Section 4 we show their optimality. Finally, Section 5 studies some of the quasiconformal consequences of Theorem 1.6.

2 Area distortion and quasiconformal mappings

The definition (1.11) of a K-quasiregular mapping says that at almost every point the partial derivatives of f are of the same order, up to the constant K. There are of course many other ways to express this property. A convenient definition, equivalent to (1.11) is

$$Df(x)^t Df(x) = J_f(x) G(x), \tag{2.1}$$

where

$$\frac{1}{K}|h|^2 \leq \langle G(x)h, h\rangle \leq K|h|^2, \quad h \in \mathbf{R}^2. \tag{2.2}$$

It follows e.g. that f satisfies the linear elliptic PDE

$$\mathrm{Div}\Big(\sigma_f(x) Df(x)^t\Big) = 0, \qquad \sigma_f(x) = G(x)^{-1}, \tag{2.3}$$

determined by its *matrix dilatation* σ_f.

These identities yield a number of different representations for planar quasiregular mappings. For later purposes it will be essential to consider these notions also in the complex analytic terms. Here recall [1] [17], that a $W^{1,2}_{\mathrm{loc}}$-mapping f is K-quasiregular if and only if it satisfies the Beltrami equation

$$\overline{\partial} f(x) = \mu(x) \partial f(x) \quad \text{a.e. } x \in \mathbf{R}^2, \tag{2.4}$$

where

$$|\mu(x)| \leq \frac{K-1}{K+1} \quad \text{a.e. } x \in \mathbf{R}^2. \tag{2.5}$$

Here $\mu = \mu_f$ is the *complex dilatation* of f; note in particular, that $\mu_f = 0$ in a domain $\Omega \subset \mathbf{R}^2$ if and only if $\overline{\partial} f(x) = 0$ there, i.e. f is complex analytic in Ω.

Furthermore, as is well known [1], [16] and [17], any $K-$quasiregular mapping f is of the form $f = h \circ g$ where h is complex analytic ($\overline{\partial}h \equiv 0$) and g is K-quasiconformal, in particular homeomorphic. Therefore many of the properties of f reduce to those of g. Hence in the rest of this section we consider only quasiconformal mappings.

It is clear from above that quasiconformal mappings have a number of connections and applications to PDE's. What make them especially useful in this setting are the various strong distortion properties of quasiconformal mappings. A particular example is Theorem 1.6, to be proven below.

On the other hand, it is equally clear that explicit and optimal distortion estimates are possible only under special normalizations of the mappings. It is for this purpose that we must introduce auxiliary subclasses.

Let Σ_K denote the class of K-quasiconformal mappings that are conformal outside the unit disk $\mathbf{D} = \{x \in \mathbf{R}^2 : |x| < 1\}$ and are normalized by

$$f(x) = x + O(1/|x|), \qquad |x| > 1. \tag{2.6}$$

The choice (2.6) is the classical one; there is a wealth of results in the theory of conformal mappings using this normalization. From there we obtain the first simple bounds.

For aesthetic reasons we denote by $dm = \frac{1}{\pi}dx$ the standard Lebesgue measure divided by π, so that the unit disk has measure $m(\mathbf{D}) = 1$.

Lemma 2.1 ([31], Theorem 1.3) *Suppose $f \in \Sigma_K$, $K \geq 1$. Then*

$$\int_{\mathbf{D}} J_f dm(x) = m(f(\mathbf{D})) \leq 1. \tag{2.7}$$

A similar identity is obtained from (2.6) via the Stokes theorem.

Lemma 2.2 *For each $f \in \Sigma_K$,*

$$\int_{\mathbf{D}} \mathrm{Tr}\,(Df(x))dm(x) = 2\Re\{\int_{\mathbf{D}} \partial f(x)dm(x)\} = 2. \tag{2.8}$$

After these preliminaries we are ready for the main result of this section.

Proof of Theorem 1.6 We shall make use of the basic arguments in [3]. We establish the claim first for open sets E and prove the general case later by approximation. Moreover, we consider only weight functions $w \geq 0$ that are bounded away from 0 and ∞ on the set E; the case of general w follows then by an obvious limiting argument.

Therefore suppose that the weight $w(x)$ and $f \in \Sigma_K$ are given, with $\overline{\partial} f(x) = 0$ for a.e. $x \in E$. Our principal goal is then to show that

$$\int_E w(x) J_f(x) dm(x) \leq \Big(\int_E w(x)^K dm(x) \Big)^{1/K}. \tag{2.9}$$

For this, let us begin by representing the $f \in \Sigma_K$ as a solution to the Beltrami equation

$$\overline{\partial} f(x) = \mu(x) \partial f(x), \quad |\mu| \leq \frac{K-1}{K+1} \chi_{\mathbf{D} \setminus E}. \tag{2.10}$$

Then for each complex number $|\lambda| < 1$ we write $\mu_\lambda(z) = \lambda \frac{K+1}{K-1} \mu(z)$, $z \in \mathbf{C}$. According to the measurable Riemann mapping theorem ([1], [17]) for each $\lambda \in \mathbf{D}$ we have a (unique) $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$-solution f_λ to

$$\overline{\partial} f_\lambda(x) = \mu_\lambda(x) \partial f_\lambda(x) \tag{2.11}$$

satisfying (2.6). In particular, with our choice of μ_λ, one has

$$\lambda = \frac{K-1}{K+1} \implies f_\lambda = f. \tag{2.12}$$

Furthermore [1], the solution $f_\lambda(x)$ depends holomorphically on the parameter λ. Since E is assumed to be open, $f_\lambda|_E$ is conformal in the classical sense and in particular f_λ is smooth, even complex analytic on E. Therefore, c.f. [16] page 69, for each fixed $x \in E$ the function $\lambda \mapsto f'_\lambda(x) = \partial f_\lambda(x)$ is holomorphic in $\mathbf{D}$. Note also that by conformality on E,

$$f'_\lambda(x) \neq 0 \quad \forall x \in E, \ \forall \lambda \in \mathbf{D}. \tag{2.13}$$

Next, let us use the concavity of the logarithm in conjunction with Jensen's inequality in the following manner: For any function $a(x) > 0$, $x \in E$, we have

$$\log \int_E a(x)\, dm(x) = \sup_p \Big[\int_E p(x) \log\Big(\frac{a(x)}{p(x)}\Big)\, dm(x) \Big]. \tag{2.14}$$

Here the supremum is taken over the functions p such that

$$\begin{aligned} &\text{i)} \quad 0 < p(x) < 1 \quad \text{a.e. } x \in E \qquad \text{and} \\ &\text{ii)} \quad \int_E p(x)\, dm(x) = 1. \end{aligned}$$

Note that the supremum is attained when $p(x) \equiv a(x)/\int_E a\, dm$.

In our case we take $a(x) = w(x)J_{f_\lambda}(x) = w(x)|f'_\lambda(x)|^2$, $x \in E$. Then (2.14) gives

$$\begin{aligned} &\log\Big(\int_E w(x)|f'_\lambda(x)|^2 dm(x)\Big) \qquad (2.15)\\ &= \sup_p \Big[\int_E p(x) \log w(x)\, dm(x) + \int_E p(x) \log\Big(\frac{|f'_\lambda(x)|^2}{p(x)}\Big)\, dm(x)\Big]. \end{aligned}$$

Here the latter integral

$$h_p(\lambda) := \int_E p(x) \log\Big(\frac{|f'_\lambda(x)|^2}{p(x)}\Big)\, dm(x) \qquad (2.16)$$

is harmonic in λ, by (2.13). Moreover, we can use Lemma 2.1 to deduce

$$h_p(\lambda) \le \log \int_E |f'_\lambda(x)|^2 \le \log\Big(\int_{\mathbf{D}} J_{f_\lambda} dm\Big) \le 0. \qquad (2.17)$$

Consequently, $\lambda \mapsto h_p(\lambda)$ is harmonic and nonpositive in $\mathbf{D}$.

We are now in position to use the Harnack's inequality. It gives

$$h_p(\lambda) \le \frac{1-|\lambda|}{1+|\lambda|} h_p(0) = \frac{1-|\lambda|}{1+|\lambda|} \int_E p(x) \log\Big(\frac{1}{p(x)}\Big)\, dm(x). \qquad (2.18)$$

Combining (2.15) and (2.18) we obtain (with $\lambda = \frac{K-1}{K+1}$) that

$$\begin{aligned} &\log\Big(\int_E w(x)J_f(x)dm(x)\Big)\\ &\le \sup_p \Big[\int_E p(x) \log w(x) + \frac{1}{K}\int_E p(x) \log\Big(\frac{1}{p(x)}\Big)\, dm(x)\Big]\\ &= \frac{1}{K} \sup_p \Big[\int_E p(x) \log\Big(\frac{w(x)^K}{p(x)}\Big)\, dm(x)\Big]\\ &= \frac{1}{K} \log\Big(\int_E w(x)^K dm(x)\Big) \qquad (2.19) \end{aligned}$$

by (2.14). Exponentiating (2.19) gives the estimate (2.9) for open sets E; this is the latter of the inequalities (1.12) in Theorem 1.6.

To deduce the first of the inequalities (1.12) for E open, one needs to use Harnack's inequality in the form

$$h_p(\lambda) \geq \frac{1+|\lambda|}{1-|\lambda|} h_p(0) \tag{2.20}$$

rather than (2.18) and then otherwise argue in a similar fashion.

Let us lastly consider the case of arbitrary measurable sets E such that $f|_E$ is conformal. Choose a decreasing sequence $\{E_n\}_{n=1}^{\infty}$ of open subsets such that $E \subset E_n$ and

$$m(E_n \setminus E) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.21}$$

Set $\mu_n = \mu \chi_{\mathbf{D} \setminus E_n}$, with μ as in (2.10), and let f_n be the corresponding quasiconformal mapping, normalized by (2.6) and satisfying the Beltrami equation $\overline{\partial} f_n(x) = \mu_n(x) \partial f_n(x)$. It follows from [1], Lemma V.B.1, p.93, that $\|\partial f_n - \partial f\|_{L^2(\mathbf{C})} \to 0$ as $n \to \infty$. Recall that on the sets E_n and E, we have $J_{f_n} = |\partial f_n|^2$ and $J_f = |\partial f|^2$ respectively. Since w can be assumed to be bounded from above on, say, E_1 we deduce

$$\int_E w(x) J_f(x) dm(x) = \lim_{n\to\infty} \int_{E_n} w(x) J_{f_n}(x) dm(x). \tag{2.22}$$

Clearly, for any $q > 0$

$$\int_E w(x)^q dm(x) = \lim_{n\to\infty} \int_{E_n} w(x)^q dm(x). \tag{2.23}$$

As the inequalities (1.12) hold for each of the open sets E_n, using (2.22) and (2.23) gives the proof of the Theorem 1.6. □

The inequalities of Theorem 1.6 are optimal, as can be shown by several examples. Indeed, let us consider the family of weights w attaining only finitely many values. Then, if a condition corresponding to (1.10) is satisfied, for any *apriori* given choice of the distribution function of w we will construct examples achieving the equality.

Example 2.1 *Suppose* $0 < w_1 < \ldots < w_n$ *and* $0 < p_j < 1,\ j = 1, \ldots, n,$ *satisfy*

$$\sum_{j=1}^{n} p_j w_j^K \leq w_1^K. \tag{2.24}$$

Then there is an open set $E \subset \mathbf{D}$ with $m(E) = \sum_{j=1}^{n} p_j$, a K-quasiconformal mapping $f \in \Sigma_K$ with $f|_E$ conformal and a weight w, positive on E, such that

$$m(\{x \in E : w(x) = w_j\}) = p_j, \quad j = 1, \ldots, n, \tag{2.25}$$

$$\int_E w(x) J_f(x) dm(x) = \Big(\int_E w(x)^K dm(x) \Big)^{1/K}. \tag{2.26}$$

To construct mappings and weights satisfying (2.25), (2.26) we need some notation. First, we may normalize the numbers w_j so that

$$\sum_{j=1}^{n} p_j w_j^K = 1. \tag{2.27}$$

Then by our assumptions $w_1 \geq 1$ and $w_{j+1} > w_j$. Hence we can find numbers $0 < R_j \leq 1$ so that

$$w_j = \Big(\prod_{l=1}^{j} R_l \Big)^{-2/K}, \quad j = 1, \ldots, n. \tag{2.28}$$

Consider next a version of the radial stretching, a basic example of a $K-$quasiconformal mapping in Σ_K,

$$f_R(x) = \begin{cases} R^{\frac{1}{K}-1} x, & 0 \leq |x| \leq R \\ x|x|^{\frac{1}{K}-1}, & R < |x| \leq 1 \\ x, & 1 < |x|. \end{cases} \tag{2.29}$$

Define also $f_R^\rho(x) = \rho f_R(\frac{x}{\rho})$, $x \in \mathbf{R}^2$.

To complete the notations let ρ_j, $j = 1, \ldots, n$, be numbers defined by the following rule,

$$R_n^2 \rho_n^2 = p_n, \quad R_j^2 \rho_j^2 - \rho_{j+1}^2 = p_j, \quad 1 \leq j \leq n-1. \tag{2.30}$$

Then by (2.28), (2.27), $\rho_1 = 1$.

After these preparations set simply

$$f = f_{R_1}^{\rho_1} \circ \ldots \circ f_{R_n}^{\rho_n}, \qquad w = \sum_{j=1}^{n} w_j \chi_{E_j} \tag{2.31}$$

where

$$E_n = \{x : |x| < \rho_n R_n\}; \quad E_j = \{x : \rho_{j+1} < |x| < \rho_j R_j\}, \quad 1 \leq j \leq n-1. \tag{2.32}$$

Then by (2.30) $m(E_j) = p_j$ for each $j = 1, \ldots, n$, (2.25) is clearly satisfied and (2.26) follows since by the chain rule $J_{f|_E} \equiv w^{K-1}$ on the set $E = \cup_{j=1}^{n} E_j$.

3 New G-closure bounds

In this section we prove new G-closure bounds. We will in fact prove bounds for the so-called overall conductivity. Standard arguments in homogenization theory yield then the corresponding statement for the homogenized conductivity (see for instance [29]).

Let us consider matrices σ of the form (1.3), (1.2) defined on a domain Ω (instead of the square Q). Then the overall, or macroscopic, conductivity σ^* is defined as follows: for any $A \in \mathcal{M}$,

$$\mathrm{Tr}[A\sigma^* A^t] = \inf_{U \in W_0^{1,2}(\Omega;\mathbf{R}^2)} \frac{1}{\mid \Omega \mid} \int_\Omega \mathrm{Tr}[(DU(x)+A)\sigma(x)(DU(x)+A)^t]dx \quad . \tag{3.1}$$

We use the following result.

Theorem 3.1 *Let $\Omega \subset \mathbf{R}^2$ be a simply connected, bounded open set with Lipschitz boundary. Let σ be a uniformly elliptic symmetric matrix with bounded measurable coefficients defined in Ω and let σ^* be the associated overall conductivity as defined above. Set $d_m = \operatorname{ess\,inf} \sqrt{\det \sigma}$. Then for any $\lambda \in (0, d_m)$, one has*

$$\frac{1}{\mathrm{Tr}(\sigma^*) + 2\lambda} = \inf_{\Psi \in T} \frac{1}{\mid \Omega \mid} \int_\Omega \frac{\mathrm{Tr}[D\Psi(x)\sigma(x)D\Psi(x)^t] - 2\lambda \det D\Psi(x)}{\det \sigma(x) - \lambda^2} dx \quad , \tag{3.2}$$

where

$$T \equiv \left\{ \Phi \in W^{1,2}(\Omega;\mathbf{R}^2) \quad , \quad \frac{1}{\mid \Omega \mid} \int_\Omega \mathrm{Tr}(D\Phi(x))dx \quad = 1 \right\} \quad . \tag{3.3}$$

For the proof the reader is referred to Proposition 2.1 in [5]. See also [29], Theorem 3.1 and [30], Section 2, Lemma 1.

We warn the reader that the mappings in T are normalized in a slightly different way with respect to the statement in [29], where $\frac{1}{|\Omega|}\int_\Omega \mathrm{Tr}(D\Phi(x))dx \quad = 2$ was used.

Let $K \geq 1$ be arbitrary but given and let Σ_K be the space of quasiconformal mappings defined in Section 2. When $\Omega = \mathbf{D}$, by Lemma (2.2) the (normalized) gradient of $f \in \Sigma_K$ belongs to T and the following corollary of Theorem 3.1 is then obtained as a consequence.

Corollary 3.1 *(Lemma 5.1 in [29]). Suppose* $K^{-1}|h|^2 \leq \langle \sigma(x)h, h\rangle \leq K|h|^2$, $h \in \mathbf{R}^2$*. Then under the assumptions of Theorem 3.1,*

$$\frac{2}{\mathrm{Tr}(\sigma^*) + 2\lambda} \leq \sup_{\Psi \in \Sigma_K} \int_{\mathbf{D}} \frac{\det D\Psi(x)}{\sqrt{\det \sigma(x)} + \lambda} dm \quad . \tag{3.4}$$

Proof. For the reader's convenience we sketch the argument from [29]. Note that we use the normalized measure $dm = dx/\pi$.

Define a new matrix function $G(x) = (\frac{\sigma(x)}{\sqrt{\det \sigma(x)}})^{-1}$ for $x \in \mathbf{D}$ and $G(x) = I$ for $x \in \mathbf{R}^2 \setminus \overline{\mathbf{D}}$. Then by the measurable Riemann mapping theorem [1], [16] there is an element $\Psi \in \Sigma_K$ with $D\Psi^t D\Psi = G \det(D\Psi)$. By the cyclic property of the Trace, one obtains

$$\mathrm{Tr}[D\Psi(x)\sigma(x)D\Psi(x)^t] = \mathrm{Tr}[G(x)\sigma(x) \det D\Psi(x)] = 2\sqrt{\det \sigma(x)} \det D\Psi(x) \quad . \tag{3.5}$$

According to Lemma 2.2, $\frac{1}{2}\Psi \in T$; substituting this test function transforms the right hand side of (3.2) to

$$\frac{[\sqrt{\det \sigma(x)} - \lambda] \det D\Psi(x)}{2[\det \sigma(x) - \lambda^2]}. \tag{3.6}$$

The claim (3.4) follows. □

Proof of Theorem 1.4. Consider Corollary 3.1 and the inequality (3.4) specialized to the case when σ has the particular form (1.3), (1.2). It follows from the proof of Corollary 3.1 that actually we can now take the supremum over $\Psi \in \Sigma_K(E)$, i.e. over the set of those $\Psi \in \Sigma_K$ that are conformal on $E = \{x \in \mathbf{D} : \sum_{i=2}^N \chi_i(x) = 1\}$. Passing to the limit as λ tends to σ_1 in (3.4), we obtain

$$\frac{2}{\mathrm{Tr}(\sigma^*) + 2\sigma_1} \leq \sup_{\Psi \in \Sigma_K(E)} \int_{\mathbf{D}} \frac{\det D\Psi(x)}{\sqrt{\det \sigma(x)} + \sigma_1} dm \quad . \tag{3.7}$$

Recall next the notation (1.5), $c_i = (\sigma_i - \sigma_1)/(\sigma_i + \sigma_1)$. With this and with (1.3), (3.7) implies

$$\frac{2}{\mathrm{Tr}(\sigma^*) + 2\sigma_1} \leq \frac{1}{2\sigma_1} \sup_{\Psi \in \Sigma_K(E)} \left(\int_{\mathbf{D}} \det D\Psi(x) dm - \sum_{1=2}^N c_i \int_{\mathbf{D}} \chi_i(x) \det D\Psi(x) dm \right). \tag{3.8}$$

Finally, recall that for any $\Psi \in \Sigma_K$, $\int_{\mathbf{D}} \det D\Psi(x) dm \leq 1$, see Lemma 2.1. Hence choosing now $w(x) = \sum_{i=2}^{n} c_i \chi_i(x)$ in (1.12), gives the desired bound for σ^*. The same bounds hold automatically for the σ_{hom}, c.f. [29]. $\square$.

Remark 3.1 *When $K = 1$ and $N > 2$, for certain values of the G-closure parameters when (1.10) is not satisfied, the bounds (1.9) cannot be optimal (see [28]). The same happens when $K > 1$. This follows using Theorem 3.1 in [29] with the choice $t(x) = \sigma_1 \chi_1(x) + (1 - \chi_1(x))\sigma_2$ and arguing as in [28].*

4 Optimality of bounds

This section is devoted to proving Theorem 1.5. Our statement, as remarked at the end of the introduction, has to be understood as optimality in the classical sense. In other words the proof requires three parts. First: for $Q = (0,1) \times (0,1)$, define a specific Q-periodic σ satisfying (1.3), (1.2) and (1.10) at almost every point of Q. Second: exhibit the weak solution to the Euler-Lagrange equations of (1.1). Third: compute the corresponding σ_{hom} and show that σ_{hom} is isotropic and it satisfies (1.9) as an equality.

There are actually several possibilities to do the constructions. One is based on structures similar to those in Example 2.1, by filling in the square by families of such concentric disks. However, we have chosen below a slightly different route which is here technically less demanding.

Proof of Theorem 1.5. We begin by performing the first part of the above program. Set h_0 to be the solution of the equation

$$\frac{1}{h_0 + \sigma_1} = \frac{1}{2\sigma_1} \left[1 - \left(\sum_{i=2}^{N} c_i^{\frac{1}{K}} p_i \right)^K \right] \tag{4.1}$$

and define

$$z = \frac{h_0 - \sigma_1}{h_0 + \sigma_1} \quad , \quad q_i = \left(\frac{z}{c_i} \right)^{\frac{1}{K}} \quad , \quad i = 2, \ldots, N \quad . \tag{4.2}$$

Note that, by (1.10), (4.1) and (4.2), each q_i belongs to $(0, 1]$. Hence, setting

$$\alpha_i = \frac{p_i}{q_i} \quad , \quad i = 2, \ldots, N \quad . \tag{4.3}$$

one has $\alpha_i > 0$ for every i. An easy calculation shows that, by the definition of h_0, one has $\sum_{i=2}^{N} \alpha_i = 1$. Now let $Q = (0,1) \times (0,1)$. We cover Q (up to a set of zero measure) with $N-1$ open sets Ω_i such that $\mid \Omega_i \mid = \alpha_i$. This can of course be done in infinitely many ways. For instance one may take Ω_i to be non overlapping rectangles of sides 1 and α_i.

Next, for $R > 0$, $0 < r < \rho$ and $y \in \mathbf{R}^2$ we define

$$B(y,R) \equiv \{x \in \mathbf{R}^2 \ : \ \mid x - y \mid < R\} \ , \ A(y,r,\rho) \equiv \{x \in \mathbf{R}^2 \ : \ r \leq \mid x - y \mid < \rho\} \ . \tag{4.4}$$

For any i there exists a countable set of non overlapping open disks $B^{(ij)} = B(x^{(ij)}, R^{(ij)})$ such that

$$\mid \Omega_i \setminus \cup_{j=1}^{\infty} B^{(ij)} \mid = 0 \quad . \tag{4.5}$$

We write $B^{(ij)}$ as the union of an inner disk and an outer annulus

$$B(x^{(ij)}, R^{(ij)}) = B(x^{(ij)}, r^{(ij)}) \cup A(x^{(ij)}, r^{(ij)}, R^{(ij)}) \quad , \quad \left(\frac{r^{(ij)}}{R^{(ij)}} \right)^2 = q_i \quad . \tag{4.6}$$

With these notation we are now ready to define our optimal conductivity, denoted by σ_{opt}. As the restriction of σ_{opt} to the disk $B^{(ij)}$ we set $(\sigma_{\text{opt}})|_{B^{(ij)}}(x) = \sigma^{(i)}(x - x^{(ij)})$, where

$$\sigma^{(i)}(x) = \sigma_i I \chi_{_{B(0,r)}}(x) + \sigma_1 R_S(x)^t \begin{pmatrix} K & 0 \\ 0 & K^{-1} \end{pmatrix} R_S(x) \chi_{_{A(0,r,R)}}(x) \tag{4.7}$$

with $r = r^{(ij)}, R = R^{(ij)}$ and the field of rotations $R_S(x)$ is given by

$$R_S(x) = \frac{x_2}{\mid x \mid} I + \frac{x_1}{\mid x \mid} J \tag{4.8}$$

where J is the (counterclockwise) rotation by 90 degrees. Outside the disks we may set, say, $\sigma_{\text{opt}} = \sigma_1 I$. Then clearly the support of phase $i \geq 2$, i.e. $\{x : \sigma_{\text{opt}}(x) = \sigma_i\}$, is $\cup_{j=1}^{\infty} B(x^{(ij)}, r^{(ij)})$ while the first phase is located, up to measure zero, in the union of the annuli $A(x^{(ij)}, r^{(ij)}, R^{(ij)})$.

Lastly we extend σ_{opt} periodically to $\mathbf{R}^2$. This concludes the first part.

We begin now the second part of the program. Our goal is to find a weak solution of the PDE

$$\text{div}[\sigma_{\text{opt}}(x)(\nabla u_0(x) + e_1)] = 0 \quad , \quad u_0 \in W_{\sharp}^{1,2}(\mathbf{R}^2, \mathbf{R}) \quad . \tag{4.9}$$

Our strategy here is to construct in each disk $B^{(ij)} = B(x^{(ij)}, R^{(ij)})$ a function of the form $u^{ij}(x) = u_0^{ij}(x) + x_1$ where $u_0^{ij} \in W_0^{1,2}(B^{(ij)}; \mathbf{R})$ and then extend this function to be x_1 outside the union of the disks. Later we show the main point namely that this "gluing" delivers a weak solution of the PDE (4.9).

Preliminary step: i-th solution in $B(y,R)$. We assume without loss of generality that $y = 0$ and that in this disk $\sigma_{\text{opt}}(x) = \sigma^{(i)}(x)$, $|x| < R$, as defined in (4.7). Then we look for the weak solution with boundary data x_1 in this specific disk. This calculation can be found for instance in [29], Subsection 6.1. We review it for the readers convenience. Suppose

$$u^{(i)}(x) = \begin{cases} a_i x_1, & 0 \le |x| < r \\ (A_i|x|^{K-1} + B_i|x|^{-K-1})x_1, & r \le |x| < R. \end{cases} \tag{4.10}$$

Then $u^{(i)}$ is a solution to the equation $\operatorname{div}[\sigma^{(i)}(x)\nabla u^{(i)}(x)] = 0$ if and only if

$$A_i = a_i \frac{\sigma_1 + \sigma_i}{2\sigma_1} \frac{1}{r^{K-1}}, \quad B_i = a_i \frac{\sigma_1 - \sigma_i}{2\sigma_1} r^{K+1}. \tag{4.11}$$

Furthermore, if the numbers h_i, $i \ge 2$, are such that

$$\frac{1}{h_i + \sigma_1} = \frac{1 - q_i^K}{2\sigma_1} + \frac{q_i^K}{\sigma_1 + \sigma_i} \tag{4.12}$$

and we make the choices

$$\left(\frac{r}{R}\right)^2 = q_i \ ; \quad a_i = \frac{1}{2\sigma_1}\left[(\sigma_1 + h_i) q_i^{\frac{K-1}{2}} + (\sigma_1 - h_i) q_i^{-\frac{K+1}{2}}\right] \tag{4.13}$$

then we have three important identities. First, for any smooth function η defined on $B(0,R)$, we have

$$\int_{B(0,R)} \langle \sigma^{(i)}(x)\nabla u^{(i)}(x), \nabla \eta(x)\rangle dx = h_i \int_{B(0,R)} \eta_{x_1}(x) dx \quad . \tag{4.14}$$

Secondly,

$$\int_{B(0,R)} \langle \sigma^{(i)}(x)\nabla u^{(i)}(x), \nabla u^{(i)}(x)\rangle dx = h_i \mid B(0,R) \mid \tag{4.15}$$

and thirdly,

$$A_i R^{K-1} + B_i R^{-K-1} = 1. \tag{4.16}$$

The last of the identities is obtained be a tedious but elementary calculation; with (4.10), (4.15) it shows in particular that $u^{(i)} - x_1 \in W_0^{1,2}(B(0,R);\mathbf{R})$. For the identities (4.14), (4.15), for small $\epsilon > 0$ we can use integration by parts in $B(0, r-\epsilon)$ and $A(r+\epsilon, R-\epsilon)$ since there the function $u^{(i)}$ is smooth satisfying $\mathrm{div}(\sigma^{(i)}(x)\nabla u^{(i)}(x)) = 0$ in the classical sense. For $|x| = r$, by (4.11) the boundary terms on the left hand side cancel as ϵ goes to zero while using Stokes theorem to the remaining boundary terms proves (4.14), (4.15).

We lastly make the observation which is crucial for the construction to succeed. Namely, with the choice of q_i in (4.2), we have

$$h_i = h_0 \ \text{ for any } \ 2 \leq i \leq N, \tag{4.17}$$

where h_0 is given by (4.1).

Definition of u in the generic disk $B^{(ij)}$. In the disk $B(x^{(ij)}, R^{(ij)})$, we define $u^{(ij)}(x) = u^{(i)}(x - x^{(ij)})$, with $R = R^{(ij)}$ in (4.10).

Definition of the solution in Q. We define $u(x)$ as follows

$$u(x) = \sum_{i=2}^{N}\sum_{j=1}^{\infty}\left(u^{(ij)}(x)\chi_{B^{(ij)}}(x) + x_1\chi_{Q\setminus \cup B^{(ij)}}(x)\right) \quad . \tag{4.18}$$

We want to show that u is a weak solution to (4.9).

By the above construction, u is continuous and belongs to $W_\sharp^{1,2}(Q;\mathbf{R}^2)$. It remains to show that for any test function $\eta \in C_\sharp^\infty(Q;\mathbf{R})$ one has

$$\int_Q \langle \sigma_{\mathrm{opt}}(x)\nabla u(x), \nabla \eta(x)\rangle dx = 0 \quad . \tag{4.19}$$

Let us use the notation $\sigma^{(ij)}$ for the restriction of σ to $B^{(ij)}$. Now, by construction

$$\int_Q \langle \sigma_{\mathrm{opt}}(x)\nabla u(x), \nabla \eta(x)\rangle dx = \sum_{i=2}^{N}\sum_{j=1}^{\infty}\int_{B^{(ij)}} \langle \sigma^{(ij)}(x)\nabla u^{(ij}(x), \nabla \eta(x)\rangle dx \quad . \tag{4.20}$$

Each integral in the above sum can be computed separately. By a linear change of coordinates (4.14) and (4.17) yield

$$\int_Q \langle \sigma_{\mathrm{opt}}(x)\nabla u(x), \nabla \eta(x)\rangle dx = h_0 \sum_{i=2}^{N}\sum_{j=1}^{\infty}\int_{B^{(ij)}} \eta_{x_1}(x) dx = h_0 \int_Q \eta_{x_1}(x) dx \quad . \tag{4.21}$$

The latter integral is obviously zero since η is Q-periodic.

Finally, using (4.15), (4.17) we see that

$$\int_Q \langle \sigma_{\rm opt}(x)\nabla u(x), \nabla u(x)\rangle dx = h_0\,. \tag{4.22}$$

Replacing x_1 by x_2 in the above calculation shows that $\sigma_{\rm hom} = h_0 I$. This completes the proof of Theorem 1.5. $\square$.

5 Quasiconformal consequences

The motivation behind Theorem 1.6, our result on distortion of weighted area under quasiconformal mappings, was to obtain new bounds for composite materials. However, once the theorem has been established, it is interesting to see that it has consequences also to quasiconformal mappings. We briefly describe here some of them; for other developments that use the theorem see [4]. Our main goal is to prove Theorem 1.7, the bound (5.6) and their optimality.

For the background, recall that Theorem 1.6 implies that $|f(E)| \le |E|^{1/K}$ if $f \in \Sigma_K$ and $f|_E$ is conformal. In [3] this was a main step to proving the optimal smoothness of $K-$quasiconformal and $K-$quasiregular mappings f, namely that

$$f \in W^{1,p}_{\rm loc} \quad \forall\, p < \frac{2K}{K-1}. \tag{5.1}$$

Here, in general, the inclusion fails for the extremal exponent $p = 2K/(K-1)$.

However, we wish to show that in special cases one can do better: Our last goal is Theorem 1.7, which proves that one has borderline integrability on any measurable set E for which $\overline{\partial} f|_E = 0$.

We first need a technical decomposition lemma.

Lemma 5.1 *Suppose f is $K-$quasiregular in $\mathbf{R}^2$. Then f admits the decomposition*

$$f = h \circ g, \tag{5.2}$$

where $g \in \Sigma_K$ and h is $K-$quasiregular in $\mathbf{R}^2$ with

$$\overline{\partial} h|_{g(\mathbf{D})} \equiv 0. \tag{5.3}$$

In particular, h is complex-analytic in the open set $g(\mathbf{D})$.

Proof. Let $\mu = \mu_f$ be the complex dilatation of f as in (2.4). Let then g be the (unique) element in Σ_K for which

$$\overline{\partial} g = \chi_{\mathbf{D}} \, \mu \, \partial g; \tag{5.4}$$

the existence of such a g follows from the measurable Riemann mapping theorem [1], [16].

We can then easily calculate the complex dilatation of the quasiregular mapping $h \equiv f \circ g^{-1}$, c.f. [16] p. 24. It follows that

$$\mu_{f \circ g^{-1}}(gx) = \frac{\mu_f(x) - \mu_g(x)}{1 - \overline{\mu_g}(x)\mu_f(x)} \frac{\partial g(x)}{\overline{\partial g(x)}} \tag{5.5}$$

From this we see that $\mu_h \circ g = \mu_f (\partial g / \overline{\partial g}) \chi_{\mathbf{R}^2 \setminus \mathbf{D}}$; hence h satisfies the required properties. □

Lemma 5.2 *Let $K > 1$ and assume that $f \in \Sigma_K$. Let $E \subset \mathbf{D}$ be a measurable set. If $\overline{\partial} f(x) = 0$ a.e. $x \in E$, then*

$$\int_E J_f(x)^p dm(x) \leq 1, \quad p = \frac{K}{K-1} \, . \tag{5.6}$$

Moreover the bound is optimal.

Proof We choose a sequence of weights such that $w_0 = 1$ and $w_n = J_f^{1/K + \ldots + 1/K^n}$, $n \geq 1$. By Theorem 1.6

$$\int_E w_n(x) J_f(x) dm(x) \leq \Big(\int_E w_n(x)^K dm(x) \Big)^{1/K} = \Big(\int_E w_{n-1}(x) J_f(x) dm(x) \Big)^{1/K} \tag{5.7}$$

for each $n \geq 1$. Using this argument inductively we arrive at

$$\int_E J_f(x)^{1 + 1/K + \ldots + 1/K^n} dm = \int_E w_n(x) J_f(x) dm \leq \Big(\int_E J_f(x) dm \Big)^{1/K^n} \leq |E|^{1/K^{n+1}} . \tag{5.8}$$

Using Fatou's lemma we can pass to the limit for $n \to \infty$ in (5.8). This proves (5.6). To prove the second part we need to give an examples of equality in (5.6). This is easy. For instance, we can take $E = B(0, R)$ with $R < 1$ and f_R the $K-$quasiconformal mapping defined in (2.29). □

Proof of Theorem 1.7 Suppose that $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ is K-quasiregular and there is a bounded measurable set $E \subset \mathbf{R}^2$ such that $\mu_f|_E \equiv 0$, or that $\overline{\partial} f|_E \equiv 0$. Our goal is then to show that $\int_E |Df(x)|^p dx < \infty$ for the exponent $p = \frac{2K}{K-1}$.

With a linear change of variables we can first assume that $E \subset \{x : |x| \leq 1/2\}$. After this we will use the decomposition $f = h \circ g$ of the above Lemma 5.1. Since h is complex analytic in a neighborhood of the set $g(E)$, $\sup_{g(E)} |h'(x)| \leq C_h < \infty$ for a constant C_h depending on the function h. Therefore we may assume that actually $f \in \Sigma_K$.

Finally, in this case it is enough to show (5.6), since $|Df(x)|^2 \leq K J_f(x)$ for a.e. x. On use of Lemma (5.2) the proof is completed. $\square$.

Theorem (1.7) is sharp. We prove this by giving the following modification of the example proving optimality of (5.6).

Example 5.1 *As in the proof of Theorem 1.5 cover the unit disk $\mathbf{D}$ by a countable set of non-overlapping open disks $B_j = B(x_j, \rho_j)$, $j \in \mathbf{N}$, such that*

$$| \mathbf{D} \setminus \cup_{j=1}^{\infty} B_j | = 0 \quad . \tag{5.9}$$

For the set E choose $E = \cup_{j=1}^{\infty} B(x_j, \rho_j R_j)$, where the $R_j < 1$ with $R_j \to 0$ so that

$$\sum_{j=1}^{\infty} \rho_j^2 \log \frac{1}{R_j} = \infty. \tag{5.10}$$

Define then a $K-$quasiconformal mapping $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ by setting

$$f(x) = \rho_j f_{R_j}((x - x_j)/\rho_j) + x_j, \quad x \in B_j \tag{5.11}$$

and by defining $f(x) = x$ for $x \notin \cup_{j=1}^{\infty} B_j$.

Clearly $f|_E$ is conformal. Moreover, for each $\varepsilon > 0$ and for each B_j,

$$\int_{B(x_j, \rho_j R_j)} J_f(x)^{(1+\varepsilon)K/(K-1)} dm = R_j^{-2\varepsilon} \rho_j^2. \tag{5.12}$$

Summing up the integrals over the disks we see from (5.10), (5.12) that

$$\int_E |Df(x)|^p dx = \infty \tag{5.13}$$

for every $p > \frac{2K}{K-1}$.

The higher integrability (5.1) and Theorem 1.7 have their counterparts for the integrability of $|Df(x)|^q$ where $q < 0$. However, as quasiregular mappings are of the form $f = h \circ g$ where h is complex analytic and g quasiconformal, the zeroes of h cause now extra singularities. Removing these we have

Corollary 5.1 *Suppose f is $K-$quasiregular in a domain $\Omega \subset \mathbf{R}^2$ and that $\overline{\partial} f|_E \equiv 0$. Then there is a discrete set $A \subset \Omega$ such that for any compact subset L of $\Omega \setminus A$*

$$\int_{E \cap L} |Df(x)|^{-2/(K-1)} dx < \infty. \tag{5.14}$$

As in Example 5.1, one can construct a $K-$quasiconformal mapping such that $\int_{E \cap L} |Df(x)|^{-q} dx = \infty$ for each $q > 2/(K-1)$.

Proof of Corollary 5.1. If $f = h \circ g$ as above and $A = g^{-1}\{x \in \Omega : h'(x) = 0\}$, then outside the discrete set $g(A)$, h' is locally bounded away from ∞ and 0. Using Theorem 1.7 for $G \equiv g^{-1}$ yields

$$\int_{E \cap L} J_g^{-1/(K-1)} dx = \int_{E \cap L} J_G^{\frac{K}{K-1}}(g) J_g dx = \int_{G(E \cap L)} J_G^{\frac{K}{K-1}} dx < \infty. \tag{5.15}$$

This proves the claim (5.14). □

Lastly, let us remark that one has similar consequences for the solutions of second order equations in divergence form. More precisely, given a domain Ω, we consider the class of symmetric matrices σ with measurable entries that satisfy

$$K^{-1}|h|^2 \leq \langle \sigma(x)h, h \rangle \leq K|h|^2, h \in \mathbf{R}^2. \tag{5.16}$$

Let us call this class $\mathcal{M}_K$. Assume that $u \in W^{1,2}(\Omega)$ is a weak solution of $\operatorname{div}[\sigma(x)\nabla u(x)] = 0$. Consider the mapping $f = (u, v)$, where $\nabla v(x) = J\sigma(x)\nabla u(x)$, then as pointed out in [18], f is $K-$ quasiregular and hence the previous results apply to the components u, v. In particular, it follows that in general $u \in W^{1,p}_{\text{loc}}$ $\forall p < 2K/(K-1)$ but $u \notin W^{1,\frac{2K}{K-1}}_{\text{loc}}$. Therefore, in this class of spaces, this is the best regularity one can obtain. In addition, thanks to the work of Alessandrini and Alessandrini and Magnanini [2], one can control the "singular set" A in terms of knowledge of the boundary data [18].

However, quite unexpectedly, we obtain an improvement of the higher integrability in the borderline case in the spirit of a *partial regularity* result.

Corollary 5.2 *Given $K \geq 1$ and given $\sigma \in \mathcal{M}_K$, suppose u is a $W^{1,2}$-solution to*

$$\operatorname{div}[\sigma(x)\nabla u(x)] = 0 \tag{5.17}$$

in a domain $\Omega \subset \mathbf{R}^2$. Let $E = \{x \in \Omega \,|\, \sigma(x) = I\}$. Then for any compact subset $L \subset \Omega$

$$\int_{E \cap L} |\nabla u(x)|^{\frac{2K}{K-1}} dx < \infty. \tag{5.18}$$

Similarly, there is a set A, discrete in Ω, such that for any compact $L \subset \Omega \setminus A$

$$\int_{E \cap L} |\nabla u(x)|^{\frac{-2}{K-1}} dx < \infty.$$

The proofs follow from Corollary 5.1 and Theorem 1.7, using the fact from [18] that $u = \Re f$, where f is $K-$quasiregular. Indeed, by construction, on the set E the mapping f satisfies the Cauchy-Riemann system.

It appears a curious fact that one has improved regularity on a special level set of σ, on $E = \{x \in \Omega : \sigma(x) = I\}$. We are unable to prove that a similar consequence holds on any other level set, even on some other level set of the form $\{x \in \Omega : \sigma(x) = cI\}$ where $c \neq 1$. This remains an interesting open question.

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