Differentiability of monotone Sobolev functions

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Abstract

We establish a sharp integrability condition on the partial derivatives of a weakly monotone Sobolev function to guarantee differentiability almost everywhere.

1 Introduction

It is well known that if a function u belongs to the Sobolev class $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$, for some $p > n$, then u is differentiable almost everywhere in Ω. This was proved by Cesari [2] for $n = 2$ and by Calderón [1] for general n. Here $W^{1,p}(\Omega)$ consists of the functions in Ω which together with their first order weak partial derivatives are p-integrable. In 1981, Stein showed that this condition can be sharpened to a very precise integrability condition. He proved that if $u \in W^{1,1}(\Omega)$ is a function whose weak partial derivatives belong to the Lorentz space $L^{n,1}(\Omega)$, then u is diffentiable almost everywhere. The notion of the Lorentz space $L^{p,q}(\Omega)$ was first introduced in [10].

In this paper we address the following question: What are the minimal integrability conditions on the partial derivatives of a homeomorphic Sobolev mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ to guarantee differentiability almost everywhere. Here $W^{1,1}(\Omega, \mathbb{R}^n)$ consists of the mappings of Ω into $\mathbb{R}^n$ whose coordinate functions belong to $W^{1,1}(\Omega)$. By the above result of Stein's, it naturally suffices to assume that $|\nabla f| \in L^{n,1}(\Omega)$. However, less is needed: it suffices to assume that $|\nabla f|$ is p-integrable, for some $p > n-1$, whereas $|\nabla f| \in L^{n-1}(\Omega)$

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is not sufficent when $n > 2$. This was proved by Väisälä [16]. Väisälä's approach is itself an n-dimensional version of a technique used by Gehring and Lehto [4] to show that a planar homeomorphism with integrable partial derivatives is differentiable a.e. Recall that each function $u \in W^{1,1}(\Omega)$ is differentiable a.e. when $\Omega \subset \mathbb{R}$. Because $L^{1,1}(\Omega) = L^1(\Omega)$, the results of Stein, Gehring and Lehto, and Väisälä suggest that the natural assumption should be that $|\nabla f| \in L^{n-1,1}(\Omega)$. We will show in this paper that this condition guarantees differentiability almost everywhere even for a larger class of mappings than the class of homeomorphisms. This will be the class of weakly monotone mappings.

Monotonicity for a continuous function u in a domain $\Omega \subset \mathbb{R}^n$ simply means that

$$\sup_{x\in B} u(x) \leq \sup_{x\in\partial B} u(x) \tag{1}$$

and

$$\inf_{x\in B} u(x) \geq \inf_{x\in\partial B} u(x) \tag{2}$$

for every ball $B \subset \Omega$. Roughly speaking, monotone functions satisfy the maximum and minimun principles in Ω. This definition of monotone function is due to Lebesgue [9].

Definition 1.1 *A real valued function $u \in W^{1,1}(\Omega)$ is said to be weakly monotone if for every ball $B \subset \Omega$ and all constants $m \leq M$ such that*

$$v := (u-m)^- + (u-M)^+ \in W_0^{1,1}(B),$$

we have

$$m \leq u(x) \leq M$$

for almost every $x \in B$.
Moreover, we say that $f \in W^{1,1}(\Omega, \mathbb{R}^m)$ is weakly monotone if its coordinate functions are weakly monotone.

Weakly monotone functions were introduced by Manfredi, [12]. They form a generalization of monotone functions in the sense of Lebesgue: a continuous function $u \in W^{1,1}(\Omega)$ is weakly monotone if and only if u satisfies conditions

(1) and (2). Especially, if $f : \Omega \to \Omega'$ is a homeomorphic mapping of the class $W^{1,1}(\Omega, \mathbb{R}^n)$, then f is a weakly monotone mapping. However is it not always true that weakly monotone functions are continuous (see [12, example 2.1]). Many solutions to partial differential equations are weakly monotone. For example, if $0 < \alpha(x) \leq \beta < \infty$ a.e. in $\mathbb{R}^2$, where α is a measurable function and β is a constant, then each $W^{1,2}$-solution to div $A(x)\nabla u(x) = 0$ for

$$A(x) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(x) \end{pmatrix}$$

are weakly monotone (see [8]).

Now we can formulate our main theorem.

Theorem 1.2 *(i) Suppose that $u \in W^{1,1}(\Omega)$ is a weakly monotone function whose weak partial derivatives belong to $L^{n-1,1}(\Omega)$. Then u is differentiable a.e.*
(ii) Suppose that $q > 1$. Then there exists a weakly monotone function $v \in W^{1,1}(B(0, \frac{1}{2}))$ such that $|\nabla v| \in L^{n-1,q}(B(0, \frac{1}{2}))$ and v is not differentiable anywhere.

Here $L^{n,q}(\Omega)$ is the Lorenz space (see Section 2) and

$$L^p(\Omega) \subset L^{n-1,1}(\Omega) \subset L^{n-1,q}(\Omega) \subset L^{n-1}(\Omega)$$

locally, if $p > n-1$ and $1 < q < n-1$ (see [17, lemma 1.8.13]). Furthermore the above inclusions are strict.

One can prove that the following inclusion is true.

$$\{u \in W^{1,1}(\Omega) : \int_\Omega |\nabla u|^n \log^{n-2+\epsilon}(e + |\nabla u|) < \infty, \epsilon > 0\} \subset L^{n-1,1}(\Omega)$$

Example 1.3 *Let $u : \Omega \to \Omega'$ be a homeomorphic mapping of the class $W^{1,1}(\Omega, \mathbb{R}^m)$ where $\Omega \subset \mathbb{R}^n$. Then the condition*

$$\int_\Omega |\nabla u_i|^n \log^\alpha(e + |\nabla u_i|) < \infty$$

for all $i = 1, .., m$ guarantees differentiability a.e. if $\alpha > n-2$ but not if $\alpha \leq n-2$.

2 Preliminaries

The notation used in the paper is standard.

Ω is an open subset of $\mathbb{R}^n$, $n \geq 2$. $B^n(x,r)$ and $B(x,r)$ will denote an n-dimensional ball centered at x with the radius r. Spheres will be denoted by $S^{n-1}(x,r)$.

The Lebesgue measure of a set E will be denoted by $|E|$. The characteristic function of a set E is χ_E. If $u : E \to \mathbb{R}$ is an integrable function on a set E with $0 < |E| < \infty$, we write

$$u_E = ⨍_E u(x)\ dx = \frac{1}{|E|} \int_E u(x)\ dx.$$

$C(a,b)$ will denote a positive constant that depends only on a, b. The value of $C(a,b)$ is not necessary the same at each occurence; it may vary even within a line.

The gradient ∇u is understood in the distributional sense. We denote $W^{1,1}(\Omega)$ the usual Sobolev space on Ω consisting of functions u such that both $u \in L^1(\Omega)$ and $|\nabla u| \in L^1(\Omega)$. $W^{1,1}(\Omega, \mathbb{R}^m)$ stands for the class of mappings $u : \Omega \to \mathbb{R}^m$ such that the coordinate functions belong to the space $W^{1,1}(\Omega)$.

Let u be a μ-measurable function defined on X, where (X, μ) is a measure space. We denote by ω the distribution function of u, namely, for $t \geq 0$ we set

$$\omega(t) = \mu(\{x \in X : u(x) > t\}).$$

Then we define the the *non-increasing rearrangement* u^* of u by setting

$$u^*(s) = \inf\{t \geq 0 : \omega(t) \leq s\}.$$

It is well known that if u and v are two μ-measurable functions on X, then

$$\int_X |u(x)v(x)| d\mu(x) \leq \int_0^{\mu(X)} u^*(s)v^*(s) ds. \tag{3}$$

Inequality (3) is due to Hardy, Littlewood and Pólya [5]. It follows immediately from the definition of $u^*(s)$ that

$$u^*(\omega(t)) \leq t. \tag{4}$$

The *Lorentz space* $L^{p,q}(X)$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, is defined as the class of all measurable functions on Ω for which the norm

$$||u||_{L^{p,q}(X)} := \begin{cases} \left(\int_0^{\mu(X)} [t^{\frac{1}{p}} u^*(t)]^q \frac{dt}{t}\right)^{\frac{1}{q}} = \left(p \int_0^\infty s^{q-1} [\omega(s)]^{\frac{q}{p}} ds\right)^{\frac{1}{q}}, & 1 \leq q < \infty \\ \sup\limits_{s>0} s[\omega(s)]^{\frac{1}{p}}, & q = \infty \end{cases}$$

is finite.
For further details about the Lorentz space, see e.g. [14] or [17].

3 The Oscillation Lemma

The aim of this section is to establish an oscillation estimate for weakly monotone functions in the Lorentz space $L^{n-1,1}$.
The elements in the Sobolev space $W^{1,1}(\Omega)$ are equivalence classes of functions which agree almost everywhere in Ω. In order to study the fine properties of a function $u \in W^{1,1}(\Omega)$, it is convenient to use the representative $\tilde{u}$, defined by the formula

$$\tilde{u}(x) = \lim_{r \to 0} \sup \fint_{B(x,r)} u(z) dz.$$

It is well known that if $v : \mathbb{R}^n \to \mathbb{R}$ is locally integrable, then

$$\lim_{r \to 0} \fint_{B(x,r)} |v(x) - v(z)| dz = 0 \tag{5}$$

for almost all $x \in \mathbb{R}^n$. Hence $\tilde{u}$ is a Borel measurable representative of u.
The function $\tilde{u}$ is a natural representative of u, for example if we know that $v = u$ a.e. and v is a continuous function, then $v \equiv \tilde{u}$.
Now we can formulate our oscillation lemma.

Lemma 3.1 *Let $u \in W^{1,1}(B(a,2r))$ be a weakly monotone function such than $|\nabla u| \in L^{n-1,1}(B(a,2r))$. Then*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C(n) r^{-\frac{1}{n-1}} \; ||\nabla u||_{L^{n,1}(B(a,2r))}$$

for all $x, y \in B(a,r)$.

If $|\nabla u| \in L^{n,1}(B(a,2r))$, then we have the following oscillation lemma. This Lemma holds without the assumption that u be weakly monotone.

Lemma 3.2 *Let $u \in W^{1,1}(B(x,R))$ be a function so that $|\nabla u| \in L^{n,1}(B(x,R))$. Then there exists a constant $C(n)$ such that*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq C(n)\ ||\nabla u||_{L^{n,1}(B(x,|x-y|))} \tag{6}$$

for all $y \in B(x,R)$.

Proof. Fix $y \in B(x,R)$ and denote $r_i = \left(\frac{|x-y|}{4}\right) 2^{-|i|}$ for all $i \in \mathbb{Z}$.
Let $x_0 = \frac{x+y}{2}$ and define recursively

$$x_i = x_{i-1} + r_{i-1}\frac{x-y}{|x-y|} \qquad \text{for } i > 0$$

and

$$x_i = x_{i+1} - r_{i+1}\frac{x-y}{|x-y|} \qquad \text{for } i < 0.$$

Then the balls $B_i = B(x_i, r_i) \subset B(x, |x-y|)$ and the points x_i satisfy

$$\begin{array}{rl}
(i) & B(x_i,r_i) \subset B(x_{i+1}, 4r_{i+1}) \quad \text{when } i \geq 0, \\
 & B(x_i,r_i) \subset B(x_{i-1}, 4r_{i-1}) \quad \text{when } i \leq 0, \\
(ii) & \sum_i \chi_{B_i} \leq 2, \\
(iii) & x_i \to x \quad \text{as } i \to \infty, \\
 & x_i \to y \quad \text{as } i \to -\infty, \\
(iv) & |x_i - x| \leq 2r_i \quad \text{when } i \geq 0 \text{ and} \\
 & |x_i - y| \leq 2r_i \quad \text{when } i \leq 0.
\end{array}$$

From properties (iii) and (iv) it follows that $u_{B_i} \to \tilde{u}(x)$ as $i \to \infty$ and $u_{B_i} \to \tilde{u}(y)$ as $i \to -\infty$.
Because

$$\tilde{u}(x) = u_{B_0} - \sum_{i=0}^{\infty}(u_{B_i} - u_{B_i+1})$$

and

$$\tilde{u}(y) = u_{B_0} - \sum_{i=0}^{-\infty}(u_{B_i} - u_{B_i-1})$$

we have

$$|\tilde{u}(x) - \tilde{u}(y)| \leq \sum_{i=0}^{\infty} |u_{B_i} - u_{B_{i+1}}| + \sum_{i=0}^{-\infty} |u_{B_i} - u_{B_{i-1}}|.$$

Applying the Poincaré inequality we obtain

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| &\leq C(n) \sum_{i=-\infty}^{\infty} r_i^{1-n} \int_{B_i} |\nabla u| \; dz \\ &= C(n) \sum_{i=-\infty}^{\infty} r_i^{1-n} \int_{\{(z,t)\in B_i \times (0,\nabla(z))\}} d(z,t) \end{aligned} \tag{7}$$

Let ω be the distribution function of $|\nabla u|$ i.e. $\omega(t) = |\{y \in B(x, |x-y|) : |\nabla u(y)| > t\}|$ for all $t > 0$. Next we split the set $\{(z,t) \in B_i \times (0, \nabla(z))\} =: E_i$ into $\{(z,t) \in E_i : \omega(t) \geq r_i^n\}$ and $\{(z,t) \in E_i : \omega(t) < r_i^n\}$.
For $i \geq 0$ from properies (ii) and (iv) it follows that

$$\begin{aligned} \sum_{i=0}^{\infty} \int_{E_i \cap \{\omega(t) \geq r_i^n\}} r_i^{1-n} \; d(z,t) &\leq 3^{n-1} \sum_{i=0}^{\infty} \int_{E_i \cap \{\omega(t) \geq 3^n |z-x|^n\}} |z-x|^{1-n} \; d(z,t) \\ &\leq 2 \cdot 3^{n-1} \int_{\cup_{i=0}^{\infty} E_i \cap \{3[\omega(t)]^{\frac{1}{n}} \geq |z-x|\}} |z-x|^{1-n} \; d(z,t) \\ &\leq 2 \cdot 3^{n-1} \int_0^{\infty} \int_{B(x, 3[\omega(t)]^{\frac{1}{n}})} |z-x|^{n-1} dz dt \\ &\leq C(n) \int_0^{\infty} [\omega(t)]^{\frac{1}{n}} dt. \end{aligned} \tag{8}$$

Similarly for $i \leq 0$ we have that

$$\sum_{i=-\infty}^{0} \int_{E_i \cap \{\omega(t) \geq r_i^n\}} r_i^{1-n} \; d(z,t) \leq C(n) \int_0^{\infty} [\omega(t)]^{\frac{1}{n}} dt. \tag{9}$$

For the $\{(z,t) \in E_i : \omega(t) < r_i^n\}$-part we have that

$$\begin{aligned} \sum_{i=-\infty}^{\infty} \int_{E_i \cap \{\omega(t) < r_i^n\}} r_i^{1-n} \; d(z,t) &\leq \int_{\cup_{i=-\infty}^{\infty} E_i} [\omega(t)]^{\frac{1-n}{n}} d(z,t) \\ &\leq \int_{B(x,|x-y|)} |\nabla(z)| [\omega(|\nabla(z)|)]^{\frac{1-n}{n}} dz \\ &\leq \int_0^{|B(x,|x-y|)|} |\nabla(t)|^* t^{\frac{1-n}{n}} dt. \end{aligned} \tag{10}$$

The last inequality follows from (3) and (4).
We obtain from (7), (8), (9) and (10) the desired inequality (6).

We will deduce Lemma 3.1 from Lemma 3.2. For this we will employ an approximation argument. It is well known that we can approximate Sobolev functions by smooth functions. If a Sobolev function is weakly monotone then it is surprising that the approximations can be in fact chosen to be "almost" monotone. Let us give a precise statement of the result that is due to Iwaniec, Koskela and Onninen [6].

Lemma 3.3 *Let $u \in W^{1,1}(B(a,R))$ be a weakly monotone function and $r < R$. Suppose that $\delta > 0$. Then there exists a number $j_0 \in \mathbb{N}$ and a sequence $\{u_j\}_{j=1}^{\infty} \subset C^{\infty}(\Omega)$ such that*

$$\begin{aligned} u_j &\to u \qquad \textit{in } \ W^{1,1}_{loc}(B(a,R)) \quad \textit{and} \\ u_j(x) &\to \tilde{u}(x) \end{aligned}$$

for all $x \in B(a,R)$.
Furthermore for all Lebesgue points $x_0, y_0 \in B(a,r)$ it holds that

$$|u_j(x_0) - u_j(y_0)| \leq \operatorname{osc}(u_j, S^{n-1}(a,t)) + 2\delta \tag{11}$$

for all $j \geq j_0$ and every $r \leq t \leq R$.

Now we are ready to prove lemma 3.1.

Proof of Lemma 3.1 Fix $t \in (r, 2r)$. Applying Lemma 3.2 on spheres for the function $u_j \in C^{\infty}(B(a,2r))$ at (11) we have that

$$|u_j(x) - u_j(y)| \leq C(n)||\nabla u_j||_{L^{n-1,1}S^{n-1}(a,t)} + 2\delta$$

for all $t \in (r, 2r)$.
Integrating over the interval $r < t < 2r$ and dividing both sides by r, we

obtain

$$
\begin{aligned}
|u_j(x)-u_j(y)| &\leq \frac{C(n)}{r}\int_r^{2r}\int_0^\infty [\mathcal{H}^{n-1}(\{x\in S^{n-1}(a,t) : |\nabla u_j| > s\})]^{\frac{1}{n-1}}\ ds\ dt \\
&\quad +2\delta \\
&= \frac{C(n)}{r}\int_0^\infty\int_r^{2r}\left(\int_{S^{n-1}(a,t)}\chi_{\{|\nabla u_j|>s\}}(y)\ d\mathcal{H}^{n-1}(y)\right)^{\frac{1}{n-1}}\ ds\ dt \\
&\quad +2\delta
\end{aligned}
$$

Then, applying Hölder's inequality, we obtain

$$
\begin{aligned}
|u_j(x)-u_j(y)| &\leq \frac{C(n)}{r}\int_0^\infty r^{\frac{n-2}{n-1}}\left(\int_r^{2r}\int_{S^{n-1}(a,t)}\chi_{\{|\nabla u_j|>s\}}(y)d\mathcal{H}^{n-1}(y)dt\right)^{\frac{1}{n-1}}ds \\
&\quad +2\delta \\
&\leq C(n)\ r^{-\frac{1}{n-1}}\int_0^\infty\left(\int_{B(a,2r)}\chi_{\{|\nabla u_j|>s\}}(y)\ dy\right)^{\frac{1}{n-1}}\ ds+2\delta \\
&= C(n)\ r^{-\frac{1}{n-1}}\ ||\nabla u_j||_{L^{n-1,1}(B(a,2r))}+2\delta
\end{aligned}
$$

By letting $j\to\infty$ and $\delta\to 0$ we conclude that

$$
|\tilde{u}(x)-\tilde{u}(y)|\leq C(n)\ r^{-\frac{1}{n-1}}\ ||\nabla u||_{L^{n-1,1}(B(a,2r))}
$$

as desired.

4 Proof of Theorem 1.2

In the proof we will use the following lemma that goes back to Stepanov [15] (see also [11]).

Lemma 4.1 *A function $u:\Omega\to\mathbb{R}$ is differentiable a.e. if and only if*

$$
\limsup_{y\to x}\frac{|u(x)-u(y)|}{|x-y|}<\infty \qquad a.e.
$$

Now (i) of Theorem 1.2 follows from Lemma 3.1 and Lemma 4.1 if

$$
\limsup_{r\to 0} r^{-\frac{n}{n-1}}||\nabla u||_{L^{n-1,1}(B(a,r))}
$$

is finite for almost every $a \in \Omega$.
The following lemma is due to Stein [13]. It is a consequence of the standard estimate for the Hardy-Littlewood maximal function: If $g \in L^1(\mathbb{R}^n)$, then

$$|\{x \in \mathbb{R}^n : \sup_{r>0} \fint_{B(x,r)} |g(y)|dy > t\}| \leq \frac{C(n)}{t} \int_{\mathbb{R}^n} |g(x)|dx. \qquad (12)$$

Lemma 4.2 *Suppose that $f \in L^{p,1}(\Omega)$, $1 < p < \infty$. Then*

$$\limsup_{r \to 0} r^{-\frac{n}{p}} ||f||_{L^{p,1}(B(x,r))} < \infty$$

for a.e. $x \in \Omega$

For the convenience of the reader we give a short proof below; Stein's proof in [13] consists of a list of references to various estimates.

Proof. Write

$$M_p(f)(x) = \sup_{r>0} r^{-\frac{n}{p}} ||f||_{L^{p,1}(B(x,r))}$$

for $x \in \Omega$.
We will show that

$$||M_p(f)||_{L^{p,\infty}(\Omega)} \leq C||f||_{L^{p,1}(\Omega)}. \qquad (13)$$

If we take inequality (13) as known, then

$$|\{x \in \Omega : \limsup_{r \to 0} r^{-\frac{n}{p}} ||f||_{L^{p,1}(B(x,r))} = \infty\}| \leq \limsup_{k \to \infty} \frac{C||f||^p_{L^{p,1}(\Omega)}}{k^p} = 0$$

as desired.
We observe that it suffices to establish inequality (13) when $f = \sum_{i=1}^h c_i \chi_{A_i}$ is a non-negative simple function (see [17, proposition 1.8.4]). We can assume that $c_1 > c_2 > ... > c_h$, $A_i \cap A_j = \emptyset$, when $i \neq j$ and $|A_i| < \infty$ for all $i \in \{1, ...h\}$.
Set $c_{h+1} = 0$. Then

$$\begin{aligned} ||f||_{L^{p,1}(\Omega)} &= \sum_{i=1}^h \int_{c_{i+1}}^{c_i} |\{x \in \Omega : f(x) > t\}|^{\frac{1}{p}} dt \\ &= \sum_{i=1}^h (c_i - c_{i+1}) |\cup_{j=1}^i A_j|^{\frac{1}{p}}. \qquad (14) \end{aligned}$$

Because $f=\sum_{i=1}^{h} c_i\chi_{A_i}=\sum_{i=1}^{h}(c_i-c_{i+1})\chi_{\cup_{j=1}^{i}A_j}$ we obtain

$$\begin{aligned} ||M_p(f)||_{L^{p,\infty}(\Omega)} &\leq \sum_{i=1}^{h}(c_i-c_{i+1})||M_p(\chi_{\cup_{j=1}^{i}A_j})||_{L^{p,\infty}(\Omega)} \\ &\leq C\sum_{i=1}^{h}(c_i-c_{i+1})|\cup_{j=1}^{i}A_j|^{\frac{1}{p}}. \end{aligned} \tag{15}$$

The last inequality follows from

$$M_p(\chi_A)(x)=\sup_{r>0} r^{-\frac{n}{p}}|A\cap B(x,r)|^{\frac{1}{p}}=(\sup_{r>0} r^{-n}|A\cap B(x,r)|^{n})^{\frac{1}{p}}$$

for arbitrary A of finite measure and inequality (12).
Now inequality (13) follows from (14) and (15).

(ii) Suppose that $q>1$. Then there exists a radial function $f\in W^{1,1}(B^{n-1}(0,1))$ such that $|\nabla f|\in L^{n-1,q}(B^{n-1}(0,1))$ and $\lim_{x\to 0} f(x)=\infty$. The existence of such a function follows from [3] or [7].
The set $\{(x_1,...,x_{n-1})\in B^{n-1}(0,\frac{1}{2}) : x_i\in\mathbb{Q}, i=1,...,n-1\}$ is dense in $B^{n-1}(0,\frac{1}{2})$. Denote those points $q_1,q_2,q_3,...$ and define $g_j(x)=2^{-j}f(x-q_j)$.
Set

$$h_k(x)=\sum_{j=1}^{k} g_j(x)$$

for all $x\in B^{n-1}(0,\frac{1}{2})$.
Because the space $W^{1,1}(B^{n-1}(0,\frac{1}{2}))$ is a Banach space and for $j<k$

$$\begin{aligned} ||h_k-h_j||_{W^{1,1}(B^{n-1}(0,\frac{1}{2}))} &= ||\sum_{i=j+1}^{k} g_i||_{W^{1,1}(B^{n-1}(0,\frac{1}{2}))} \\ &\leq \sum_{i=j+1}^{k}||g_i||_{W^{1,1}(B^{n-1}(0,\frac{1}{2}))} \\ &\leq ||f||_{W^{1,1}(B^{n-1}(0,1))}\sum_{i=j+1}^{k}2^{-i}, \end{aligned}$$

we have that the sequence $(h_j)_j$ converges to some h in $W^{1,1}(B^{n-1}(0,\frac{1}{2}))$. Furthermore $L^{n-1,q}(B^{n-1}(0,\frac{1}{2}))$ is a Banach space [14, chapter v, theorem 3.22.]

and thus the same argument as above shows that $|\nabla h| \in L^{n-1,q}(B^{n-1}(0, \frac{1}{2}))$. Since $h_j \to h$ in $L^1(B^{n-1}(0, \frac{1}{2}))$, we may choose a subsequence denoted again by h_j, so that $h_j(x) \to h(x)$ a.e. Thus h is finite almost everywhere but also infinite on a dense set, and so h is everywhere discontinuous. It follows from the construction that also every function that coincides with h a.e. is everywhere discontinuous.

Finally set

$$v(x_1, ..., x_n) = h(x_1, ..., x_{n-1})$$

for all $x \in B^n(0, \frac{1}{2})$.

It follows that the function v is weakly monotone and non-differentiable (even discontinuous) everywhere in $B^n(0, \frac{1}{2})$. The proof is complete.

In 1981, Stein [13] proved that if $f \in W^{1,1}(\Omega)$ and $|\nabla f| \in L^{n,1}$ then f is differentiable a.e. We can give an alternate proof of the first part of Theorem 1.2 based on this fact on spheres and some tools from nonlinear potential theory, along the lines of the argument in [16]. This approach is however technical and involves the notion of trace. Notice that our function u need not be continuous.

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