REMOVABLE SETS FOR CONTINUOUS SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We show that sets of $n-p+\alpha(p-1)$ Hausdorff measure zero are removable for α-Hölder continuous solutions to quasilinear elliptic equations similar to the p-Laplacian. The result is optimal. We also treat larger sets in terms of a growth condition. In particular, our results apply to quasiregular mappings.

1. Introduction

Throughout this paper we let Ω be an open set in $\mathbf{R}^n$ and $1 < p < \infty$ a fixed number. Continuous solutions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{1.1}$$

are called *$\mathcal{A}$-harmonic* in Ω; here $\mathcal{A}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is assumed to verify for some constants $0 < \lambda \le \Lambda < \infty$:

$$\begin{aligned} &\text{the function } x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbf{R}^n\,, \text{ and} \\ &\text{the function } \xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for a.e. } x \in \mathbf{R}^n\,; \end{aligned} \tag{1.2}$$

for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$

$$\mathcal{A}(x,\xi)\cdot\xi \ \ge \lambda|\xi|^p\,, \tag{1.3}$$

$$|\mathcal{A}(x,\xi)| \ \le \ \Lambda|\xi|^{p-1}, \tag{1.4}$$

$$(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta))\cdot(\xi-\zeta) \ > \ 0 \tag{1.5}$$

whenever $\xi \ne \zeta$. A prime example of the operators is the p-Laplacian

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

in this case, the continuous solutions of (1.1) are called p-harmonic functions. The main result in this paper is the following theorem.

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1.6. Theorem. *Let $E \subset \Omega$ be closed and $s > 0$. Suppose that u is a continuous function in Ω, $\mathcal{A}$-harmonic in $\Omega \setminus E$ such that*

$$|u(x_0) - u(y)| \le C|x_0 - y|^{(s+p-n)/(p-1)} \tag{1.7}$$

for all $y \in \Omega$ and $x_0 \in E$. If E is of s-Hausdorff measure zero, then u is $\mathcal{A}$-harmonic in Ω.

Since sets of p-capacity zero are removable for bounded $\mathcal{A}$-harmonic functions, Theorem 1.6 is interesting for $s > n - p$ only. Kilpeläinen, Koskela, and Martio [KKM] had a special version of Theorem 1.6, where u was assumed to be flat on E and Hausdorff measure was replaced by a Minkowski content type condition.

1.8. Corollary. *Suppose that $u \in C^{0,\alpha}(\Omega)$, $0 < \alpha \le 1$, is $\mathcal{A}$-harmonic in $\Omega \setminus E$. If E is a closed set of $n-p+\alpha(p-1)$ Hausdorff measure[1] zero , then u is $\mathcal{A}$-harmonic in Ω.*

The following theorem shows that Corollary 1.8 is optimal. Before stating the theorem, we recall that there is a constant κ, $0 < \kappa = \kappa(n, p, \lambda, \Lambda) \le 1$, such that every $\mathcal{A}$-harmonic function h in Ω verifies the local Hölder continuity estimate

$$\operatorname{osc}(h, B(x,r)) \le c\Big(\frac{r}{R}\Big)^{\kappa} \operatorname{osc}(h, B(x,R)) \tag{1.9}$$

for each $0 < r < R$ and $B(x,R) \subset \Omega$ [HKM, 6.6]. For smooth $\mathcal{A}$, in particular for the p-Laplacian, we may choose $\kappa = 1$ (see e.g. [K, 2.3]).

1.10. Theorem. *Let κ be as above and $0 < \alpha < \kappa$. Suppose that $E \subset \Omega$ is a closed set with positive $n - p + \alpha(p - 1)$ Hausdorff measure[1]. Then there is $u \in C^{0,\alpha}(\Omega)$ which is $\mathcal{A}$-harmonic in $\Omega \setminus E$, but does not have an $\mathcal{A}$-harmonic extension to Ω.*

For the p-Laplacian we have the following sharp result.

1.11. Corollary. *Let $0 < \alpha < 1$. A closed set E is removable for α-Hölder continuous p-harmonic functions if and only if E is of $n - p + \alpha(p - 1)$ Hausdorff measure[1] zero.*

Carleson [C] proved Corollary 1.11 for the Laplacian ($p = 2$). As to the quasilinear case, Heinonen and Kilpeläinen [HK, 4.5] proved Corollary 1.8 with $\alpha = 1$, and Trudinger and Wang [TW] proved it under the assumption that u has an $\mathcal{A}$-superharmonic extension to Ω, which assumption can be dispensed with for small α. However, in the general situation the growth condition of Theorem 1.6 yields a more useful result, since $\mathcal{A}$-harmonic functions are not in general in $C^{0,\alpha}$ for α close to 1. Koskela and Martio [KM2] proved a weaker version of Corollary 1.13 and 1.8, where Minkowski content is used in place of Hausdorff measure. Buckley and Koskela [BK] also established very special cases of Corollary 1.8. In [K] there is a weaker version of Theorem 1.10.

A mapping $f\colon \Omega \to \mathbf{R}^n$ is called *quasiregular* if $f \in W^{1,n}_{\mathrm{loc}}(\Omega)$ and there is a constant K such that

$$|f'(x)|^n \le K J_f(x)$$

for a.e. $x \in \Omega$; here $J_f(x)$ is the Jacobian determinant of f at x. The coordinate functions of a quasiregular map f satisfy an equation of type (1.1) with $p = n$ (cf. [HKM, Ch. 14], whence we have:

[1] Assume, of course that $\alpha \ge (p - n)/(p - 1)$.

1.12. Corollary. *Let $E \subset \Omega$ be a closed set of s-Hausdorff measure zero, $0 < s \le n$. Suppose that $f\colon \Omega \to \mathbf{R}^n$ is a continuous mapping quasiregular in $\Omega \setminus E$. If*

$$|f(x_0) - f(y)| \le C|x_0 - y|^{s/(n-1)}$$

for all $y \in \Omega$ and $x_0 \in E$, then f is quasiregular in Ω.

1.13. Corollary. *Suppose that $f \in C^{0,\alpha}(\Omega)$ is quasiregular in $\Omega \setminus E$. If E is a closed set of $\alpha(n-1)$-Hausdorff measure zero, then f is quasiregular in Ω.*

Koskela and Martio [KM1] showed that sets whose Minkowski dimension is less than αn are removable for α-Hölder continuous quasiregular mappings provided that $\alpha < 1 - 1/n$, and the same for sets of αn-Hausdorff measure zero if $\alpha \le 1/n$.

Our method of proof combines some ideas from [K], [L], and [TW]. We use solutions of equations

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu\,,$$

where μ is a nonnegative Radon measure from $W^{-1,p'}_{\text{loc}}(\Omega)$, i.e. $u \in W^{1,p}_{\text{loc}}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu$$

for all $\varphi \in C_0^\infty(\Omega)$. In particular, we prove the following theorem that improves the main theorem in [K].

1.14. Theorem. *Let κ be the number given by (1.9). Suppose that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a solution of*

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu\,,$$

where μ is a nonnegative Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with

$$\mu(B(x,r)) \le M r^{n-p+\alpha(p-1)} \tag{1.15}$$

whenever $B(x, 3r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$. Moreover, $\kappa(n,p,1,1) = 1$, that is, in the case of the p-Laplacian any $\alpha < 1$ will do.

Theorem 1.14 is the best possible (see [KM, 4.18], [K, 2.7]).

Finally, we remark here that Corollary 1.11 is not true when $\alpha = 1$. The problem for which sets are removable for Lipschitz continuous p-harmonic functions is more delicate. David and Mattila [DM] treated the case $n = p = 2$: a compact set E of finite 1-Hausdorff measure is removable for Lipschitz continuous harmonic functions if and only if E is purely unrectifiable. The other cases remain open.

2. Proof of Theorem 1.6

We need a potential theoretic version of the obstacle problem. Suppose that ψ is a continuous function on Ω and let the *balayage* $\hat{R}^\psi = \hat{R}^\psi(\Omega)$ be the pointwise infimum of all supersolutions[2] u to (1.1) that lie above ψ in Ω. Similarly, let $\underline{\hat{R}}^\psi = \underline{\hat{R}}^\psi(\Omega)$ be the pointwise supremum of all subsolutions that lie below ψ in Ω. Then $\hat{R}^\psi \ge \psi$ is a continuous supersolution in Ω and $\mathcal{A}$-harmonic in $\{\hat{R}^\psi > \psi\}$; similar statements hold for $\underline{\hat{R}}^\psi$. For a more thorough discussion see [HKM, Ch. 9]. Next we show the following estimate for the balayage; see [L] for a related result.

[2] i.e. $u \in W^{1,p}_{\text{loc}}(\Omega)$ and $-\operatorname{div} \mathcal{A}(x, \nabla u) \ge 0$ in Ω

2.1. Lemma. *Let $K \subset \Omega$ be compact. Suppose that ψ is a continuous function with*

$$|\psi(x) - \psi(y)| \le M|x-y|^\alpha \text{ for all } x \in K \text{ and } y \in \Omega,$$

where $M > 0$ and $\alpha > 0$. Let $u = \hat{R}^\psi$ and

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)\,.$$

Then

$$\mu\big(B(x,r)\big) \le c\, r^{n-p+\alpha(p-1)}$$

for all $r < r_0 = \frac{1}{64}\operatorname{dist}(K, \partial\Omega)$ and $x \in K$, here $c = c(n,p,\lambda,\Lambda,M,\alpha) > 0$.

Proof. Write

$$I = \{x \in \Omega \colon \psi(x) = u(x)\}$$

for the contact set.

First, let $x_0 \in I$. We assume, as we may, that $u(x_0) = 0 = \psi(x_0)$. If $r \le \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)$ and

$$\gamma_0 = \operatorname{osc}(\psi, B(x_0, 8r)),$$

then $(u-\gamma_0)^+$ is a subsolution and $u+\gamma_0$ a nonnegative supersolution in $B(x_0,8r)$. Hence we deduce from the weak Harnack inequalities [HKM, 3.34 and 3.59] that

$$\begin{aligned}
\sup_{B(x_0,r)} (u-\gamma_0) &\le c \left(\fint_{B(x_0,2r)} |(u-\gamma_0)^+|^{p-1}\,dx \right)^{1/(p-1)} \\
&\le c \left(\fint_{B(x_0,2r)} (u+\gamma_0)^{p-1}\,dx \right)^{1/(p-1)} \\
&\le c \inf_{B(x_0,2r)} (u+\gamma_0) \\
&\le c\gamma_0\,.
\end{aligned}$$

Keeping in mind that $u \ge \psi \ge -\gamma_0$ we conclude

$$\operatorname{osc}(u, B(x_0,r)) \le c\gamma_0 = c\operatorname{osc}(\psi, B(x_0,8r))\,. \tag{2.2}$$

Let $r \le \frac{1}{32}\operatorname{dist}(x_0,\partial\Omega)$ and let $\eta \in C_0^\infty(B(x_0,2r))$ be a usual nonnegative cut-off function with $\eta = 1$ in $B(x_0,r)$ and $|\nabla\eta| \le 2/r$. Then we obtain by applying the Caccioppoli estimate [HKM, 3.29] to $u - \sup_{B(x_0,4r)} u$ and (2.2) that

$$\begin{aligned}
\mu(B(x_0,r)) &\le \int_{B(x_0,2r)} \eta^p\,d\mu = p\int_{B(x_0,2r)} \eta^{p-1}\mathcal{A}(x,\nabla u)\cdot\nabla\eta\,dx \\
&\le c\Big(\int_{B(x_0,2r)} |\nabla u|^p\eta^p\,dx\Big)^{(p-1)/p}\Big(\int_{B(x_0,2r)} |\nabla\eta|^p\,dx\Big)^{1/p} \\
&\le c\,r^{n-p}\operatorname{osc}(u, B(x_0,2r))^{p-1} \\
&\le c\,r^{n-p}\operatorname{osc}(\psi, B(x_0,16r))^{p-1}\,.
\end{aligned}$$

Now if $x_0 \in I$ is such that

$$\operatorname{dist}(x_0, K) \le r \le 2r_0\,,$$

we have the estimate

$$\mu(B(x_0,r)) \le c\, r^{n-p+\alpha(p-1)}\,, \tag{2.3}$$

where $c = c(n,p,M) > 0$.

Finally, for $x_0 \in K$ and $r < r_0$, there are two alternatives. Either $B(x_0,r)\cap I = \emptyset$ and thus $\mu(B(x_0,r)) = 0$, or there is $x \in B(x_0,r)\cap I$. In this latter case

$$\mu(B(x_0,r)) \le \mu(B(x,2r)) \le c\, r^{n-p+\alpha(p-1)}$$

by (2.3). The lemma is proven. □

Remark. Using (1.9) and (2.2), one can easily prove that if $\psi \in C^{0,\alpha}(\Omega)$, then $\hat{R}^\psi \in C^{0,\beta}(\Omega)$, where $\beta = \min(\alpha,\kappa)$ and $\kappa > 0$ is the constant such that (1.9) holds. (see e.g. [HKM, 6.47]).

Proof of Theorem 1.6. Fix a regular set $D \subset\subset \Omega$, for instance a ball. Let $v = \hat{R}^u = \hat{R}^u(D)$ and

$$\mu = -\operatorname{div} \mathcal{A}(x,\nabla v).$$

Let $K \subset E$ be compact. Since sets of $n-p$ Hausdorff measure zero ($p \le n$) are known to be removable for bounded $\mathcal{A}$-harmonic functions (see e.g. [HKM]), we need only consider the case, where $\alpha = (s+p-n)/(p-1) > 0$. Since $s = n-p+\alpha(p-1)$ we infer from (1.7) and Lemma 2.1 that

$$\mu\big(B(x,r)\big) \le c\, r^s$$

for all $r \le r_0$ and $x \in K$. Because $\mathcal{H}^s(K) = 0$, we may cover K by balls $B(x_j,r_j)$ so that

$$\mu(K) \le \sum_j \mu\big(B(x_j,r_j)\big) \le c \sum_j r_j^s < \varepsilon\,,$$

where $\varepsilon > 0$ is given. Consequently, $\mu(E) = 0$ and therefore $\mu = 0$, which means that v is $\mathcal{A}$-harmonic in D [M, 3.19].

Next let $w = \underline{\hat{R}}^u(D)$. We similarly find that w is $\mathcal{A}$-harmonic in D. Since $v = u = w$ on ∂D by [HKM, 9.26], we have that $v = w$ in D by the uniqueness of $\mathcal{A}$-harmonic functions. Since

$$w \le u \le v = w,$$

u is $\mathcal{A}$-harmonic in D and the theorem follows. □

3. Proof of Theorems 1.14 and 1.10

We recall that κ is the constant such that (1.9) holds for every $\mathcal{A}$-harmonic function h in Ω. Then

$$\int_{B(x,r)} |\nabla h|^p\,dx \le c\Big(\frac{r}{R}\Big)^{n-p+p\kappa} \int_{B(x,R)} |\nabla h|^p\,dx\,, \tag{3.1}$$

for each $0 < r < R$ with $B(x,R) \subset \Omega$; here $c = c(n,p,\lambda,\Lambda) > 0$ (see e.g. [K, 2.1]).

The following lemma provides the key estimate.

3.2. Lemma. *Let $u \in W^{1,p}(B(x_0,R))$ be a solution of*

$$-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu\,,$$

where μ is a nonnegative Radon measure such that

$$\mu(B(x_0,r))\le c_0\,r^{n-p+\alpha(p-1)}$$

for all $0<r\le R$. Then for each $0<r<R$ and $\varepsilon>0$ we have

$$\int_{B(x_0,r)}|\nabla u|^p\,dx\le c_1\Big(\big(\frac{r}{R}\big)^{n-p+p\kappa}+\varepsilon\Big)\int_{B(x_0,R)}|\nabla u|^p\,dx+c_2\,R^{n-p+p\alpha}\,,$$

where $c_1=c_1(n,p,\lambda,\Lambda)>0$ and $c_2=c_2(n,p,\lambda,\Lambda,\alpha,c_0,\varepsilon)>0$.

Proof. There is no loss of generality in assuming that $r<R/2$. Let h be the $\mathcal{A}$-harmonic function in $B(x_0,R)$ with $u-h\in W_0^{1,p}(B(x_0,R))$. Then

$$\begin{aligned}\lambda\int_{B(x_0,r)}|\nabla u|^p\,dx&\le\int_{B(x_0,r)}\mathcal{A}(x,\nabla u)\cdot\nabla u\,dx\\&=\int_{B(x_0,r)}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\big)\cdot\big(\nabla u-\nabla h\big)\,dx\\&\quad+\int_{B(x_0,r)}\mathcal{A}(x,\nabla h)\cdot(\nabla u-\nabla h)\,dx+\int_{B(x_0,r)}\mathcal{A}(x,\nabla u)\cdot\nabla h\,dx\\&\le\int_{B(x_0,R)}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\big)\cdot\big(\nabla u-\nabla h\big)\,dx\\&\quad+\Lambda\int_{B(x_0,r)}|\nabla h|^{p-1}|\nabla u|+|\nabla h|\,|\nabla u|^{p-1}\,dx\end{aligned}\tag{3.3}$$

where we used the structural assumptions (1.3)-(1.5). Since h is $\mathcal{A}$-harmonic with $h-u\in W_0^{1,p}(B(x_0,R))$ and thus quasiminimizes the p-Dirichlet integral, we have by using Adams' inequality (see [AH, Thm 7.2.2] or [Z, Thm 4.7.2]) that

$$\begin{aligned}\int_{B(x_0,R)}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\big)\cdot\big(\nabla u-\nabla h\big)\,dx&=\int_{B(x_0,R)}(u-h)\,d\mu\\&\le c\,R^{(p-1)(n-p+\alpha p)/p}\Big(\int_{B(x_0,R)}|\nabla u-\nabla h|^p\,dx\Big)^{1/p}\\&\le c\,R^{n-p+\alpha p}+\frac{\lambda}{2}\varepsilon\int_{B(x_0,R)}|\nabla u|^p\,dx\,,\end{aligned}$$

where we also used Young's inequality. The remaining integrals on the right of (3.3) do not exceed

$$\begin{aligned}&\frac{\lambda}{2}\int_{B(x_0,r)}|\nabla u|^p\,dx+c\int_{B(x_0,r)}|\nabla h|^p\,dx\\&\le\frac{\lambda}{2}\int_{B(x_0,r)}|\nabla u|^p\,dx+c\big(\frac{r}{R}\big)^{n-p+p\kappa}\int_{B(x_0,R)}|\nabla h|^p\,dx\\&\le\frac{\lambda}{2}\int_{B(x_0,r)}|\nabla u|^p\,dx+c\big(\frac{r}{R}\big)^{n-p+p\kappa}\int_{B(x_0,R)}|\nabla u|^p\,dx\,,\end{aligned}$$

where we also employed (3.1) and the quasiminimizing property of $\mathcal{A}$-harmonic functions. Plugging these estimates in (3.3) we arrive at

$$\int_{B(x_0,r)} |\nabla u|^p\,dx \le c\,R^{n-p+\alpha p} + \varepsilon \int_{B(x_0,R)} |\nabla u|^p\,dx + c\Big(\frac{r}{R}\Big)^{n-p+p\kappa} \int_{B(x_0,R)} |\nabla u|^p\,dx\,.$$

The lemma follows. □

Proof of Theorem 1.14. If $B(x_0, 4R) \subset \Omega$, then by appealing to [G, Lemma III.2.1, p. 86] Lemma 3.2 yields

$$\int_{B(x_0,r)} |\nabla u|^p\,dx \le c\Big(\frac{r}{R}\Big)^{n-p+p\alpha}$$

for $r < R$. Thus $u \in C^{0,\alpha}(\Omega)$ by the Dirichlet growth theorem [G, Theorem III.1.1, p. 64]. □

Proof of Theorem 1.10. Let κ be the number as in Theorem 1.14. Let $K \subset E$ be compact with $\mathcal{H}^{n-p+\alpha(p-1)}(K) > 0$. Frostman's lemma ([AH, 5.1.12], [C]) gives us a nonnegative Radon measure μ living on K with $\mu(K) > 0$ and $\mu(B(x,r)) \le r^{n-p+\alpha(p-1)}$. Any solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ to

$$-\operatorname{div}\mathcal{A}(x,\nabla u) = \mu$$

is $\mathcal{A}$-harmonic in $\Omega \setminus E$ [M, 3.19] and $u \in C^{0,\alpha}(\Omega)$ by Theorem 1.14, but u fails to have an $\mathcal{A}$-harmonic extension to Ω, since $\mu(K) > 0$. □

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