Mappings of finite distortion: Discreteness and openness

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1 Introduction

This paper is a crucial part of our program to establish the fundamentals of the theory of mappings of finite distortion [11], [1], [12], [17] which form a natural generalization of the class of quasiregular mappings, also called mappings of bounded distortion. The results of this paper give sharp criteria for topological properties, such as openness, for a mapping of finite distortion. The theory of mappings of bounded distortion is by now well understood, see the monographs by Yu. G. Reshetnyak [26], by S. Rickman [27] and by T. Iwaniec and G. Martin [13]. The motivation for relaxing the boundedness of the distortion partially arises from the fundamental works of J. Ball [2], [3] on nonlinear elasticity. We study mappings $f = (f_1, ..., f_n) : \Omega \to \mathbb{R}^n$ in the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^n)$, where Ω is a connected, open subset of $\mathbb{R}^n$ with $n \geq 2$. Thus, for almost every $x \in \Omega$, we can speak of the linear transformation $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$, called differential of f at x. Its norm is defined by $|Df(x)| = \sup\{|Df(x)h| : h \in S^{n-1}\}$. We shall often identify $Df(x)$ with its matrix, and denote by $J(x, f) = J_f(x) = \det Df(x)$ the Jacobian determinant.

Definition 1.1. A Sobolev mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ is said to have *finite distortion* if there is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that

$$|Df(x)|^n \leq K(x)J(x, f) \qquad \text{a.e.} \tag{1.1}$$

We call (1.1) the distortion inequality for f. Notice that, unless we put any extra conditions on K, we only require that $J(x, f) \geq 0$ a.e. and that the differential Df vanishes a.e. in the zero set of the Jacobian determinant

$J(x,f)$. It is worth recalling that the smallest such function K, referred to as *outer dilatation*, is then defined by

$$K_O(x,f) = \begin{cases} \frac{|Df(x)|^n}{J(x,f)} & \text{if } J(x,f) \neq 0, \\ 1 & \text{if } J(x,f) = 0. \end{cases} \tag{1.2}$$

Geometrically this means that, at the points where $J(x,f) > 0$, the differential takes the unit ball to an ellipsoid E and we have $K_O(x,f) = \text{vol } B_O \,/\, \text{vol } E$, where B_O is the ball circumscribed about E.

Let us begin by recalling some of the known results on mappings of finite distortion which are relevant for our discussion. A mapping in the Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$ with finite distortion $K \in L^\infty(\Omega)$ is called a quasiregular mapping or a mapping of bounded distortion. This class of mappings can be traced back to the work of Yu. G. Reshetnyak [25]. Reshetnyak proves the remarkable result that a mapping of bounded distortion is continuous and either constant or open and discrete. For an exposition of the theory of mappings of bounded distortion we refer the reader to the monographs by Yu. G. Reshetnyak [26], by S. Rickman [27] and by T. Iwaniec and G. Martin [13]. Here continuity means that f has a continuous representative. Openness of a continuous mapping f requires that the image of each open set be open and the discreteness that the preimage of any point in $\mathbb{R}^n$ be an isolated set of points in Ω. Thus Reshetnyak's result gives topological conclusions from analytic assumptions.

V. Gol'dstein and S. K. Vodop'yanov showed later in [8] that even mappings of finite distortion in the Sobolev class $W^{1,n}(\Omega, \mathbb{R}^n)$ are continuous. Regarding discreteness and openness, the uniform boundedness of the distortion in the planar case was relaxed to the (local) integrability of the distortion for Sobolev mappings $f \in W^{1,2}(\Omega, \mathbb{R}^2)$ by T. Iwaniec and V. Šverák [16]. In higher dimensions, the analog of this holds when $K_O \in L^p(\Omega)$ for some $p > n-1$ and $f \in W^{1,n}(\Omega, \mathbb{R}^n)$. It fails if $p < n-1$, [3], and it remains unknown in the critical case of $p = n-1$. For the positive results see the papers [10] by J. Heinonen and P. Koskela and [19], [20] by J. Manfredi and E. Villamor. Notice that in all these results we assume that the partial derivatives of f are n-integrable.

The natural Sobolev setting for mappings of finite distortion is in the space $W^{1,n}(\Omega, \mathbb{R}^n)$, largely due to the wish to integrate the Jacobian determinant by parts. However, matters are quite complicated if one does not know a priori that the Jacobian is locally integrable or, even if so, whether it coincides with the so-called distributional Jacobian. The first regularity results below the natural setting were recently established by T. Iwaniec, P. Koskela and G. Martin [11]. Assuming that $J_f \in L^1(\Omega)$ and $e^{\lambda K} \in L^1(\Omega)$ for some sufficiently large $\lambda = \lambda(n)$ they proved, among other things, that in fact $f \in W^{1,n}(\Omega, \mathbb{R}^n)$. It then follows that f is continuous and either constant or open and discrete. Also see [1] for further developments. The standing conjecture is that one can take $\lambda = \lambda(n) = 1$ as the critical exponent for the regularity conclusions; it is known that the L^n-integrability of the differential fails for any $\lambda < 1$. The relevant examples are homeomorphic maps in $W^{1,1}(\Omega, \mathbb{R}^n)$ and, therefore, have locally integrable Jacobian determinants.

Very recently, T. Iwaniec, P. Koskela and J. Onninen (c.f. [12]) verified that mappings with exponentially integrable distortion and integrable Jacobian determinant are always continuous.

Theorem 1.2. *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy the distortion inequality*

$$|Df(x)|^n \leq K(x)J(x,f) \qquad a.e.$$

in Ω, where $K \geq 1$ and $\exp(\lambda K)$ is integrable for some $\lambda > 0$. If the Jacobian determinant of f is integrable, then f is continuous.

One consequence of our current work is that we also have discreteness and openness for non-constant mappings as above.

Theorem 1.3. *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy the distortion inequality*

$$|Df(x)|^n \leq K(x)J(x,f) \qquad a.e.$$

in Ω, where $K \geq 1$ and $\exp(\lambda K)$ is integrable for some $\lambda > 0$. If the Jacobian determinant of f is integrable, then f is continuous and either constant or both discrete and open. Conversely, there is a non-constant, continuous mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with integrable Jacobian determinant and of distortion K with $\exp(\lambda K/\log^2(1+K))$ integrable for some λ that is neither open nor discrete.

We will obtain Theorem 1.3 as a corollary to our more general results. Theorem 1.3 is new even in the plane; see the work of G. David [5] for existence questions. For notational simplicity we do not formulate our results here in the ultimate generality. See Sections 2 and 3 for even sharper results.

Theorem 1.4. *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy*

$$\lim_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx = 0. \tag{1.3}$$

If f has finite distortion $K \in L^p(\Omega)$ for some $p > n-1$, then f is continuous and either constant or both discrete and open. Conversely, there is a continuous, non-constant $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with integrable Jacobian, of finite distortion K with $\exp(\lambda K/\log^2(1+K))$ integrable for some λ, with the Sobolev regularity

$$\limsup_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx < \infty$$

and so that f is neither open nor discrete.

Above, the assumptions in the first part of Theorem 1.4 guarantee that the Jacobian determinant of f is (locally) integrable and that, in fact, the point-wise Jacobian coincides with the so-called distributional Jacobian. This fact plays a fundamental role in the proof. The continuity of f in Theorem 1.4 is from [12].

A reader familiar with discrete and open mappings recognizes by Theorem 1.4 that a mapping f satisfying our assumptions has to be sense-preserving, that is, the topological degree is always strictly positive. This is indeed part of our argument of proof for Theorem 1.4

Theorem 1.5. *Let $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ satisfy*

$$\lim_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx = 0.$$

If f has finite distortion, then f is continuous and sense-preserving. Conversely, there is a continuous $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with integrable Jacobian, with $J(x, f) > 0$ a.e., of finite distortion K with $\exp(\lambda K/\log^2(1+K))$ integrable for some λ, with

$$\limsup_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon}\, dx < \infty,$$

and so that f is not sense-preserving.

Theorem 1.5 gives very sharp criteria for one to conclude from analytic assumptions that a mapping is sense-preserving. Observe from the example giving the sharpness that the sign of the Jacobian determinant need not have any global topological meaning, even for mappings with partial derivatives in weak–L^n.

Our proofs are based on the following ingredients. First of all, our assumptions guarantee that the Jacobian of f is (locally) integrable and that the point-wise Jacobian coincides with the distributional Jacobian. This does not only hold for f but also for certain modifications to f. Using this we show that f preserves the divergence of smooth vector fields in a certain distributional sense. This then results in a (weak) change of variables formula that allows us to conclude that f is sense-preserving. Here we wish to acknowledge the important contributions of T. Iwaniec and C. Sbordone [15], V. Šverák [28], and of S. Müller, Qi Tang and B. S. Yan [23] that gave us crucial ideas. The rest of the proof of discreteness and openness follows ideas of J. Manfredi and E. Villamor [19], [20] that are a refinement of the original argument of Yu. G. Reshetnyak [25]. Also see the work of S. K. Vodop'yanov [30]. The example to show sharpness is based on ideas in a construction by T. Iwaniec and G. Martin [14] and in the modification of this construction by J. Malý [18]. We however need to substantially improve on these previous examples.

In the next part, [17], of our program on mappings of finite distortion, we will study the distortion of sets of measure zero under mappings of finite distortion.

The paper is organized as follows: theorems giving sufficient conditions for a mapping to be sense-preserving are proven in Section 2 and the results concerning discreteness and openness in Section 3. In Section 4 we construct a mapping that shows that our results are sharp in the above mentioned sense.

2 Sense-preserving mappings

We consider a function space $X(\Omega)$ such that if $g, h : \Omega \to [0, \infty]$ are measurable, $h \in X(\Omega)$ and $g \le ch$ for some $0 < c < \infty$, then also $g \in X(\Omega)$. Furthermore, we assume that if $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, $|Df| \in X(\Omega)$ and $J_f \ge 0$ a.e., then $J_f \in L^1(\Omega)$ and the distributional Jacobian $\operatorname{Det} Df$ equals J_f in Ω. This means that

$$\int_\Omega f_1(x) J(x, (\varphi, f_2, \dots, f_n))\, dx = -\int_\Omega \varphi(x) J(x, f)\, dx$$

for all $\varphi \in C_c^\infty(\Omega)$. It suffices, for example, to require that (1.3) holds (see [9, Corollary 4.1]). Note that then $|Df|$ lies in a space that is between $L^n \log^{-1} L$ and $\cap_{\alpha<-1} L^n \log^\alpha L$ (see e.g. [12, Section 2]).

We call $f : \Omega \to \mathbb{R}^n$ *sense-preserving* if $\deg(f, \Omega', y_0) > 0$ for all domains $\Omega' \subset\subset \Omega$ and all $y_0 \in f(\Omega') \setminus f(\partial\Omega')$, where $\deg(f, \Omega', y_0)$ is the topological degree of f at y_0 with respect to Ω'. For the definition of the topological degree see e.g. [7].

If A is a real $n \times n$ matrix, we denote the cofactor matrix of A by $\operatorname{cof} A$. Then the entries of $\operatorname{cof} A$ are $a_{ij} = (-1)^{i+j} \det A_{ij}$, where A_{ij} is the ijth minor of A, and $\operatorname{cof} A$ is the transpose of the adjungate $\operatorname{adj} A$ of A.

Theorem 2.1. *Suppose that $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ is continuous, $|Df| \in X(\Omega)$ and $J_f \ge 0$ a.e. and let $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then*

$$\operatorname{div}\big((V \circ f) \operatorname{cof} Df\big) = \big((\operatorname{div} V) \circ f\big) J_f \tag{2.1}$$

holds in the sense of distributions in Ω, i.e.

$$\int_\Omega \langle V(f(x)) \operatorname{cof} Df(x),\, \nabla\varphi(x)\rangle\, dx = -\int_\Omega (\operatorname{div} V)(f(x)) J_f(x) \varphi(x)\, dx \tag{2.2}$$

for all $\varphi \in C_c^\infty(\Omega)$.

Proof. It suffices to show that (2.1) holds for any $\Omega' \subset\subset \Omega$.

Consider first the case $V = (v, 0, \dots, 0)$, where $v \in C^1(\mathbb{R}^n)$. Since a general C^1-function can be written, on a bounded set, as a difference of two C^1-functions whose first partial derivatives are nonnegative (take e.g. $v_+(x) = v(x) + \sup\{|\partial_1 v(x)| : x \in f(\Omega')\}x_1$ and $v_- = v_+ - v$), we may, by linearity of (2.1) with respect to V, assume that $\partial_1 v \geq 0$ on $f(\Omega')$. Denote $g = (v \circ f, f_2, \dots, f_n)$. Then $g \in W^{1,1}(\Omega, \mathbb{R}^n)$, $|Dg| \in X(\Omega)$ and $J_g(x) = \partial_1 v(f(x)) J_f(x) \geq 0$ a.e. $x \in \Omega'$, whence $J_g \in L^1(\Omega')$ and $\operatorname{Det} Dg = J_g$ in Ω'. Now, for any $\varphi \in C_c^\infty(\Omega')$, we have

$$\begin{aligned}\int_{\Omega'} \langle V(f(x)) \operatorname{cof} Df(x), \nabla\varphi(x)\rangle \, dx &= \int_{\Omega'} g_1(x) J(x, (\varphi, g_2, \dots, g_n)) \, dx \\ &= -\int_{\Omega'} \varphi(x) J(x, g) \, dx \\ &= -\int_{\Omega'} (\operatorname{div} V)(f(x)) J_f(x) \varphi(x) \, dx\end{aligned}$$

which means that (2.1) holds for $V = (v, 0, \dots, 0)$ in Ω'.

A similar argument also applies to $V = (0, \dots, v, \dots, 0)$, and the general case follows by the coordinate decomposition of V. $\square$

Theorem 2.2. *Let Ω be bounded and suppose that $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f \in C(\overline{\Omega}, \mathbb{R}^n) \cap W^{1,n-1}(\Omega, \mathbb{R}^n)$ and $f(\partial\Omega) \cap \operatorname{spt} \operatorname{div} V = \emptyset$. Then there is $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ in $\operatorname{spt}((\operatorname{div} V) \circ f)$ and*

$$-\int_{\Omega} \langle V(f(x)) \operatorname{cof} Df(x), \nabla\varphi(x)\rangle \, dx = \int_{\mathbb{R}^n} \operatorname{div} V(y) \deg(f, \Omega, y) \, dy. \tag{2.3}$$

Proof. To choose φ, take an open set $U' \subset \mathbb{R}^n \setminus f(\partial\Omega)$ such that $\operatorname{spt} \operatorname{div} V \subset U'$ and $\overline{U'} \cap f(\partial\Omega) = \emptyset$. Then $U := f^{-1}(U') \subset\subset \Omega$ is open and contains $\operatorname{spt}((\operatorname{div} V) \circ f)$. Now choose $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ in U.

If f is smooth, then the classical degree theory yields that (see e.g. [7, Exercise 1.5])

$$\int_{\Omega} (\operatorname{div} V)(f(x)) J_f(x) \, dx = \int_{\mathbb{R}^n} \operatorname{div} V(y) \deg(f, \Omega, y) \, dy. \tag{2.4}$$

Since (2.2) holds for smooth mappings, it remains to use the assumption that $\varphi = 1$ on the set where $(\operatorname{div} V)(f(x)) \neq 0$ to conclude that (2.3) holds for all smooth f.

In the general case, we find a sequence (f_j) of smooth mollifications of f that converges to f both uniformly in $\overline{\Omega}$ and in $W^{1,n-1}(G)$, where $G \subset\subset \Omega$ is an open set containing $\operatorname{spt} \varphi$. By uniform convergence and by the choice of U, we have for large j that $f_j(\partial\Omega) \cap \operatorname{spt} \operatorname{div} V = \emptyset$, $\varphi = 1$ on the set where $(\operatorname{div} V)(f_j(x)) \neq 0$, and $\deg(f_j, \Omega, y) = \deg(f, \Omega, y)$ for all $y \in \operatorname{spt} \operatorname{div} V$. The claim follows now from by applying (2.3) to the mappings f_j and letting j tend to infinity. $\square$

Theorem 2.3. *Let Ω be bounded and suppose that f belongs to $C(\overline{\Omega}, \mathbb{R}^n) \cap W^{1,n-1}(\Omega, \mathbb{R}^n)$, $J_f \in L^1(\Omega)$ and that (2.1) holds in the sense of distributions for each $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. If $\eta \in C_c^2(\mathbb{R}^n)$ is such that $f(\partial\Omega) \cap \operatorname{spt} \eta = \emptyset$, then*

$$\int_\Omega \eta(f(x)) J_f(x)\, dx = \int_{\mathbb{R}^n} \eta(y) \deg(f, \Omega, y)\, dy. \tag{2.5}$$

Proof. Let $u \in C^2(\mathbb{R}^n)$ be a solution of Poisson's equation $\triangle u = \eta$, that is, $\operatorname{div} \nabla u = \eta$, and denote $V = \nabla u$. Now the claim follows from (2.1) and Theorem 2.2. □

Theorem 2.4. *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be continuous. Suppose that $J_f \in L^1(\Omega)$, $J_f \geq 0$ a.e. and that (2.1) holds in the sense of distributions for each $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. If f has finite distortion, then f is sense-preserving.*

Proof. Let $\Omega' \subset\subset \Omega$ and take $y_0 \in f(\Omega') \setminus f(\partial\Omega')$. We take an open ball B centered at y_0 such that $\overline{B} \cap f(\partial\Omega') = \emptyset$ and a nonnegative smooth function η with support in B such that $\eta(y_0) > 0$. Then by Theorem 2.3 and properties of degree (it is constant on B)

$$\deg(f, \Omega', y_0) \int_B \eta(y)\, dy = \int_{\Omega'} \eta(f(x)) J_f(x)\, dx. \tag{2.6}$$

Denote

$$G = \{x \in \Omega' : \ \eta(f(x)) > 0\}.$$

It follows from (2.6) that $\deg(f, \Omega', y_0) \geq 0$. Suppose that $\deg(f, \Omega', y_0) = 0$. Then $J_f = 0$ a.e. on G and since f has finite dilatation, it follows that $|Df| = 0$ a.e. on G. Hence f and thus also $\eta \circ f$ are locally constant on G. Since $\eta \circ f = 0$ on ∂G, we deduce that $\eta \circ f = 0$ on G. This contradiction shows that $\deg(f, \Omega', y_0) > 0$. □

Since, by [12, Theorem 1.3], a mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ of finite distortion satisfying (1.3) is continuous (i.e. has continuous representative), we obtain the first part of Theorem 1.5 as a corollary to Theorems 2.1 and 2.4.

According to [11, Section 7], a mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ with exponentially integrable dilatation and with $J_f \in L^1(\Omega)$ satisfies (1.3), whence we have the following corollary to Theorem 1.5.

Corollary 2.5. *Suppose that $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ has finite distortion K with*

$$\int_\Omega \exp\bigl(\lambda K(x)\bigr)\, dx < \infty$$

for some $\lambda > 0$ and assume that $J_f \in L^1(\Omega)$. Then f is continuous and sense-preserving.

Similarly, we get the first parts of Theorems 1.3 and 1.4 as corollaries to Theorem 3.1 below. The example of Section 4 gives the second parts of Theorems 1.3, 1.4, and 1.5.

3 Discreteness and openness

Theorem 3.1. *Let $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ be continuous. Suppose that $J_f \in L^1(\Omega)$, $J_f \geq 0$ a.e. and that (2.1) holds in the sense of distributions for each $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. If f has finite distortion $K \in L^p(\Omega)$ for some $p > n-1$, then f is either constant or both discrete and open.*

Proof. Suppose that f is not constant. By Theorem 2.4, f is sense-preserving. It suffices to prove that f is *light* (i.e. $f^{-1}(y)$ is totally disconnected for all $y \in \mathbb{R}^n$, that is, $f^{-1}(y)$ does not contain an arc) since a sense-preserving light mapping is both discrete and open (see [29]).

We will prove that for all $y \in \mathbb{R}^n$ there is $s \in (n-1, n)$ such that the s-capacity of $f^{-1}(y)$, denoted by $\operatorname{cap}_s f^{-1}(y)$, equals zero. This means that

$$\inf \left\{ \int_{\mathbb{R}^n} |\nabla g|^s \right\} = 0$$

where the infimum is taken over all test functions $g \geq 0$ such that $g \in L^{ns/(n-s)}(\mathbb{R}^n)$, $|\nabla g| \in L^s(\mathbb{R}^n)$ and $f^{-1}(y) \subset \operatorname{int}\{g \geq 1\}$ (for more about capacity see e.g. [6]). Then the Hausdorff dimension of $f^{-1}(y)$ is smaller than or equal to $n-s < 1$, and thus $f^{-1}(y)$ is totally disconnected. By considering the translation $f - y$ we may assume that $y = 0$, and that $0 \in f(\Omega)$. Since our argument is local in nature, we can assume that $f \in W^{1,n-1}(\Omega, \mathbb{R}^n)$, $K \in L^p(\Omega)$ and that $f(\Omega) \subset B(0, e^{-e}) = \Omega'$.

Suppose that Φ is a positive C^2-smooth n-superharmonic function (i.e. $\operatorname{div}(|\nabla\Phi|^{n-2}\nabla\Phi) \leq 0$) on the ball Ω' such that $\Phi \geq \delta > 0$ and such that

$$V = \frac{|\nabla\Phi|^{n-2}\nabla\Phi}{\Phi^{n-1}}$$

is in the class $C^1(\Omega', \mathbb{R}^n)$ with bounded partial derivatives. Since Φ is n-superharmonic, it follows that

$$\operatorname{div} V \leq (1-n)\frac{|\nabla\Phi|^n}{\Phi^n}.$$

Substituting this into (2.1) we obtain

$$\begin{aligned}
\operatorname{div}\left(\frac{|\nabla\Phi \circ f|^{n-2}\nabla\Phi \circ f}{(\Phi \circ f)^{n-1}} \operatorname{cof} Df\right) &= \operatorname{div}\big((V \circ f) \operatorname{cof} Df\big) \\
&= \big((\operatorname{div} V) \circ f\big) J_f \\
&\leq (1-n)\frac{|\nabla\Phi \circ f|^n}{(\Phi \circ f)^n} J_f
\end{aligned}$$

in the sense of distributions. Hence for any $\eta \in C_c^1(\Omega)$, $\eta \geq 0$, we have

$$\begin{aligned}
&n \int_\Omega \left\langle \frac{|(\nabla\Phi \circ f)(x)|^{n-2}(\nabla\Phi \circ f)(x)}{(\Phi \circ f)(x)^{n-1}} \operatorname{cof} Df(x),\ \eta(x)^{n-1}\nabla\eta(x) \right\rangle dx \\
&\geq (n-1) \int_\Omega \frac{|(\nabla\Phi \circ f)(x)|^n}{(\Phi \circ f)(x)^n} J_f(x)\eta(x)^n \, dx.
\end{aligned}$$

It follows that

$$\begin{aligned}&\int_\Omega \frac{|(\nabla\Phi\circ f)(x)|^n}{(\Phi\circ f)(x)^n}\, J_f(x)\eta(x)^n\,dx\\&\le \frac{n}{n-1}\int_\Omega \frac{|(\nabla\Phi\circ f)(x)|^{n-1}}{(\Phi\circ f)(x)^{n-1}}\,|\operatorname{cof} Df(x)|\eta(x)^{n-1}|\nabla\eta(x)|\,dx.\end{aligned}$$

Since

$$|\operatorname{cof} Df(x)| \le c(n)|Df(x)|^{n-1} \le c(n)\big(K(x)J_f(x)\big)^{(n-1)/n}$$

we have

$$\begin{aligned}&\int_\Omega \frac{|(\nabla\Phi\circ f)(x)|^n}{(\Phi\circ f)(x)^n}\, J_f(x)\eta(x)^n\,dx\\&\le c(n)\int_\Omega \bigg(\frac{|(\nabla\Phi\circ f)(x)|^n}{(\Phi\circ f)(x)^n}\, J_f(x)\eta(x)^n\bigg)^{(n-1)/n} K(x)^{(n-1)/n}|\nabla\eta(x)|\,dx.\end{aligned}$$

Here, and subsequently, $c(n)$ denotes a constant depending only on n which might differ from occurrence to occurrence. Using the Hölder inequality we obtain

$$\int_\Omega \frac{|(\nabla\Phi\circ f)(x)|^n}{(\Phi\circ f)(x)^n}\, J_f(x)\eta(x)^n\,dx \le c(n)\int_\Omega K(x)^{n-1}|\nabla\eta(x)|^n\,dx. \tag{3.1}$$

Now choose $s\in(n-1,n)$ such that $s/(n-s)\le p$. The Hölder inequality, chain rule, and equation (3.1) yield

$$\begin{aligned}&\int_\Omega |\nabla(\log\Phi\circ f)(x)|^s\eta(x)^s\,dx\\&\le \bigg(\int_\Omega |\nabla(\log\Phi\circ f)(x)|^n\,\frac{\eta(x)^n}{K(x)}\,dx\bigg)^{s/n}\bigg(\int_\Omega K(x)^{s/(n-s)}\,dx\bigg)^{(n-s)/n}\\&\le \bigg(\int_\Omega \frac{|(\nabla\Phi\circ f)(x)|^n}{(\Phi\circ f)(x)^n}\,\frac{|Df(x)|^n}{K(x)}\,\eta(x)^n\,dx\bigg)^{s/n}\bigg(\int_\Omega K(x)^{s/(n-s)}\,dx\bigg)^{(n-s)/n}\\&\le c(n)\bigg(\int_\Omega K(x)^{n-1}|\nabla\eta(x)|^n\,dx\bigg)^{s/n}\bigg(\int_\Omega K(x)^{s/(n-s)}\,dx\bigg)^{(n-s)/n}.\end{aligned} \tag{3.2}$$

Next we will employ the family Φ_a of functions of [20] that approximates $\log(1/|x|)$ as $a\to 0$, and use the functions $\log\Phi_a\circ f$ as test functions for the s-capacity of $f^{-1}(0)$.

Lemma 3.2. *For each $0<a<e^{-e}$ there exists a function $\Phi_a:\Omega'\to\mathbb{R}$ with the following properties:*

(i) $\Phi_a\in C^2(\Omega')$,

(ii) $\Phi_a(y) \geq e$ *for every* $y \in \Omega'$,

(iii) Φ_a *is radial,*

(iv) $\Phi_a'(r) = \Phi_a'(|y|) \leq 0$,

(v) Φ_a *is* n*-superharmonic,*

(vi) $\log(1/a) \leq \Phi_a(y) \leq \log(1/a) + 1/2 + \log 2$ *for every* $|y| \leq a$,

(vii) $\Phi_a(y) = \log(1/|y|)$ *for* $a \leq |y| \leq e^{-e}$,

(viii) $|\nabla \Phi_a(y)|^{n-2} \nabla \Phi_a(y) \in C^1(\Omega', \mathbb{R}^n)$.

Since f is non-constant in Ω, there exists $\bar{x} \in \Omega$ such that $|f(\bar{x})| = 2b$ for some $b > 0$. Set $\Omega_b = f^{-1}(B(0,b))$. Then Ω_b is open and contains $f^{-1}(0)$. Let U be a component of Ω_b such that $U \cap f^{-1}(0) \neq \emptyset$ and K a compact subset of $U \cap f^{-1}(0)$. To show that $\operatorname{cap}_s f^{-1}(0) = 0$ it suffices to show that $\operatorname{cap}_s K = 0$.

Since $\bar{x} \notin U$, there exists $x_0 \in \partial U \cap \Omega \subset \partial \Omega_b \cap \Omega$. By continuity $|f(x_0)| = b$, and there exists $r_0 > 0$ such that $|f(x)| > b/2$ for every $x \in B(x_0, r_0)$. Choose a ball $B \subset\subset B(x_0, r_0) \cap U$. From the property (vii) it follows that for $x \in B$

$$\log \Phi_a(f(x)) = \log\log(1/|f(x)|) < \log\log(2/b) \tag{3.3}$$

whenever $a < b/2$. On the other hand, by property (vi), we have that for $x \in K$

$$\log \Phi_a(f(x)) = \log \Phi_a(0) \geq \log(1/a).$$

Choose $\eta \in C_c^\infty(U)$ such that $\eta \geq 0$ and $\eta \geq 1$ in K. Set

$$V_a(x) = \frac{\eta(x) \log \Phi_a(f(x))}{\log(1/a)}.$$

Now $V_a \geq 0$, $V_a \geq 1$ in K and V_a is absolutely continuous on almost all lines parallel to the coordinate axes with the gradient

$$\nabla V_a(x) = \frac{1}{\log(1/a)} \Big(\nabla(\log \Phi_a \circ f)(x) \eta(x) + (\log \Phi_a \circ f)(x) \nabla \eta(x) \Big).$$

Moreover, $V_a \in L^{ns/(n-s)}(\mathbb{R}^n)$ since V_a is continuous and has a compact support. In order to use the functions V_a as test functions to the s-capacity of K we have to show that the gradient of V_a is in $L^s(U)$ and estimate its $L^s(U)$-norm. Using (3.2) we obtain

$$\begin{aligned} \int_U |\nabla V_a(x)|^s \, dx \leq & \frac{c(n)}{\log(1/a)^s} \\ & \cdot \Bigg[\Big(\int_U K(x)^{n-1} |\nabla \eta(x)|^n \, dx \Big)^{s/n} \Big(\int_U K(x)^{s/(n-s)} \, dx \Big)^{(n-s)/n} \\ & \quad + \int_U |(\log \Phi_a \circ f)(x)|^s |\nabla \eta(x)|^s \, dx \Bigg]. \end{aligned} \tag{3.4}$$

The first term in the right hand side of (3.4) is bounded independently of a. The second term is finite, and we will show that it is also bounded independently of a. We will use the following standard consequence of the Poincaré inequality.

Lemma 3.3. *Let $G \subset \mathbb{R}^n$ be a domain with smooth boundary and $B \subset G$ an open ball. Then there exists a constant C such that for all $u \in W^{1,s}(G)$ we have*

$$\int_G |u(x)|^s\,dx \le C\left(\int_G |\nabla u(x)|^s\,dx + \int_B |u(x)|^s\,dx\right).$$

Applying Lemma 3.3 to a domain $G \subset\subset \Omega$ for which $\operatorname{spt}\eta \subset G$ and to the function $\log \Phi_a \circ f$ we get

$$\begin{aligned}
&\int_U |(\log \Phi_a \circ f)(x)|^s |\nabla \eta(x)|^s\,dx \le c(\eta, s)\int_G |(\log \Phi_a \circ f)(x)|^s\,dx \\
&\le c(\eta, s, C)\left(\int_G |\nabla(\log \Phi_a \circ f)(x)|^s\,dx + \int_B |(\log \Phi_a \circ f)(x)|^s\,dx\right).
\end{aligned} \tag{3.5}$$

By (3.3) the second term on the right hand side of (3.5) is bounded independently of $a < b/2$. This is also the case with the first term: Choose a nonnegative $\eta_G \in C_c^\infty(\Omega)$ for which $\eta_G|G \ge 1$. Then by (3.2) we have

$$\begin{aligned}
&\int_G |\nabla(\log \Phi_a \circ f)(x)|^s\,dx \le \int_\Omega |\nabla(\log \Phi_a \circ f)(x)|^s \eta_G(x)^s\,dx \\
&\le c(n)\left(\int_\Omega K(x)^{n-1}|\nabla \eta_G(x)|^n\,dx\right)^{s/n}\left(\int_\Omega K(x)^{s/(n-s)}\,dx\right)^{(n-s)/n} \\
&\le c(n, s, \eta_G)\left(\int_\Omega K(x)^{n-1}\,dx\right)^{s/n}\left(\int_\Omega K(x)^{s/(n-s)}\,dx\right)^{(n-s)/n}.
\end{aligned} \tag{3.6}$$

Thus also the second term in the right hand side of (3.4) is bounded independently of a, whence

$$\int_U |\nabla V_a(x)|^s\,dx \to 0$$

as $a \to 0$. This implies that $\operatorname{cap}_s K = 0$. $\square$

4 An example

We will construct a continuous mapping $f : Q_0 = [0,1]^n \to \mathbb{R}^n$, $n \ge 2$, which has the following properties:

(a) $f \in W^{1,1}(Q_0, \mathbb{R}^n)$, f is differentiable almost everywhere, and

$$\sup_{0<\epsilon<1} \epsilon \int_{Q_0} |Df(x)|^{n-\epsilon}\,dx < \infty. \tag{4.1}$$

(b) The Jacobian determinant $J_f(x)$ is strictly negative for almost every $x \in Q_0$, and

$$\int_{Q_0} |J_f(x)| \, dx < \infty. \tag{4.2}$$

(c) The dilatation $K(x) = \frac{|Df(x)|^n}{|J_f(x)|}$ is finite almost everywhere and there exists $\lambda > 0$ such that

$$\int_{Q_0} \exp\left(\frac{\lambda K(x)}{\log^2(1+K(x))}\right) dx < \infty. \tag{4.3}$$

(d) f does not satisfy Lusin's condition N: there is a set $N \subset Q_0$ of measure zero so that $f(N)$ has positive measure.

(e) f is neither open nor discrete.

(f) f fixes the boundary ∂Q_0 and thus $\deg(f, \partial Q_0, y) = 1$ for all $y \in \operatorname{int} Q_0$.

Let us next describe how to obtain a mapping as referred to in Theorems 1.3, 1.4, and 1.5 using f. Let $Q \subset \mathbb{R}^n$ be any cube with sides parallel to coordinate axes. By scaling, shifting and changing the sign of the first coordinate function of the mapping f, we get a continuous mapping $f_Q : Q \to \mathbb{R}^n$ for which $J_{f_Q} > 0$ a.e. in Q, (4.1), (4.2) and (4.3) hold and $f_Q|Q(x) = (-x_1, x_2, \dots, x_n)$ whence $\deg(f_Q, \partial Q, y) = -1$ for all $y \in f_Q(Q) \setminus f_Q(\partial Q)$.

Consider a finite collection $\mathcal{Q}$ of closed cubes Q with pairwise disjoint interiors and sides parallel to coordinate axes such that $\Omega \subset \cup_{Q \in \mathcal{Q}} Q$ and $Q' \subset \Omega$ for some $Q' \in \mathcal{Q}$. Define g to be f_Q in each $Q \in \mathcal{Q}$. Then $g : \Omega \to \mathbb{R}^n$ is a continuous mapping such that $J_g > 0$ a.e. in Ω, (4.1), (4.2) and (4.3) (and (d)) hold with f replaced by g and $\deg(g, \partial Q', y) = -1$ for all $y \in g(Q') \setminus g(\partial Q') \neq \emptyset$. Thus g is not sense-preserving. Moreover, by (e), g is neither open nor discrete.

We now move on to the construction of f after introducing some notations and stating a preliminary lemma. Besides the usual euclidean norm $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$ we will use the cubic norm $\|x\| = \max_i |x_i|$. Using the cubic norm, the x_0-centered closed cube with edge length $2r > 0$ and sides parallel to coordinate axes can be represented in the form

$$Q(x_0, r) = \{x \in \mathbb{R}^n : \ \|x - x_0\| \leq r\}$$

We then call r the radius of Q. Let us denote $cQ(x_0, r) = Q(x_0, cr)$ if $c > 0$. We will use the notation $a \lesssim b$ if there is a constant $c > 0$ (not depending on (integration) variables or summation indices) such that $a \le cb$, and we write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

We will be dealing with radial stretchings that map cubes $Q(0,r)$ onto cubes.

Lemma 4.1. *Let $\rho : (0,\infty) \to (0,\infty)$ be a strictly monotone, differentiable function. Then for the mapping*

$$f(x) = \frac{x}{\|x\|}\,\rho(\|x\|), \qquad x \neq 0$$

we have

$$|Df(x)|/c(n) \le \max\left\{\frac{\rho(\|x\|)}{\|x\|},\ |\rho'(\|x\|)|\right\} \le c(n)|Df(x)|$$

and

$$J_f(x)/c(n) \le \frac{\rho'(\|x\|)\rho(\|x\|)^{n-1}}{\|x\|^{n-1}} \le c(n)J_f(x)$$

where $c(n)$ depends only on n.

Proof. An elementary reasoning shows that for the mapping

$$g(x) = \frac{x}{|x|}\,\rho(|x|)$$

we have

$$|Dg(x)| = \max\left\{\frac{\rho(|x|)}{|x|},\ |\rho'(|x|)|\right\}$$

and

$$J_g(x) = \frac{\rho'(|x|)\rho(|x|)^{n-1}}{|x|^{n-1}}.$$

The Lemma follows by considering the decomposition $f = h^{-1} \circ g \circ h$, where

$$h(x) = \frac{\|x\|}{|x|}\,x$$

(i.e. h is the 'natural' stretching that maps each cube $Q(0,r)$ onto the ball $\overline{B}(0,r)$). □

In the following, we will construct a sequence of continuous, piecewise continuously differentiable mappings $f_k : Q_0 \to \mathbb{R}^n$. First we introduce a sequence of compact sets in Q_0 whose intersection is a Cantor type set.

The unit cube Q_0 is first divided into 2^n cubes with radius $1/4$, which are each in turn divided into a subcube with radius $(1/4)/2$ and a difference of two cubes which we refer to as an annulus. The family $\mathcal{Q}_1$ consists of these 2^n subcubes. The remainder of the construction is then self-similar.

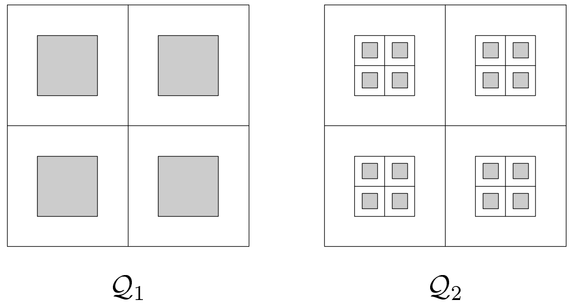

Figure 1: Families $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

The subcube is divided into 2^n cubes which are each in turn divided into a subcube with radius $4^{-2}/2$ and an annulus. The family $\mathcal{Q}_2$ consists of these 2^{2n} subcubes (see Figure 1). Continuing this way, we get the families $\mathcal{Q}_k,\ k = 1, 2, 3, \dots$, for which the radius of $Q \in \mathcal{Q}_k$ is $r(Q) = 4^{-k}/2$ and the number of cubes in $\mathcal{Q}_k$ is $\#\mathcal{Q}_k = 2^{nk}$.

We are now ready to define the mappings f_k. Define $f_0 = \mathrm{id}$. We will give a mapping f_1 that leaves the boundaries $\partial(2Q),\ Q \in \mathcal{Q}_1$ fixed, turns each annulus $2Q \setminus Q$ inside out and stretches the cube Q so that f_1 is continuous (see Figure 2). The Jacobian determinant J_{f_1} will be negative in each annulus $2Q \setminus Q$ and positive in each cube Q. Next, f_2 equals f_1 in the annulae $2Q \setminus Q,\ Q \in \mathcal{Q}_1$, turns each annulus $2Q \setminus Q,\ Q \in \mathcal{Q}_2$, inside out, stretches the cube Q and shifts the image so that f_2 is continuous. Moreover, J_{f_2} is negative a.e. in $Q_0 \setminus \bigcup_{Q \in \mathcal{Q}_2} Q$ and positive in $\bigcup_{Q\in\mathcal{Q}_2} Q$. We will then continue in this manner.

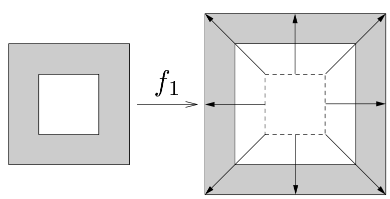

Figure 2: The mapping f_1 acting on $2Q,\ Q \in \mathcal{Q}_1$.

To be precise, let $f_0 = \mathrm{id}|Q_0$ and for $k = 1, 2, 3, \dots$ define

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \notin \bigcup_{Q\in\mathcal{Q}_k} 2Q, \\ f_{k-1}(z(Q)) + a_k \dfrac{x - z(Q)}{\|x - z(Q)\|} \Big(\log\log \dfrac{1}{\|x-z(Q)\|}\Big)^{1/\log(2k)} & \\ & \text{if } x \in 2Q \setminus Q,\ Q \in \mathcal{Q}_k, \\ f_{k-1}(z(Q)) + b_k(x - z(Q)) & \text{if } x \in Q,\ Q \in \mathcal{Q}_k, \end{cases}$$

where $z(Q)$ is the center of the cube Q and a_k and b_k are chosen so that f_k is continuous and fixes the boundary ∂Q_0:

$$\begin{aligned} a_1 &= 1/\big(4(\log\log 4)^{1/\log 2}\big), \\ b_1 &= 2(\log\log 8/\log\log 4)^{1/\log 2}, \end{aligned}$$

and, for $k = 2, 3, 4, \dots,$

$$a_k\Big(\log\log\frac{1}{4^{-k}/2}\Big)^{1/\log(2k)} = b_k \cdot 4^{-k}/2 \quad \text{and} \tag{4.4}$$

$$a_k\Big(\log\log\frac{1}{4^{-k}}\Big)^{1/\log(2k)} = b_{k-1}4^{-k}. \tag{4.5}$$

Remark. The ratio of the outer radius and the inner radius of the image annulus in the level k is

$$\frac{a_k\Big(\log\log\frac{1}{4^{-k}/2}\Big)^{1/\log(2k)}}{a_k\Big(\log\log\frac{1}{4^{-k}}\Big)^{1/\log(2k)}} = \Big(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}}\Big)^{1/\log(2k)}$$

that has the limit 1 as $k \to \infty$, i.e. the volume of the image annulus is small compared to the volume of the cube $f_k(Q)$ for large k.

Next we will show that

$$a_k \approx 2^{-k}. \tag{4.6}$$

By (4.4) $a_k \approx b_k 4^{-k}$, whence it is enough to show that

$$b_k \approx 2^k. \tag{4.7}$$

It follows from (4.4) and (4.5) that

$$b_k = 2b_{k-1}\Big(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}}\Big)^{1/\log(2k)}$$

for all $k = 2, 3, 4, \dots$. Then

$$b_k \approx 2^k \prod_{j=1}^{k}\Big(\frac{\log\log 2^{2j+1}}{\log\log 2^{2j}}\Big)^{1/\log(2j)}.$$

For (4.7) it suffices to show that the product

$$\prod_{k=1}^{\infty}\Big(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}}\Big)^{1/\log(2k)}$$

converges. This happens if, and only if,

$$\sum_{k=1}^{\infty} \log \left(\left(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}} \right)^{1/\log(2k)} \right) \tag{4.8}$$

converges. Let us estimate the terms of this sum. Since $\log t \approx t-1$ for t close to 1, we have

$$\begin{aligned} \log \left(\left(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}} \right)^{1/\log(2k)} \right) &= \frac{1}{\log(2k)} \; \log\left(\frac{\log\log 2^{2k+1}}{\log\log 2^{2k}} \right) \\ &\approx \frac{1}{\log(2k)} \; \frac{\log\log 2^{2k+1} - \log\log 2^{2k}}{\log\log 2^{2k}} \\ &= \frac{1}{\log(2k)} \; \frac{\log(1+\frac{1}{2k})}{\log(2k\log 2)} \\ &\approx \frac{1}{k\log^2(2k)}, \end{aligned}$$

whence (4.8) converges.

Since

$$|f_{k+1}(x) - f_k(x)| \lesssim a_k(\log\log(2\cdot 4^k))^{1/\log(2k)} \approx 2^{-k}$$

the sum

$$\sum_{k=1}^{\infty} |f_{k+1}(x) - f_k(x)|$$

and thus the sequence (f_k) converges uniformly. Hence the limit $f = \lim_{k\to\infty} f_k$ is continuous. Clearly f is differentiable almost everywhere, its Jacobian determinant is strictly negative almost everywhere, and f is absolutely continuous on almost all lines parallel to coordinate axes.

To finish the proof of the properties (a)–(c) we next use Lemma 4.1 to estimate $|Df(x)|$ and $|J_f(x)|$ at $x \in \operatorname{int}(2Q \setminus Q)$, $Q \in \mathcal{Q}_k$, $k = 1, 2, 3, \ldots$. Denote $r = \|x - z(Q)\| \approx 4^{-k}$ and $\rho(r) = a_k(\log\log(1/r))^{1/\log(2k)}$. Since

$$|\rho'(r)| = \frac{1}{\log(2k)\cdot\log(1/r)\cdot\log\log(1/r)} \frac{\rho(r)}{r} \lesssim \frac{\rho(r)}{r}$$

we have

$$\begin{aligned} |Df(x)| &\approx \frac{\rho(r)}{r} = \frac{a_k}{r}(\log\log(1/r))^{1/\log(2k)} \\ &\approx 2^k(\log(2k))^{1/\log(2k)} \approx 2^k \end{aligned} \tag{4.9}$$

and

$$
\begin{aligned}
|J_f(x)| &\approx \Big(\frac{\rho(r)}{r}\Big)^{n-1}|\rho'(r)| \\
&= \Big(\frac{\rho(r)}{r}\Big)^{n}\frac{1}{\log(2k)\cdot\log(1/r)\cdot\log\log(1/r)} \\
&\approx 2^{kn}\ \frac{1}{k\log^2(2k)}.
\end{aligned} \tag{4.10}
$$

Equations (4.9) and (4.10) yield

$$
K(x) = \frac{|Df(x)|^n}{|J_f(x)|} \approx k(\log(2k))^2. \tag{4.11}
$$

The measure of $\bigcup_{Q\in\mathcal{Q}_k} 2Q$ is $(2\cdot 4^{-k})^n 2^{nk} \approx 2^{-kn}$ and so for $0<\epsilon<1$

$$
\epsilon\int_{Q_0}|Df(x)|^{n-\epsilon}\,dx \lesssim \epsilon\sum_{k=1}^{\infty}2^{-kn}2^{k(n-\epsilon)} \le \epsilon\sum_{k=0}^{\infty}2^{-\epsilon k} = \frac{\epsilon}{1-2^{-\epsilon}} \le C
$$

where $C<\infty$ is a constant that does not depend on ϵ. This proves (4.1), and it follows that $f\in W^{1,1}(Q_0,\mathbb{R}^n)$. Alternatively, the fact that $f\in W^{1,1}(Q_0,\mathbb{R}^n)$ can also be seen without using the absolute continuity on almost all lines from the above calculations because they show that the sequence (f_k) converges in $W^{1,1}(Q_0,\mathbb{R}^n)$. Similarly we prove (4.2) and (4.3):

$$
\int_{Q_0}|J_f(x)|\,dx \lesssim \sum_{k=1}^{\infty}2^{-kn}\frac{2^{kn}}{k(\log(2k))^2} \lesssim \sum_{k=2}^{\infty}\frac{1}{k(\log k)^2} < \infty.
$$

By (4.11) there is a constant $1\le c<\infty$ such that $K(x)\le ck(\log k)^2$ in $\operatorname{int}(2Q\setminus Q)$, $Q\in\mathcal{Q}_k$, for $k\ge 2$, and since $t\mapsto t/\log^2(1+t)$ is increasing for large t,

$$
\begin{aligned}
&\int_{Q_0}\exp\left(\frac{\lambda K(x)}{\log^2(1+K(x))}\right)\,dx \lesssim \sum_{k=3}^{\infty}2^{-kn}\exp\left(\frac{\lambda ck(\log k)^2}{\log^2(1+ck(\log k)^2)}\right) \\
&\le \sum_{k=3}^{\infty}2^{-kn}\exp(\lambda ck) = \sum_{k=3}^{\infty}(e^{c\lambda - n\log 2})^k < \infty
\end{aligned}
$$

if we choose $\lambda>0$ such that $\lambda c < n\log 2$.

We prove the property (d) by showing that

$$
Q_0 \subset f\Big(\bigcap_{k=1}^{\infty}\bigcup_{Q\in\mathcal{Q}_k}Q\Big).
$$

From the construction it follows that for each $k=1,2,3,\dots$

$$
f_k\Big(\bigcup_{Q\in\mathcal{Q}_k}Q\Big) \subset f_k\Big(\bigcup_{Q\in\mathcal{Q}_{k+1}}2Q\Big) \subset f_{k+1}\Big(\bigcup_{Q\in\mathcal{Q}_{k+1}}Q\Big).
$$

Since $Q_0 \subset f_1\big(\bigcup_{Q\in\mathcal{Q}_1} Q\big)$, denoting

$$H_k = \bigcup_{Q\in\mathcal{Q}_k} Q$$

we have $Q_0 \subset f_k(H_k) \subset f_l(H_k)$ for all $l \geq k \geq 1$. Now (H_k) is a decreasing sequence of compact sets, whence

$$Q_0 \subset \bigcap_{k=1}^{\infty} \bigcap_{l\geq k} f_l(H_k) \subset \bigcap_{k=1}^{\infty} f(H_k) \subset f\Big(\bigcap_{k=1}^{\infty} H_k\Big)$$

Notice that f is not open: it follows from the construction that $f(\partial Q_0) = \partial Q_0 \subset f(\operatorname{int} Q_0)$ whence $f(Q_0) = f(\operatorname{int} Q_0)$. Because $f(Q_0)$ is a nonempty compact set, $f(\operatorname{int} Q_0)$ is not open. To prove non-discreteness of f let

$$G_k = \bigcup_{l\geq k} f\Big(\bigcup_{Q\in\mathcal{Q}_l} \operatorname{int} 2Q \setminus Q\Big).$$

Then the sets G_k are dense and open, and by the Baire category theorem their intersection is nonempty. But if $y \in \cap_k G_k$, then $f^{-1}(y)$ is an infinite compact set and thus it is not discrete.

The property (f) is clear from the construction.

Remark. $\dim_{\mathcal{H}}\big(\bigcap_{k=1}^{\infty} \bigcup_{Q\in\mathcal{Q}_k} Q\big) = n/2$ (see e.g. [21, Theorem 4.14]).

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