Weierstrass points of extremal surfaces in genus 2

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Abstract. A closed Riemann surface of a given genus is extremal, when it contains a hyperbolic disc with greatest possible radius. In genus two the maximal radius is $\operatorname{arcosh}(1+2\cos(\pi/9))$, attained when the surface admits a regular $2\pi/3$-angled 18-gon as a fundamental polygon. We develop here a geometric approach for locating on such an extremal surface the Weierstrass points as intersection points of certain geodesics. Girondo and Gonzales-Diez determined in [1] the Weierstrass points of genus two extremal surfaces using mainly algebraic considerations.

1. Introduction. Let X be a closed Riemann surface of genus two. By Haas and Susskind [2] the hyperelliptic involution j of X maps every simple closed geodesic α of X onto itself. Furthermore, the involution reverses the orientation of α if and only if α is a non-dividing geodesic. Weierstrass points are the fixed points of j, so that every simple closed geodesic α has exactly two Weierstrass points when α is non-dividing, and none when α divides the surface X. Also the restriction of j on the finite intersection $\alpha \cap \beta$ of two simple closed geodesics α and β is an involution, so that when $\alpha \cap \beta$ has an odd number of points, one of these must be a Weierstrass point. As there is always a shortest closed geodesic through two given Weierstrass points, we can specify any Weierstrass point as the intersection point of two simple closed geodesics. We use this to determine geometrically the Weierstrass points of extremal surfaces of genus two. We would like to thank Dr. Ari Lehtonen and Dr. Jouni Parkkonen, who helped us to draw the diagrams.

2. Basic facts. Let P be a regular hyperbolic 18-gon with identified side-pairs, so that it is a fundamental polygon for a surface X of genus 2. The different identification patterns are listed in Jørgensen and Näätänen [3], cf. also [4]. An isometric automorphism of the Riemann surface X maps geodesics to geodesics, thus preserving the set W of Weierstrass points.It follows that any isometry of the fundamental polygon P compatible with given identifications maps W onto itself. We use this to facilitate the search of the six Weierstrass points:

(i) When P admits a hyperbolic reflection, the set W lies symmetrically with respect to the mirror line.

(ii) If an admissible rotation R of P fixes at least four boundary points under the identifications, then by a lemma of Hurwitz it must be a lift of the hyperelliptic involution j. In this case the centre 0 is a Weierstrass point and the other Weierstrass points, i.e. points fixed by R under the identifications, must lie on the boundary of P.

If 0 is not a Weierstrass point, an admissible rotation R of P defines a cyclic permutation on W.

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3. Geodesics as axes of side-pairings. As the covering group of X does not contain elliptic transformations, a simple trace estimation shows that a side-pairing must always cross over at least two sides. Furthermore, is is easy to see that for a side-pairing over more than two sides the axis of the identification mapping joins both identified sides within the fundamental 18-gon, so that in this case its axis determines a simple closed geodesic on the surface X as the orthogonal arc joining the identified sides in P. For a side-pairing over exactly two sides the situation is slightly different, as described by the theorem below:

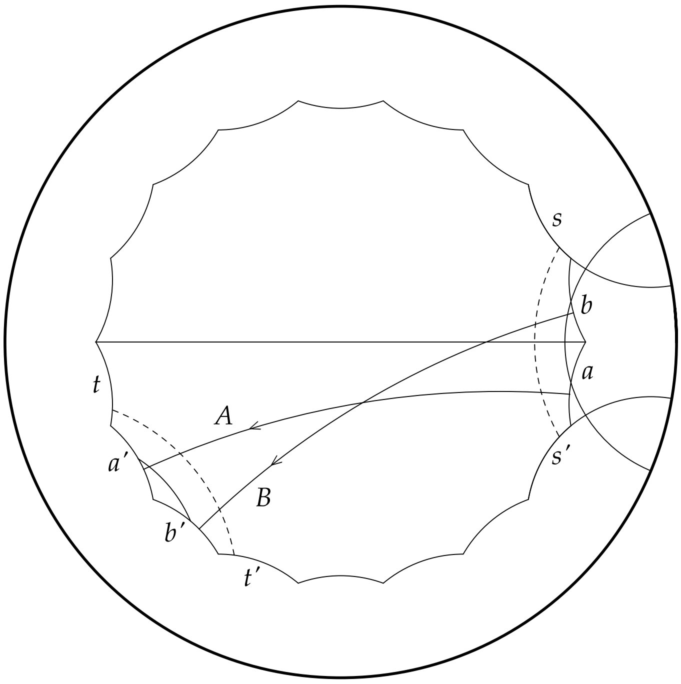

Fig.1

Theorem. *If two adjacent sides a and b of the regular 18-gon P are identified with another pair of adjacent sides a' and b', also the sides s, s' contiguous to $a \cup b$ as well as the sides t, t' contiguous to $a' \cup b'$ are identified.*

Conversely, if identified sides s, s' have exactly two adjacent sides a and b between them, the sides a and b are identified with a pair of adjacent sides a' and b' lying between another pair of identified sides t, t'.

In both cases the axes of the two side-pairings between s and s' and between t and t' correspond to the same simple closed geodesic on the surface X, represented in P by two geodesic arcs joining the mid-points of the sides a and b, respectively the mid-points of the sides a' and b'.

Proof. The covering group of the surface X contains no elliptic elements, so that when mappings A, B identify two adjacent sides a, b with another pair a', b' of adjacent sides, the sides a', b' must have the same cyclic order on the boundary of P as the sides a, b (cf. Figure 1). Thus the two geodesic arcs connecting the mid-points of a and b, resp. the mid-points of a' and b', form together a simple closed geodesic of the surface X represented by the axes of the conjugate mappings

$S = A^{-1}B$ and $T = BA^{-1}$ identifying the sides s and s' contiguous to $a \cup b$, resp. the sides t and t' contiguous to $a' \cup b'$.

Conversely, if the sides s and s' identified by a covering transformation S have only two adjacent sides a and b between them, the mid-points of a and b divide the closed geodesic orthogonal to the extended sides s and s' in two equal halves, the other half joining in P the mid-points of a pair a', b' identified to a, b.

4. Locating the Weierstrass points. The eight different identification patterns 18:1 - 18:8 determined in [3] (see also [4]), are given below in Figure 2, where geodesics are drawn as solid lines. Identification over more than two sides is shown by the common orthogonal of the identified sides, whereas a dashed line indicates a side-pairing over exactly two sides. Except for 18:8, all Weierstrass points are determined as intersection points of simple closed geodesics.

To give some details, in the case 18:1 the simple closed geodesics given by the axes of the side-pairings $(2, 6)$ and $(3, 7)$ intersect at a Weierstrass point. By the above Theorem the two geodesic arcs connecting the mid-points of the sides 2 and 3, respectively the sides 7 and 6, form together a simple closed geodesic intersecting the geodesics given by the axes of the side-pairings $(2, 6)$ and $(3, 7)$ at further two Weierstrass points. The remaining three Weierstrass points lie symmetrically with respect to the dividing (9,18)-axis.

The cases 18:2 - 18:7 are rather similar, the corresponding configurations exhibiting varying degrees of symmetry. It should be noted that in the cases 18:1 - 18:7 there are no Weierstrass points on the boundary of P.

In the case 18:8 five mid-points of sides are fixed by the rotation R of order 2 around 0, so that by 2(ii) these together with the origin 0 are the six Weierstrass points.

References

[1] E. Girondo and G. González-Diez, *Genus two extremal surfaces: Extremal discs, isometries and Weierstrass points*, A manuscript.

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[3] T. Jørgensen and M. Näätänen, *Surfaces of genus 2: Generic fundamental polygons*, Quart. J. Math. Oxford Ser.(2) **33** (1982), 451–461.

[4] M. Näätänen and T. Kuusalo, *On arithmetic genus 2 subgroups of triangle groups*, Contemp. Math. **201** (1997), 21–28.

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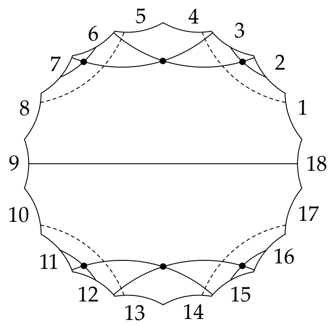
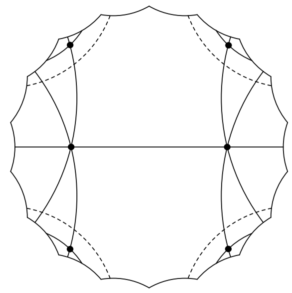
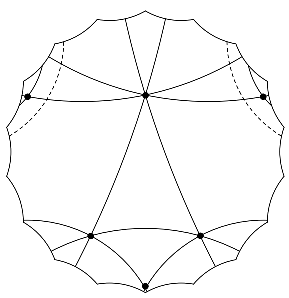


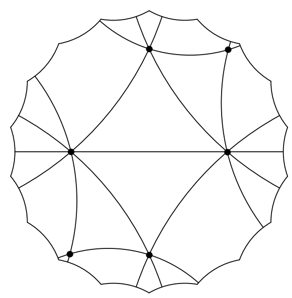
18:1



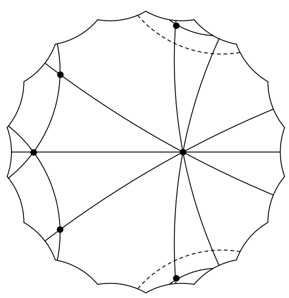
18:2



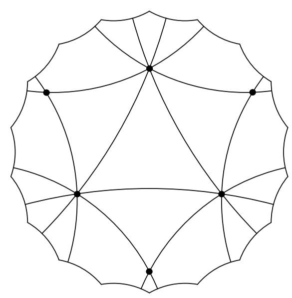
18:3



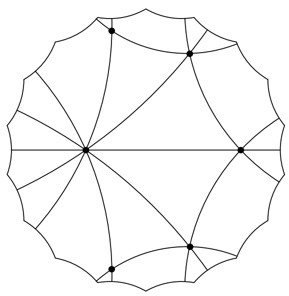
18:4



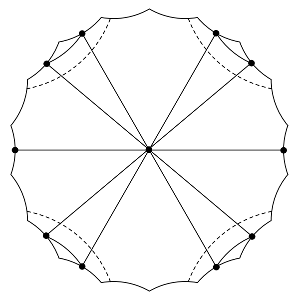
18:5



18:6



18:7



18:8

Fig. 2