Failure of the condition N below $W^{1,n}$

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1 Introduction

Suppose that f is a continuous mapping from an open bounded subset Ω of $\mathbb{R}^n$, $n \geq 2$, into $\mathbb{R}^n$. We consider the following (Lusin) condition N: if $E \subset \Omega$, $\mathcal{L}^n(E) = 0$, then $\mathcal{L}^n(f(E)) = 0$. Physically, this condition requires that there is no creation of matter under the deformation f of the n-dimensional body Ω. This is a natural requirement as the N-property with differentiability a.e. is sufficient for validity of various change-of-variable formulas, including the area formula, and the condition N holds for a homeomorphism f if and only if f maps measurable sets to measurable sets.

If $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, we can ask which integrability conditions on the derivative Df guarantee the condition N. According to a classical result by Marcus and Mizel [11], f satisfies the condition N if $|Df| \in L^p(\Omega)$ for some $p > n$. If $|Df| \in L^p(\Omega)$ with $p \leq n$, the condition N may fail; see e.g. [10] or [15] for examples.

It is then interesting to ask for the size of the exceptional set, i.e. the set outside of which the condition N holds. It is worth pointing out that if f is any mapping in the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^n)$, then f satisfies the condition N outside a set of measure zero (see e.g. [14] for references). Malý and Martio proved in [10] (also see [8]), as a substantial improvement on this fact, that if $f \in W^{1,n}(\Omega, \mathbb{R}^n)$, then the condition N holds for f outside of a set that has Hausdorff dimension zero.

It seems that very little is known about the size of the exceptional set below $W^{1,n}$. It would be natural to expect that the Hausdorff dimension of the exceptional set decreases as the exponent p increases to n. In this paper we will prove, however, that if $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ so that (1.1) holds, and thus in particular when $|Df| \in \cap_{p<n} L^p(\Omega)$, one cannot, in general, find an exceptional set of Hausdorff dimension smaller than n.

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Theorem 1.1. *Let $s \in (0,\infty)$. Then there exists a homeomorphism $f : Q_0 \to Q_0$ from the unit cube $Q_0 = [0,1]^n$ onto itself such that the following properties hold:*

(a) *f fixes the boundary ∂Q_0.*

(b) *$f \in W^{1,1}(Q_0, \mathbb{R}^n)$, f is differentiable almost everywhere, and*

$$\sup_{0<\epsilon\leq n-1} \epsilon \int_{Q_0} |Df(x)|^{n-\epsilon}\, dx < \infty. \tag{1.1}$$

(c) *The Jacobian determinant $J(x,f)$ is strictly positive for almost every $x \in Q_0$ and $J(\cdot, f) \in L^1(Q_0)$.*

(d) *If $E \subset Q_0$ with $\mathcal{H}^h(E) = 0$, where $h(t) = t^n \log^s \log(4 + 1/t)$, then there exists a compact set $F \subset Q_0 \setminus E$ such that $\mathcal{L}^n(f(F)) > 0$.*

If we put certain additional topological or analytical assumptions on f, then a lower degree of integrability of $|Df|$ is needed for the condition N. Reshetnyak proved in [16] that for a homeomorphism it suffices to assume that $|Df| \in L^n(\Omega)$. There are several generalizations of this result: instead of homeomorphicity it suffices to assume that f is continuous and open [10], or that $J(x,f) > 0$ almost everywhere [2] (see also [17] and [12]). For further results of this type see the survey paper [9]. Recently Kauhanen, Koskela and Malý [5] proved that, for a topologically sense-preserving (and thus in particular for homeomorphical) mapping f, it suffices to assume that

$$\lim_{\epsilon\to 0+} \epsilon \int_{\Omega} |Df|^{n-\epsilon} = 0. \tag{1.2}$$

The construction of f of Theorem 1.1 is a refinement of our construction in [5], where a similar mapping was constructed to show that the condition N may fail under condition (1.1). For the convenience of the reader we will present a complete construction here. The basic idea of the construction comes from Ponomarev [15]; also see [14]. Ponomarev gave an example of a homeomorphism f with $|Df| \in \cap_{p<n} L^p(\Omega)$ such that f creates matter. He, however, did not consider the size of the exceptional set and worked only on the L^p-scale.

2 About the function spaces

In this section we briefly review some function spaces that we dealt with in the introduction; for a more detailed discussion see [4]. Let us define $BL^{n)}(\Omega)$ as the collection of all measurable functions u with

$$||u||_{n)} = \sup_{0<\epsilon\leq n-1} \Big(\epsilon \int_{\Omega} |u|^{n-\epsilon}\Big)^{1/(n-\epsilon)} < \infty.$$

Then $BL^{n)}(\Omega)$ is a Banach space and

$$VL^{n)}(\Omega) = \left\{ u \in BL^{n)}(\Omega) : \lim_{\epsilon \to 0^+} \epsilon \int_\Omega |u|^{n-\epsilon} = 0 \right\}$$

is a closed subspace. The following inclusions hold (see [3]):

$$L^n(\Omega) \subsetneq L^n \log^{-1} L(\Omega) \subsetneq VL^{n)}(\Omega) \subsetneq BL^{n)}(\Omega) \subsetneq \bigcap_{\alpha<-1} L^n \log^\alpha L(\Omega)$$
$$\subsetneq \bigcap_{p<n} L^p(\Omega),$$

where the space $L^n \log^\alpha L(\Omega)$ consists of all measurable functions u with

$$\int_\Omega |u|^n \log^\alpha(e + |u|) < \infty.$$

One question that Theorem 1.1 and the other results mentioned in the introduction leave unanswered is the size of the exceptional set for continuous mappings $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with $|Df|$, for example, in $L^n \log^{-1} L(\Omega)$ or $VL^{n)}(\Omega)$.

3 Proof of Theorem 1.1

Let us begin with some notation. Besides the usual euclidean norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ we will use the cubic norm $\|x\| = \max_i |x_i|$. Using the cubic norm, the x_0-centered closed cube with edge length $2r > 0$ and sides parallel to coordinate axes can be represented in the form

$$Q(x_0, r) = \{x \in \mathbb{R}^n : \ \|x - x_0\| \le r\}.$$

We then call r the radius of Q. $c(a, b, \ldots)$ denotes a positive constant depending only on $a, b, \ldots$ which might differ from occurrence to occurrence. We write $x \approx c(a, b, \ldots)y$ if $x \le c(a, b, \ldots)y$ and $y \le c(a, b, \ldots)x$.

We will be dealing with radial stretchings that map cubes $Q(0, r)$ onto cubes. The following lemma can be verified by an elementary calculation.

Lemma 3.1. *Let $\rho : (0, \infty) \to (0, \infty)$ be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\rho(\|x\|), \qquad x \neq 0$$

we have

$$|Df(x)| \approx c(n) \max\left\{ \frac{\rho(\|x\|)}{\|x\|}, \ |\rho'(\|x\|)| \right\}$$

and

$$J(x, f) \approx c(n) \frac{\rho'(\|x\|)\rho(\|x\|)^{n-1}}{\|x\|^{n-1}}.$$

We will first give two Cantor set constructions in Q_0. Our mapping f will be defined as a limit of a sequence of piecewise continuously differentiable homeomorphisms $f_k : Q_0 \to Q_0$, where each f_k maps the k:th step of the first Cantor set construction onto the second one. Then f maps the first Cantor set onto the second one. Choosing the Cantor sets so that the Hausdorff h measure $\mathcal{H}^h$ of the first one is positive and finite and so that the second one has positive measure, we get the property (d).

Let $V \subset \mathbb{R}^n$ be the set of all vertices of the cube $Q(0,1)$. Then the sets $V^k = V \times \cdots \times V$, $k = 1, 2, \ldots$, will serve as the sets of indices for our construction (with the exception of the subscript 0 used in Step 0). If $w \in V^{k-1}$, we denote

$$V^k[w] = \{v \in V^k : \ v_j = w_j, \ j = 1, \ldots, k-1\}.$$

Let $z_0 = (\frac{1}{2}, \ldots, \frac{1}{2})$, $r_0 = \frac{1}{2}$ and denote

$$\psi(k) = \frac{1}{2} \frac{\log^{s/n} 2}{\log^{s/n}(k+2)}.$$

For $v \in V^1 = V$ let $z_v = z_0 + \frac{1}{4}v$, $P_v = Q(z_v, \frac{1}{4})$ and $Q_v = Q(z_v, \psi(1)2^{-1})$. If $k = 2, 3, 4, \ldots$ and $Q_w = Q(z_w, r_{k-1})$, $w \in V^{k-1}$, is a cube from the previous step of construction, then Q_w is divided into 2^n subcubes P_v, $v \in V^k[w]$, with centers z_v and radius $r_{k-1}/2$ and inside them we pick concentric cubes Q_v, $v \in V^k[w]$, with radius

$$r_k = \psi(k) 2^{-k}.$$

Note that $r_k < r_{k-1}/2$ for all k. Thus, denoting $v = (v_1, \ldots, v_k)$, we have

$$z_v = z_w + \tfrac{1}{2} r_{k-1} v_k = z_0 + \tfrac{1}{2} \sum_{j=1}^{k} r_{j-1} v_j \,,$$
$$P_v = Q(z_v, r_{k-1}/2), \quad Q_v = Q(z_v, r_k).$$

See Figure 1. The number of cubes in the family $\{Q_v : v \in V^k\}$, $k = 1, 2, 3, \ldots$, is $\#V^k = 2^{nk}$. The Hausdorff h measure of the resulting Cantor set

$$C = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v$$

is positive and finite basically since for large k we have

$$\begin{aligned} \sum_{v \in V^k} h(\operatorname{diam} Q_v) &\approx c(n,s) 2^{nk} r_k^n \log^s \log(1/r_k) \\ &\approx c(n,s) \psi(k)^n \log^s \big(\log 2^k + \log \psi(k)^{-1}\big) \\ &\approx c(n,s) \log^{-s} k \log^s k \approx c(n,s). \end{aligned} \tag{3.1}$$

In lack of a convenient reference we give a short proof for this fact.

Lemma 3.2. $0 < \mathcal{H}^h(C) < \infty$.

Proof. We mimic the proof in [13, Section 4.10]. The estimate $\mathcal{H}^h(C) < \infty$ directly follows from (3.1). Let us justify $\mathcal{H}^h(C) > 0$.

Let $\{U_j\}$ be an open covering of C. It suffices to show that

$$\sum_j h(\operatorname{diam} U_j) \geq c(n,s). \tag{3.2}$$

Since C is compact, we can assume that $\{U_j\}$ is finite. For each j choose $x_j \in U_j \cap C$ (the intersection can be assumed to be non-empty) and denote $B_j = B(x_j, \operatorname{diam} U_j)$. Then there is k_0 such that for all $k \geq k_0$ every cube Q_v, $v \in V^k$, is contained in some B_j, and because $h(2t) \leq 2^n h(t)$ for $t \geq 0$, we have that

$$\sum_j h(\operatorname{diam} U_j) \geq 2^{-n} \sum_j h(\operatorname{diam} B_j).$$

Therefore it suffices to prove (3.2) for the family $\{B_j\}$ in the place of $\{U_j\}$. We claim that for any open ball B with center in C and any fixed ℓ,

$$\sum_{v \in V^k,\ Q_v \subset B} h(\operatorname{diam} Q_v) \leq c(n,s) h(\operatorname{diam} B). \tag{3.3}$$

This gives (3.2), since for $k \geq k_0$, by (3.1),

$$\begin{aligned}
\sum_j h(\operatorname{diam} B_j) &\geq c(n,s) \sum_j \sum_{v \in V^k,\ Q_v \subset B_j} h(\operatorname{diam} Q_v) \\
&\geq c(n,s) \sum_{v \in V^k} h(\operatorname{diam} Q_v) \\
&\approx c(n,s).
\end{aligned}$$

To prove (3.3), suppose that $Q_v \subset B$ for some $v \in V^\ell$, and let m be the smallest integer for which $Q_w \subset B$ for some $w \in V^m$. Then $m \leq \ell$. Denote

$$W = \{w \in V^m : \ Q_w \cap B \neq \emptyset\}.$$

By concidering the geometry of the construction of C, it is evident that $\#W \leq c = c(n,s)$. Thus

$$\begin{aligned}
h(\operatorname{diam} B) &\geq \frac{1}{c} \sum_{w \in W} h(\operatorname{diam} Q_w) \\
&\approx c(n,s) \sum_{w \in W} \sum_{v \in V^\ell,\ Q_v \subset Q_w} h(\operatorname{diam} Q_v) \\
&\geq c(n,s) \sum_{v \in V^\ell,\ Q_v \subset B} h(\operatorname{diam} Q_v),
\end{aligned}$$

where the second step can be proven similarly as (3.1). □

The second Cantor set construction is similar to the first one except that at this time we denote the centers by z'_v and the cubes by P'_v, Q'_v, $v \in V^k$, with

$$z'_v = z'_w + \tfrac{1}{2} r'_{k-1} v_k = z_0 + \tfrac{1}{2} \sum_{j=1}^{k} r'_{j-1} v_j \,,$$

$$P'_v = Q(z'_v, r'_{k-1}/2), \quad Q'_v = Q(z'_v, r'_k).$$

We set $r'_0 = 1/2$ and

$$r'_k = \varphi(k) 2^{-k}$$

for $k = 1, 2, 3, \ldots$, where

$$\varphi(k) = \frac{1}{4}\Big(1 + \frac{\log\log 4}{\log\log(k+4)}\Big).$$

Note that $r'_k < r'_{k-1}/2$ for each k. We have

$$\mathcal{L}^n\Big(\bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v\Big) = \lim_{k\to\infty} \mathcal{L}^n\Big(\bigcup_{v \in V^k} Q_v\Big) = \lim_{k\to\infty} 2^{nk}(2r'_k)^n = 2^{-n} > 0.$$

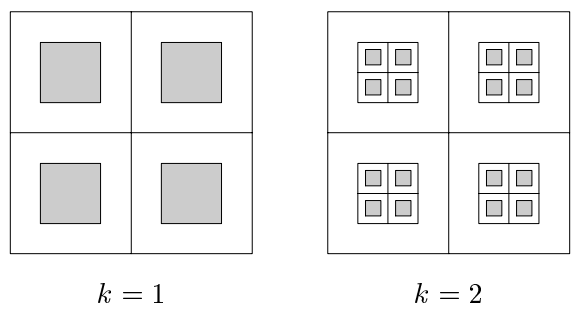

Figure 1: Cubes Q_v, $v \in V^k$.

We are now ready to define the mappings f_k. Define $f_0 = \mathrm{id}$. We will give a mapping f_1 that stretches each cube Q_v, $v \in V^1$, homogeneously so that $f_1(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f_1 is defined to be an appropriate radial map with respect to z_v and z'_v in the image in order to make f_1 a homeomorphism. The general step is the following: If $k > 1$, f_k is defined as f_{k-1} outside the union of all cubes Q_w, $w \in V^{k-1}$. Further, f_k remains equal to f_{k-1} at the centers of cubes Q_v, $v \in V^k$. Then f_k stretches each cube Q_v, $v \in V^k$, homogeneously so that $f(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f is defined to be an appropriate radial map with respect to z_v in preimage and z'_v in image to make f_k a homeomorphism (see Figure 2). Notice that the Jacobian determinant $J(x, f_k)$ will be strictly positive almost everywhere in Q_0.

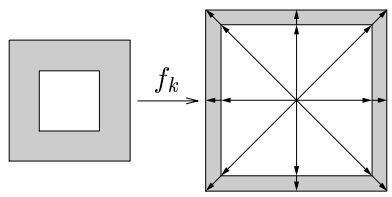

Figure 2: The mapping f_k acting on P_v, $v \in V^k$.

To be precise, let $f_0 = \mathrm{id}|Q_0$ and for $k = 1, 2, 3, \ldots$ define

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \notin \bigcup_{v \in V^k} P_v, \\ f_{k-1}(z_v) + a_k(x - z_v) + b_k \dfrac{x - z_v}{\|x - z_v\|} & \\ & \text{if } x \in P_v \setminus Q_v, \quad v \in V^k, \\ f_{k-1}(z_v) + c_k(x - z_v) & \text{if } x \in Q_v, \quad v \in V^k. \end{cases}$$

where a_k, b_k and c_k are chosen so that f_k maps each Q_v onto Q'_v, is continuous and fixes the boundary ∂Q_0:

$$\begin{aligned} a_k r_k + b_k &= r'_k \\ a_k r_{k-1}/2 + b_k &= r'_{k-1}/2 \\ c_k r_k &= r'_k \end{aligned} \tag{3.4}$$

Clearly the limit $f = \lim_{k\to\infty} f_k$ is differentiable almost everywhere, its Jacobian determinant is strictly positive almost everywhere, and f is absolutely continuous on almost all lines parallel to coordinate axes. Continuity of f follows from the uniform convergence of the sequence (f_k): for any $x \in Q_0$ and $l \geq j \geq 1$ we have

$$|f_l(x) - f_j(x)| \leq c(n) r'_j \to 0$$

as $j \to \infty$.

It is easily seen that f is a one-to-one mapping of Q_0 onto Q_0. Since f is continuous and Q_0 is compact, it follows that f is a homeomorphism.

To finish the proof of the properties (b) and (c), we next estimate $|Df(x)|$ and $J(x, f)$ at x in the interior of the annulus $P_v \setminus Q_v$, $v \in V^k$, $k = 1, 2, 3, \ldots$. Denote $r = \|x - z_v\| \approx c(n, s) r_k$. In the annulus

$$f(x) = f_{k-1}(z_v) + \big(a_k \|x - z_v\| + b_k\big) \frac{x - z_v}{\|x - z_v\|}$$

whence denoting $\rho(r) = a_k r + b_k$ we have by Lemma 3.1 (it is easy to check that $b_k > 0$ for large k)

$$|Df(x)| \approx c(n, s)\big(a_k + b_k / r_k\big)$$

and

$$J(x,f) \approx c(n,s)a_k(a_k + b_k/r_k)^{n-1}.$$

From the equations (3.4) it follows that

$$a_k = \frac{r'_{k-1}/2 - r'_k}{r_{k-1}/2 - r_k} = \frac{\varphi(k-1) - \varphi(k)}{\psi(k-1) - \psi(k)} \approx c(n,s)\frac{\varphi'(k)}{\psi'(k)} \approx c(n,s)\frac{\log^{s/n} k}{\log^2 \log k}$$

and

$$a_k + b_k/r_k = r'_k/r_k = \frac{\varphi(k)}{\psi(k)} \approx c(n)\frac{1}{\psi(k)} \approx c(n,s)\log^{s/n} k.$$

Therefore

$$|Df(x)| \approx c(n,s)\log^{s/n} k$$

and

$$J(x,f) \approx c(n,s)\frac{\log^s k}{\log^2 \log k}.$$

The measure of $\bigcup_{v\in V^k}(P_v \setminus Q_v)$ is

$$\begin{aligned}
2^{nk}\left(r_{k-1}^n - (2r_k)^n\right) &\approx c(n)\big(\psi(k-1)^n - \psi(k)^n\big) \\
&\approx c(n,s)\left(\frac{1}{\log^s(k-1)} - \frac{1}{\log^s k}\right) \\
&\approx -c(n,s)\frac{d}{dk}\left(\frac{1}{\log^s k}\right) \\
&\approx c(n,s)\frac{1}{k\log^{1+s} k}
\end{aligned}$$

and so for $0 < \epsilon \le n-1$

$$\begin{aligned}
\epsilon\int_{Q_0} |Df(x)|^{n-\epsilon}\,dx &\approx c(n,s)\epsilon\sum_{k=2}^{\infty}\frac{1}{k\log^{1+s} k}\log^{(s/n)(n-\epsilon)} k \\
&= c(n,s)\epsilon\sum_{k=2}^{\infty}\frac{1}{k\log^{1+(s/n)\epsilon} k} \qquad (3.5)\\
&\approx c(n,s)\epsilon\int_2^{\infty}\frac{dt}{t\log^{1+(s/n)\epsilon} t} \\
&\approx c(n,s).
\end{aligned}$$

This proves (1.1), and it follows that $f \in W^{1,1}(Q_0,\mathbb{R}^n)$. Similarly we prove the property (c):

$$\begin{aligned}
\int_{Q_0} J(x,f)\,dx &\approx c(n,s)\epsilon\sum_{k=4}^{\infty}\frac{1}{k\log^{1+s} k}\,\frac{\log^s k}{\log^2\log k} \qquad (3.6)\\
&\approx c(n,s)\int_4^{\infty}\frac{dt}{t\log t\log^2\log t} < \infty.
\end{aligned}$$

Let us now prove the property (d). Since $\mathcal{H}^h$ is Borel regular, we find a Borel set $E' \supset E \cap C$ such that $\mathcal{H}^h(E') = 0$. Now $C \setminus E'$ is a Borel set, whence there is a compact set $F \subset C \setminus E'$ such that $\mathcal{H}^h(F) > 0$. Denote

$$W^k = \{v \in V^k : \ Q_v \cap F \neq \emptyset\}.$$

Then there exists a constant $c > 0$ such that

$$\frac{\#W^k}{\#V^k} \geq c \tag{3.7}$$

for all k, since otherwice we would have, similarly as in (3.1),

$$\sum_{v \in W^k} h(\operatorname{diam} Q_v) \approx c(n,s) \#W^k \, 2^{nk} = c(n,s) \frac{\#W^k}{\#V^k} \to 0$$

as $k \to \infty$ for k in some subsequence $(k) \subset \mathbb{N}$, whence we would have $\mathcal{H}^h(F) = 0$, a contradiction. Since F is compact, we have that

$$F = \bigcap_{k=1}^{\infty} \bigcup_{v \in W^k} Q_v,$$

whence (3.7) yields

$$\begin{aligned}
\mathcal{L}^n(f(F)) &= \mathcal{L}^n\Big(f\Big(\bigcap_{k=1}^{\infty} \bigcup_{v \in W^k} Q_v\Big)\Big) = \mathcal{L}^n\Big(\bigcap_{k=1}^{\infty} f\Big(\bigcup_{v \in W^k} Q_v\Big)\Big) \\
&= \lim_{k \to \infty} \mathcal{L}^n\Big(\bigcup_{v \in W^k} Q'_v\Big) \approx c(n) \lim_{k \to \infty} \#W^k (\varphi(k) 2^{-k})^n \\
&\geq c(n) \lim_{k \to \infty} \#W^k \, 2^{-nk} = c(n) \lim_{k \to \infty} \frac{\#W^k}{\#V^k} \geq c(n) c > 0.
\end{aligned}$$

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References

[1] K. Falconer, *Fractal geometry. Mathematical foundations and applications.* John Wiley & Sons, 1990.

[2] V. Gol'dstein and S. Vodopyanov, *Quasiconformal mappings, and spaces of functions with first generalized derivatives.* (Russian) Sibirsk. Mat. Zh. **17** (1976), no. 3, 515–531.

[3] L. Greco, *A remark on the equality* $\det Df = \operatorname{Det} Df$. Differential Integral Equations **6** (1993), 1089–1100.

[4] T. Iwaniec, P. Koskela and J. Onninen, *Mappings of finite distortion: Monotonicity and continuity.* Preprint 216, Department of Mathematics, University of Jyväskylä, 2000.

[5] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Condition N.* Preprint 228, Department of Mathematics, University of Jyväskylä, 2000.

[6] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: Discreteness and openness.* Preprint 226, Department of Mathematics, University of Jyväskylä, 2000.

[7] J. Kauhanen, P. Koskela and J. Malý, *On functions with derivatives in a Lorentz space.* Manuscripta Math. **100** (1999), no. 1, 87–101.

[8] J. Malý, *The area formula for* $W^{1,n}$*-mappings.* Comment. Math. Univ. Carolinae **35** (1994), no. 2, 291–298.

[9] J. Malý, *Sufficient Conditions for Change of Variables in Integral.* Proceedings Analysis & Geometry, to appear.

[10] J. Malý and O. Martio, *Lusin's condition (N) and mappings of the class* $W^{1,n}$. J. Reine Angew. Math. **458** (1995), 19–36.

[11] M. Marcus and V. Mizel, *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems.* Bull. Amer. Math. Soc. **79** (1973), no. 4, 790–795.

[12] O. Martio and W. Ziemer, *Lusin's condition (N) and mappings with non-negative Jacobians.* Michigan Math. J. **39** (1992), 495–508.

[13] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, 1995.

[14] S. Müller and S. Spector, *An existence theory for nonlinear elasticity that allows for cavitation.* Arch. Rational Mech. Anal. **131** (1995), 1–66.

[15] S. Ponomarev, *Property N of homeomorphisms of the class* W_p^1. (Russian) Sibirsk. Mat. Zh. **28** (1987), 140–148.

[16] Yu. Reshetnyak, *Some geometrical properties of functions and mappings with generalized derivatives.* (Russian) Sibirsk. Math. Zh. **7** (1966), 886–919.

[17] V. Šverák, *Regularity properties of deformations with finite energy.* Arch. Ration. Mech. Anal. **100** (1988), 105–127.