Minimal assumptions for the integrability of the Jacobian

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1 Introduction

Throughout this paper we assume that Ω is a bounded open set in $\mathbf{R}^n$ and that $f\colon \Omega \to R^n$, $n \geq 2$, is an orientation-preserving mapping, that is, the Jacobian $J = J(x, f) = \det Df$ is nonnegative for almost every $x \in \Omega$. We warn the reader that the word "orientation-preserving" is partially misleading here in the sense that there exist a continuous, orientation-preserving mapping f with

$$\sup_{0<\epsilon\leq n-1} \epsilon \int_\Omega |Df|^{n-\epsilon}\, dx < \infty \tag{1}$$

that has a strictly negative topological degree. On the other hand, under assumptions very close to (1), e.g. (2) below, an orientation preserving mapping has a non-negative degree. For these results see [11].

S. Müller proved in [16] that if $|Df| \in L^n(\Omega)$, then $J \in L\log L(K)$ for any compact $K \subset \Omega$. The result of T. Iwaniec and C. Sbordone [9] on the integrability of the Jacobian is in a sense a dual to S. Müller's result: if $|Df| \in L^n(\log L)^{-1}(\Omega)$, then $J \in L^1_{\mathrm{loc}}(\Omega)$. As a matter of fact $J \in L\log\log L(K)$ for any compact $K \subset \Omega$, [15]. It was also shown in [9] that a Sobolev mapping f has an integrable Jacobian if $|Df|$ belongs to the *grand* Lebesgue space $GL^n(\Omega)$, that is, if (1) holds. The phenomenon of the improved integrability of the Jacobian was then investigated by several researchers; see [5] and the references therein.

Recently, L. Greco [4] proved a conjecture in [5] by relaxing the known sufficient conditions for the integrability of the Jacobian. He proved, among the other things,

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that for an orientation-preserving mapping f on Ω, if $|Df| \in L^{\Phi}(\Omega)$, then $J \in L^{\Psi}_{\text{loc}}(\Omega)$, whenever the *Orlicz function* Φ satisfies the following two assumptions:

$$\text{(i)} \qquad \frac{d}{dt}\left(\frac{\Phi(t)}{t^{n-1+\delta}}\right) \geq 0, \text{ for some } 0 < \delta < 1 \text{ and all } t > t_0 > 0;$$

$$\text{(ii)} \qquad \tilde{\Phi}(N) = \int_1^N \frac{\Phi(t)}{t^{n+1}}\, dt \to \infty, \text{ as } N \to \infty.$$

and the Orlicz function Ψ is defined as

$$\Psi(t) = \Phi(t^{1/n}) + nt \int_0^{t^{1/n}} \frac{\Phi(s)}{s^{n+1}}\, ds,$$

and that this result is optimal. We note here that $\Psi(t) \succ t$, that is, the integrability for the Jacobian is above the L^1-degree. We also remark that the grand Lebesgue space $GL^n(\Omega)$ is not contained in any allowable Orlicz space $L^{\Phi}(\Omega)$ and that $L^{\Phi}(\Omega)$, for $\Phi(t) = t^n(\log(e+t))^{-1}$, is essentially the largest Orlicz space contained in $GL^n(\Omega)$, see [8].

Our first result in this note extends and unifies the sufficient conditions for the integrability of the Jacobian of an orientation-preserving mapping.

Theorem 1.1 *Suppose that Φ satisfies the assumptions* (i) *and* (ii). *Let $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ be an orientation-preserving mapping and $|Df| \in GL^{\Phi}(\Omega)$, that is,*

$$\limsup_{N\to\infty} \frac{1}{\tilde{\Phi}(N)} \int_{\{|Df|\leq N\}} \Phi(|Df|)\, dx < \infty.$$

Then $J(x, f) \in L^1_{\text{loc}}(\Omega)$.

In Section 2 below we show that $GL^{\Phi}(\Omega)$ coincides with $GL^n(\Omega)$ when $\Phi(t) = t^n$, and thus Theorem 1.1 can be considered as an analog of the result of Iwaniec and Sbordone mentioned above. Some further properties of the spaces $GL^{\Phi}(\Omega)$ are also given in Section 2. To see that Theorem 1.1 does not follow from Greco's result, simply consider the cavitating map $f_0(x) = \frac{x}{||x||}$ that collapses the unit ball to the sphere. Clearly $|Df_0| \notin L^{\Phi}(\Omega)$ for any allowable Φ, but, on the other hand, we show below by a simple computation that $|Df_0| \in GL^{\Phi}(\Omega)$ for each allowable Φ.

Before commenting on the sharpness of Theorem 1.1, let us continue with the problem of the relation between the point-wise Jacobian J and the distributional determinant $\mathrm{Det}Df$ introduced by J. M. Ball [2]. We write $\det Df = \mathrm{Det}Df$ if the point-wise Jacobian coincides with the distributional Jacobian, that is, if,

$$\int_{\Omega} f_1(x) J(x, (\phi, f_2, \ldots, f_n))\, dx = -\int_{\Omega} \phi(x) J(x, f)\, dx$$

for each test function $\phi \in C_0^\infty(\Omega)$. According to the results of [9], $\det Df = \mathrm{Det} Df$ holds when f is orientation-preserving if $|Df| \in L^n(\log L)^{-1}(\Omega)$. This requirement was then relaxed in [3] to the assumption that

$$\lim_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx = 0 \tag{2}$$

and in [4] to $|Df| \in L^\Phi(\Omega)$, where Φ satisfies the assumptions (i) and (ii). These two conditions are incomparable. Our second result extends and unifies these conclusions.

Theorem 1.2 *Suppose that Φ satisfies the assumptions* (i) *and* (ii). *Let $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ be an orientation-preserving mapping such that $|Df| \in VL^\Phi(\Omega)$, that is,*

$$\lim_{N \to \infty} \frac{1}{\tilde{\Phi}(N)} \int_{\{|Df| \le N\}} \Phi(|Df|)\, dx = 0.$$

Then $\det Df = \mathrm{Det} Df$.

We show in Section 2 that the condition $|Df| \in VL^\Phi(\Omega)$ is equivalent to (2) when $\Phi(t) = t^n$.

As a corollary to Theorem 1.1 and Theorem 1.2 we obtain the following result that is also contained in the work [4] of Greco.

Corollary 1.3 *Let Φ satisfy assumptions* (i) *and* (ii). *Let $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ be an orientation-preserving mapping such that $|Df| \in L^\Phi(\Omega)$. Then $J(x, f) \in L^1_{\mathrm{loc}}(\Omega)$ and* $\det Df = \mathrm{Det} Df$.

Let us discuss the sharpness in Theorem 1.1 and in Theorem 1.2. First of all, the desired results hold in L^n and fail in L^p for all $p < n$. Thus the assumption (i) that

$$\left(\frac{\Phi(t)}{t^{n-1+\delta}} \right)' \ge 0$$

is, in practice, harmless. We will prove below that we can let $\delta = 0$ when $n > 2$. Let us then consider (ii). Given an increasing function Φ with

$$\int_1^\infty \frac{\Phi(t)}{t^{n+1}}\, dt < \infty,$$

under mild additional conditions on Φ we construct in Section 4 (Example 1) a mapping

$$f(x) = \frac{x}{\psi(||x||)},$$

where

$$\psi(t) = o(t)$$

when $t \to 0$ so that $|Df| \in L^{\Phi}(B^n)$ and $J(x, f)$ is not locally integrable. This shows that (ii) is crucial, as expected in [5]. Secondly, Example 2 in Section 4 shows that the L^1-integrability of the Jacobian as stated in Theorem 1.1 cannot be improved: Suppose we are given an Orlicz function Φ which satisfies the assumptions (i) and (ii). Then for each function Θ such that

$$\lim_{t \to \infty} \frac{\Theta(t)}{t} = \infty,$$

there exists an orientation-preserving mapping f with $|Df| \in GL^{\Phi}(\Omega)$, whose Jacobian fails to belong to $L^{\Theta}_{\text{loc}}(\Omega)$. Finally, regarding sharpness in Theorem 1.2, consider the cavitating map $f_0(x) = \frac{x}{||x||}$ that collapses the unit ball to the sphere. Computing in spherical coordinates, we see that

$$\int_{\{|Df_0| \leq N\}} \Phi(|Df_0|)\, dx = \omega_{n-1} \int_{1/N}^{1} t^{n-1} \Phi(1/t)\, dt =$$

$$= \omega_{n-1} \int_1^N \Phi(t) t^{-1-n} dt \; = \omega_{n-1} \tilde{\Phi}(N).$$

Thus the limit in Theorem 1.2 exists (and equals ω_{n-1}) for each allowable Φ, but, nevertheless, the distributional Jacobian is the delta-function and the point-wise Jacobian the zero-function. Theorem 1.2 then states that $\det Df = \operatorname{Det} Df$ as soon as

$$\int_{\{|Df| \leq N\}} \Phi(|Df|)\, dx = o(\int_{\{|Df_0| \leq N\}} \Phi(|Df_0|)\, dx)$$

for an allowable Φ.

The above cavitating map f_0 also shows that the grand Orlicz space $GL^{\Phi}(\Omega)$ is pretty large: f_0 is in $GL^{\Phi}(\Omega)$ for all Φ satisfying the assumptions (ii), but an elementary computation reveals that it fails to belong to any $L^{\Phi}(\Omega)$.

The proofs of the results in [9] and [3] and are based on new estimates in the Hodge Decomposition, see also [7] and [10] for this nice method. Greco uses in [4] ideas from a [1] paper by Acerbi and Fusco which do not as such seem to powerful enough for our setting. Instead of this we use a method of J. L. Lewis [13]; in fact while we began this work we were unaware of Greco's paper [4] and found out of his work only after we had proved Theorem 1.1 and Theorem 1.2. In both methods one constructs a Lipschitz continuous function by using a point-wise inequality for Sobolev functions in terms of the maximal function of the gradient, see also [14], [18] and [12].

2 Orlicz Spaces

In this section we give the definitions of the Orlicz space $L^\Phi(\Omega)$ and the grand Orlicz spaces $GL^\Phi(\Omega)$ and $VL^\Phi(\Omega)$.

A continuous and strictly increasing function $\Phi\colon [0,\infty] \to [0,\infty]$ with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$ is called an Orlicz function. The Orlicz space $L^\Phi(\Omega)$ is made up of all measurable functions u on Ω such that

$$\int_\Omega \Phi(k^{-1}|u(x)|)\, dx < \infty$$

for some $k = k(u) > 0$. The space $L^\Phi(\Omega)$ is a complete linear metric space [17]. In general the Luxemburg functional

$$||u||_\Phi = \inf\{k > 0\colon \int_\Omega \Phi(k^{-1}|u|) \le \Phi(1)\}$$

need not be a norm, but it is if Φ is convex. In this case $L^\Phi(\Omega)$ is a Banach space.

We are interested in the Orlicz functions $\Phi(t)$ which grow at ∞ a little bit slower than t^n. More precisely, Φ satisfies for some $t_0 > 0$ and $0 < \delta < 1$,

$$\text{(i)} \qquad \frac{d}{dt}\left(\frac{\Phi(t)}{t^{n-1+\delta}}\right) \ge 0, \ \text{ for } t > t_0 > 0$$

$$\text{(ii)} \qquad \tilde{\Phi}(N) = \int_1^N \frac{\Phi(t)}{t^{n+1}}\, dt \to \infty, \ \text{ as } N \to \infty.$$

A prime example of such Orlicz functions is $\Phi(t) = t^n(\log^+ t ... \log^{k+} t)^{-1}$.

The grand Orlicz space $GL^\Phi(\Omega)$ consists of all measurable functions u on Ω such that

$$\limsup_{N\to\infty} \frac{1}{\tilde{\Phi}(N)} \int_{\{|u|\le N\}} \Phi(|u|)\, dx < \infty,$$

and $VL^\Phi(\Omega)$ consists of all functions u such that

$$\lim_{N\to\infty} \frac{1}{\tilde{\Phi}(N)} \int_{\{|u|\le N\}} \Psi(|u|)\, dx = 0.$$

The reason for calling them grand Orlicz spaces is that when $\Phi(t) = t^n$, $GL^\Phi(\Omega)$ and $VL^\Phi(\Omega)$ coincide with the grand Lebesgue spaces $GL^n(\Omega)$ and $VL^n(\Omega)$, respectively. These are defined as in [9] by

$$\begin{aligned} GL^n(\Omega) &= \left\{u \in \cap_{0<\epsilon\le n-1} L^{n-\epsilon}(\Omega)\colon \sup\nolimits_{0<\epsilon\le n-1} \epsilon \int_\Omega |u|^{n-\epsilon}\, dx < \infty\right\}; \\ VL^n(\Omega) &= \left\{u \in \cap_{0<\epsilon\le n-1} L^{n-\epsilon}(\Omega)\colon \lim_{\epsilon\to 0} \epsilon \int_\Omega |u|^{n-\epsilon}\, dx = 0\right\}. \end{aligned}$$

Proposition 2.1 *Let $\Phi(t) = t^n$. Then $GL^\Phi(\Omega) = GL^n(\Omega)$ and $VL^\Phi(\Omega) = VL^n(\Omega)$.*

Proof. We only prove that $VL^\Phi(\Omega) = VL^n(\Omega)$. The other equality can be proven in the same way. If $u \in VL^n(\Omega)$, we observe from the trivial inequality

$$\frac{1}{\log N} \int_{|u|\le N} |u|^n \, dx \le \frac{e}{\log N} \int_\Omega |u|^{n-1/\log N} \, dx$$

that $u \in VL^\Phi(\Omega)$. Thus, $VL^n(\Omega) \subset VL^\Phi(\Omega)$. For the other direction, let $u \in VL^\Phi(\Omega)$. Then for $\delta > 0$, there exists $N = N(\delta) > 0$ such that for all $t > N$ we have

$$\int_{|u|<t} |u|^n \, dx \le \delta \log t.$$

It follows from

$$\begin{aligned} \epsilon \int_\Omega |u|^{n-\epsilon} \, dx &= \epsilon \int_{|u|\le N} |u|^{n-\epsilon} \, dx + \epsilon \int_{|u|>N} |u|^{n-\epsilon} \, dx \\ &\le \epsilon N^{n-\epsilon}|\Omega| + \epsilon^2 \int_N^\infty t^{-1-\epsilon} \int_{|u|<t} |u|^n \, dxdt \\ &\le \epsilon N^{n-\epsilon}|\Omega + \delta\epsilon^2 \int_N^\infty t^{-1-\epsilon} \log t \, dt \\ &\le \epsilon N^{n-\epsilon}|\Omega| + \delta \left(\frac{\epsilon \log N}{N^\epsilon} + \frac{1}{N^\epsilon} \right) \\ &\le \epsilon N^{n-\epsilon}|\Omega| + 2\delta \end{aligned}$$

that $u \in GL^n(\Omega)$, and the proposition follows.

The following proposition shows that the grand Orlicz spaces $GL^\Phi(\Omega)$ and $VL^\Phi(\Omega)$ are ordered.

Proposition 2.2 *Let Φ_1 and Φ_2 be two Orlicz functions satisfying the assumption* (ii). *Suppose that $\Phi_2(t) = \phi(t)\Phi_1(t)$ and that $\phi(t)$ decreases to 0 as t increases to ∞. Then $GL^{\Phi_1}(\Omega) \subset GL^{\Phi_2}(\Omega)$ and $VL^{\Phi_1}(\Omega) \subset VL^{\Phi_2}(\Omega)$*

Proof. We only prove that $GL^{\Phi_1}(\Omega) \subset GL^{\Phi_2}(\Omega)$. Let $u \in GL^{\Phi_1}(\Omega)$. Then for $N > 1$,

$$\begin{aligned} \int_{|u|\le N} \Phi_2(u) \, dx &\le \int_0^N \frac{d}{dt}(-\phi(t)) \int_{|u|\le t} \Phi_1(u) \, dxdt \\ &\le M \int_0^N \frac{d}{dt}(-\phi(t))\tilde{\Phi}_1(t) \, dt \\ &\le 2M \int_0^N \frac{\Phi_2(t)}{t^{n+1}} \, dt, \end{aligned}$$

where we used the trivial inequality

$$\int_0^N \frac{\Phi_2(t)}{t^{n+1}}\,dt \geq \phi(N)\int_0^N \frac{\Phi_1(t)}{t^{n+1}}\,dt.$$

This shows the proposition.

The inclusions of Proposition 2.2 are strict (cf. the proof of Proposition 2.1 in [8]).

In the following, we will assume that

$$\Phi(t) = \frac{\Phi(t_0)}{t_0^{n+1}}t^{n+1}$$

for all $t \leq t_0$. We may do this because the spaces $GL^{\Phi}(\Omega)$ and $VL^{\Phi}(\Omega)$ remain unchanged. We define

$$\tilde{\Phi}(t) = \int_0^N \frac{\Phi(t)}{t^{n+1}}\,dt.$$

3 Proofs of Theorem 1.1 and Theorem 1.2

The inequalities of the following lemma are well-known; the proof relies on an argument due to L. I. Hedberg [6].

Lemma 3.1 *Let $v \in W^{1,q}(\mathbf{R}^n)$, $1 < q < \infty$, and let x and y be Lebesgue points of v such that $x \in B_0 = B(x_0, r)$. Then*

$$|v(x) - v_{B_0}| \leq crM(|Dv|\chi_{2B_0})(x); \tag{3}$$

$$|v(x) - v(y)| \leq c|x-y|(M(|Dv|)(x) + M(|Dv|)(y)), \tag{4}$$

where $c = c(n) > 0$, χ_E is the characteristic function of set E, v_{B_0} is the average integral of v over B_0, and

$$Mh(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |h|\,dy$$

is the Hardy-Littlewood maximal function of h.

We are now ready to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $B \subset \Omega$ be a ball such that $4B \subset \Omega$, and let $\phi \in C_0^\infty(2B)$ be nonnegative. Define a function u on $\mathbf{R}^n$ by setting $u = (f_1 - (f_1)_{2B})\phi$ in $2B$ and 0 outside $2B$. Denote for $\lambda > 0$,

$$F_\lambda = \{x \in 2B\colon\ M(|Du|)(x) \leq \lambda\} \cap \{x \in 2B\colon x \text{ is Lebesgue point of } u\}.$$

We claim that

$$\tilde{u}_\lambda = \begin{cases} u(x) & \text{on } F_\lambda \\ 0 & \text{on } \mathbf{R}^n \setminus 2B \end{cases}$$

is Lipschitz continuous with constant $c\lambda$ for some $c = c(n) \geq 1$. Indeed, suppose that $x, y \in F_\lambda$. Then it follows from (2) in Lemma 3.1 that

$$\begin{aligned} |u(x) - u(y)| &\leq c|x-y|(M(|Du|)(x) + M(|Du|)(y)) \\ &\leq c\lambda|x-y|. \end{aligned}$$

If $x \in F_\lambda, y \in \mathbf{R}^n \setminus 2B$, then let $r = 2\text{dist}(x, \mathbf{R}^n \setminus 2B)$. Since $u = 0$ outside $2B$, we have by the Sobolev inequality that

$$|(u)_{B(x,r)}| \leq cr \frac{1}{|B(x,r)|} \int_{B(x,r)} |Du|\, dx \leq c\lambda r,$$

and by inequality (1) in Lemma 3.1 that

$$|u(x) - (u)_{B(x,r)}| \leq c\lambda r.$$

Thus,

$$|u(x)| \leq c\lambda \text{dist}(x, \mathbf{R}^n \setminus 2B),$$

which shows the claim. By the classical McShane extension theorem, we extend $\tilde{u}_\lambda$ to a Lipschitz continuous function u_λ in $\mathbf{R}^n$ with the same Lipschitz constant such that $u_\lambda(x) = \tilde{u}_\lambda(x) = u(x)$ for $x \in F_\lambda$, $u_\lambda(x) = 0$ for $x \in \mathbf{R}^n \setminus 2B$ and $|Du_\lambda(x)| \leq c\lambda$ for a.e. $x \in \mathbf{R}^n$. We deduce from

$$\int_{2B} J(x, (u_\lambda, f_2, \ldots, f_n))\, dx = 0$$

that

$$\left| \int_{F_\lambda} J(x, (u, f_2, \ldots, f_n))\, dx \right| \leq c\lambda \int_{2B \setminus F_\lambda} |Df|^{n-1}\, dx. \tag{5}$$

We estimate the right hand side of (5)):

$$\begin{aligned} \lambda \int_{2B \setminus F_\lambda} |Df|^{n-1}\, dx &\leq \lambda \int_{cM(M(|Df|\chi_{4B}))>\lambda} |Df|^{n-1}\, dx & (6) \\ &\leq \lambda^{1-\delta} \int_{cM(M(|Df|\chi_{4B}))>\lambda} M(M(|Df|\chi_{4B}))^{n-1+\delta}\, dx \\ &\leq c\lambda^{1-\delta} \int_{2cM(|Df|\chi_{4B})>\lambda} M(|Df|\chi_{4B})^{n-1+\delta}\, dx & (7) \\ &\leq c\lambda^{1-\delta} \int_{\{x \in 4B : 4c|Df|>\lambda\}} |Df|^{n-1+\delta}\, dx. & (8) \end{aligned}$$

Here (6) follows from the estimate

$$|Du| \leq |Df_1|\phi + |f_1 - (f_1)_{2B}||D\phi| \leq cM(|Df|\chi_{4B}),$$

obtained from Lemma 3.1, and (7) and (8) follow from the inequality

$$\int_{M(g)>\lambda} M(g)^q \, dx \leq c \int_{2g>\lambda} |g|^q \, dx, \text{ for all } g \in L^q(\mathbf{R}^n), q > 1.$$

This inequality is verified as follows:

$$\begin{aligned} \int_{M(g)>\lambda} M(g)^q \, dx &= q \int_\lambda^\infty t^{q-1} |\{M(g) > t\} \, dt + \lambda^q |\{M(g) > \lambda\}| \\ &\leq c \int_\lambda^\infty t^{q-2} \int_{2|g|>t} |g| \, dx dt + c\lambda^{q-1} \int_{2|g|>\lambda} |g| \, dx \\ &\leq c \int_{2|g|>\lambda} |g|^q \, dx. \end{aligned}$$

We note here that we may assume $\delta = 0$, if $n > 2$. Combining (3) and (6), we obtain that

$$\begin{aligned} |\int_{F_\lambda} \phi J(x,f) \, dx &+ \int_{F_\lambda} (f_1 - (f_1)_{2B}) J(x, (\phi, f_2, \ldots, f_n)) \, dx| \\ &\leq c\lambda^{1-\delta} \int_{\{x \in 4B : c|Df|>\lambda\}} |Df|^{n-1+\delta} \, dx. \end{aligned} \tag{9}$$

This inequality holds for all $\lambda > 0$, and by multiplying it by

$$\frac{1}{\lambda^{1-\delta}} \frac{d}{dt} \left(\frac{\Phi(t)}{t^{n-1+\delta}} \right),$$

and integrating over $(0, N)$, we obtain by changing the order of integration that

$$\begin{aligned} &|\int_{M(|Df|)\leq N} \phi J(x,f)(\Psi(N) - \Psi(M(|Df|)) \, dx \\ &+ \int_{M(|Df|)\leq N} (f_1 - (f_1)_{2B}) J(x, (\phi, f_2, \ldots, f_n))(\Psi(N) - \Psi(M(|Df|)) \, dx| \\ &\leq c \int_{4B} |Df|^{n-1+\delta} \Theta(\min(c|Df|, N)) \, dx, \end{aligned}$$

where

$$\Psi(t) = \frac{\Phi(t)}{t^n} + (1-\delta)\tilde{\Phi}(t), \text{ and } \Theta(t) = \frac{\Phi(t)}{t^{n-1+\delta}}.$$

Divide both sides by $\Psi(N)$ and let $N \to \infty$. Taking into account the fact that $J(x,f) \geq 0$, we have by the monotone convergence theorem and the dominated convergence theorem that

$$\begin{aligned}
\left|\int_{2B}\phi J(x,f)\,dx\right. &+ \left.\int_{2B}(f_1-(f_1)_{2B})J(x,(\phi,f_2,\dots,f_n))\,dx\right| \\
&\le \limsup_{N\to\infty}\frac{c}{\Psi(N)}\int_{4B}|Df|^{n-1+\delta}\Theta(\min(c|Df|,N))\,dx \\
&\le \limsup_{N\to\infty}\frac{c}{\tilde{\Phi}(N)}\int_{\{x\in 4B: c|Df|\le N\}}\Phi(c|Df|)\,dx \qquad (10)
\end{aligned}$$

where in the last step we used the inequality

$$\limsup_{N\to\infty}\frac{\Theta(N)}{\Psi(N)}\int_{c|Df|>N}|Df|^{n-1+\delta}\,dx \le \limsup_{N\to\infty}\frac{c}{\tilde{\Phi}(N)}\int_{\{c|Df|\le N\}}\Phi(c|Df|)\,dx.$$

This follows from the estimate

$$\begin{aligned}
\int_{|g|>N}|g|^{n-1+\delta}\,dx &\le \int_N^\infty \frac{d}{dt}\left(\frac{-1}{\Theta(t)}\right)\int_{|g|<t}\Phi(|g|)\,dxdt \\
&\le (M+O(1))\int_N^\infty \frac{d}{dt}\left(\frac{-1}{\Theta(t)}\right)\tilde{\Phi}(t)\,dt \\
&= \frac{M+O(1)}{1-\delta}\frac{\Psi(N)}{\Theta(N)}, \qquad (11)
\end{aligned}$$

that holds for all $g\in GL^{\Phi}(4B)$, where $O(1)\to 0$ as $N\to\infty$, and

$$M=\limsup_{N\to\infty}\frac{1}{\tilde{\Phi}(N)}\int_{|g|\le N}\Phi(|g|)\,dx.$$

Now the theorem follows from (8) by replacing cf by f.

Proof of Theorem 1.2. We infer from the proof of inequality (9) that

$$\lim_{N\to\infty}\frac{\Theta(N)}{\Psi(N)}\int_{|g|>N}|g|^{n-1+\delta}\,dx=0$$

for $g\in VL^{\Phi}(\Omega)$. Then it follows from (8) that

$$\int_{2B}\phi J(x,f)\,dx+\int_{2B}(f_1-(f_1)_{2B})J(x,(\phi,f_2,\dots,f_n))\,dx=0,$$

which concludes the proof of the theorem.

4 Examples

Example 1. The necessity of (ii). Let Φ be a strictly increasing, differentiable function satisfying

$$\int_1^\infty \frac{\Phi(t)}{t^{n+1}}\,dt<\infty, \qquad (12)$$

$$C_1\frac{\Phi(t)}{t} \leq \frac{d}{dt}\Phi(t) \leq C_2\frac{\Phi(t)}{t} \tag{13}$$

for all $t > 0$. We note here that assumption (i) is equivalent to the first inequality of (13) with $C_1 = n-1+\delta$, which also implies the Δ_2-regularity of the inverse function Φ^{-1} of Φ [5], that is,

$$\Phi^{-1}(2^{C_1}t) \leq 2\Phi^{-1}(t). \tag{14}$$

Set

$$\rho(t) = t\Phi^{-1}\left(\Phi(1/t)/(\int_{1/t}^{\infty}\Phi(s)s^{-1-n}\,ds)^{1/2}\right).$$

It then follows from (12) and (14) that

$$\lim_{t\to 0}\rho(t) = \infty$$

and from (12) and (13) that

$$\rho'(t) \leq C_3\rho(t)/t.$$

Set

$$f(x) = \frac{\rho(||x||)x}{||x||}.$$

It follows that $|Df(x)| \leq C_4\rho(t)/t$ and we conclude using (12) that $|Df| \in L^{\Phi}(B^n)$. On the other hand, f maps homeomorphically $B^n \setminus \{0\}$ onto a domain of infinite volume, and thus $J_f \notin L^1(B^n)$.

Example 2. Sharpness of Theorem 1.1. Suppose we are given an Orlicz function Φ which satisfies the assumptions (i) and (ii). Let Θ be such that

$$\lim_{t\to\infty}\frac{\Theta(t)}{t} = \infty. \tag{15}$$

We construct an orientation-preserving mapping F with $|DF| \in GL^{\Phi}(\Omega)$, whose Jacobian fails to belong to $L^{\Theta}_{\mathrm{loc}}(\Omega)$. We only give the example for the case $n = 2$; the general case can be handled in much the same way. Let $B = \{x \in \Omega : |x-a| < r\}$ and $1/2 < \alpha < 1$. Consider the orientation preserving mapping

$$f(x) = a + \frac{r^{\alpha}(x-a)}{|x-a|^{\alpha}}.$$

It follows easily that

$$|Df(x)| = \beta\frac{r^{\alpha}}{|x-a|^{\alpha}}$$

where $\beta = (2 - 2\alpha + \alpha^2)^{1/2}$, and that

$$\det Df = (1-\alpha)\frac{r^{2\alpha}}{|x-a|^{2\alpha}}.$$

Then for $N > 0$,

$$\int_{\{x \in B : |Df| \le N\}} \Phi(|Df|)\, dx = \frac{\pi \beta^{2/\alpha} r^2}{\alpha} \int_\beta^N \frac{\Phi(t)\, dt}{t^{2/\alpha+1}} \le 8\pi r^2 \tilde{\Phi}(N),$$

and, for every $k > 1$, we have

$$\int_B \Theta\left(\frac{\det Df}{k}\right) dx = \frac{\pi r^2}{\alpha} \int_1^\infty \frac{\Theta(\frac{1-\alpha}{k} t)}{t^{1+1/\alpha}}\, dt \ge \frac{\pi r^2 (1-\alpha)}{k^{1/\alpha} e} \int_1^\infty \frac{\Theta(t)\, dt}{t^{1+1/\alpha}}.$$

Set

$$\phi(\alpha) = (1-\alpha) \int_1^\infty \frac{\Theta(t)\, dt}{t^{1+1/\alpha}}.$$

We have

$$\lim_{\alpha \to 1} \phi(\alpha) = \infty. \tag{16}$$

Indeed, it follows from (15) that for each $M > 0$ there exists $t_0 = t_0(M) > 1$ such that $\Theta(t) \ge Mt$ for $t \ge t_0$. Splitting the defining integral for ϕ as $\int_1^\infty = \int_1^{t_0} + \int_{t_0}^\infty$, we have

$$\lim_{\alpha \to 1} \phi(\alpha) \ge M,$$

and hence (16) follows. Now we construct an orientation preserving mapping $F : \Omega \to \mathbf{R}^n$ using f. First, we choose mutually disjoint disks $B_j = \{x \in \Omega : |x - a_j| < r_j\} \subset \Omega, j = 1, 2, ...,$ and numbers $0 < \alpha_j < 1$ converging to 1, such that

$$\sum_{j=1}^\infty r_j^2 = \infty, \text{ but } \sum_{j=1}^\infty \phi(\alpha_j) r_j^2 = \infty.$$

This is possible because of (16). We can certainly assume that $\cup_{j=1}^\infty B_j$ is compactly contained in Ω. Put

$$f_j(x) = a_j + \frac{r_j^{\alpha_j}(x - a_j)}{|x - a_j|^{\alpha_j}}$$

for $x \in B_j, j = 1, 2, ...$ Then define

$$F(x) = \begin{cases} f_j(x) & \text{if } x \in B_j \\ x & \text{if } x \in \Omega \setminus \cup B_j. \end{cases}$$

We then find that $F \in GL^{\Phi}(\Omega)$. On the other hand, for each $k \geq 1$ we have

$$\int_{B_j} \Theta\left(\frac{\det DF}{k}\right)\, dx \geq \frac{\pi r_j^2}{e k^{1/\alpha_j}} \phi(\alpha_j).$$

Hence

$$\int_{\Omega} \Theta\left(\frac{\det DF}{k}\right)\, dx \geq \sum_{j=1}^{\infty} \int_{B_j} \Theta\left(\frac{\det DF}{k}\right)\, dx = \infty.$$

This shows that $\det DF \notin L^{\Theta}_{\mathrm{loc}}(\Omega)$, as desired.

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