NON-UNIQUENESS OF SRB-MEASURES FOR COUPLED MAP LATTICES

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Abstract. We consider coupled map lattices close to an uncoupled expanding map and show that typically they have infinite number of SRB-measures. In particular, we give a counter example to the Bricmont-Kupiainen conjecture.

1. Introduction

The question of the uniqueness of SRB-measures (Sinai, Ruelle, Bowen) for coupled map lattices has been around for over a decade. Bunimovich and Sinai [BS] studied expanding maps of the unit interval with a special diffusive coupling over one-dimensional lattice $\mathbb{Z}$. They showed that the system has an invariant Gibbs state whose projections onto finite-dimensional subsystems are absolutely continuous with respect to the corresponding Lebesgue measure. In [BK1] Bricmont and Kupiainen proved the existence of a SRB-measure for analytic expanding circle maps in the regime of small analytic coupling over d-dimensional lattice $\mathbb{Z}^d$ and conjectured the uniqueness of this SRB-measure. In [BK2] they extended the existence result for special Hölder continuous functions. They also proved that the SRB-measure is unique in the class of measures for which the logarithm of the density is Hölder continuous. In [J] it was shown that all these results remain true if one replaces the circle by any compact Riemannian manifold. Jiang and Pesin [JP] considered weakly coupled Anosov maps and proved the existence and uniqueness of certain equilibrium states which they called SRB-measures.

The answer to the question of the uniqueness of SRB-measures for coupled map lattices may depend on the definition of the SRB-measure. For finite-dimensional expanding systems one definition for a SRB-measure is that it is an invariant measure which is absolutely continuous with respect to the Lebesgue measure. According to another definition it is an equilibrium state for a certain potential function obtained from the derivative of the map. For finite-dimensional systems the uniqueness of this equilibrium state implies the equivalence between these definitions since the equilibrium state is ergodic and equivalent to the Lebesgue measure. Thus the ergodic decomposition and the mutual singularity of ergodic measures implies the uniqueness of the SRB-measure.

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For coupled map lattices the absolute continuity with respect to the Lebesgue measure of the whole infinite system does not give a good definition for a SRB-measure. The reason for this is that the natural candidates for uncoupled expanding maps are typically singular with respect to the Lebesgue measure. A reasonable condition, used by Bricmont and Kupiainen in [BK1,BK2], is to demand that all the projections onto finite-dimensional subsystems are absolutely continuous with respect to the corresponding Lebesgue measure. Jiang and Pesin in turn used an extension of the equilibrium state definition. In statistical physics the equilibrium states are the natural measures since the potential is the primary (physical) object to study. However, one might argue that, although being extremely useful, potentials and equilibrium states are secondary concepts in the theory of dynamical systems. So the absolute continuity is perhaps more fundamental property and we will concentrate on the SRB-measures á la Bricmont and Kupiainen.

In this paper we will construct a coupled map lattice which has infinite number of SRB-measures in the sense of Bricmont and Kupiainen (see Theorem 3.4). We also argue that our example is not just a curious artificial system but it manifests a typical behaviour.

2. Preliminaries

Our main motivation comes from the well-known projection results in $\mathbb{R}^n$ stating that the projections of a Radon measure μ onto almost all m-planes are absolutely continuous with respect to the m-dimensional Lebesgue measure provided that the m-energy of μ is finite [M, Theorem 9.7]. Our strategy is to use the fact that expanding maps have small invariant sets (and measures) in the sense that their dimensions are less than the dimension of the ambient manifold. For example, the $\frac{1}{3}$-Cantor set is invariant under the map $x \mapsto 3x \bmod 1$. If one takes a finite n-fold product of these Cantor-sets, one will obtain a set which is invariant under the corresponding n-fold product map. Of course, the dimension of this product set is less than n, and so the natural Hausdorff measure living on the set, although being invariant, is not a SRB-measure since it is not absolutely continuous with respect to the n-dimensional Lebesgue measure. However, as n grows, the dimension of the product Cantor set grows. In particular, for each integer m one can find n such that the dimension of the n-fold Cantor set is greater than m. By the above mentioned projection result typical projections of the n-fold Hausdorff measure onto m-dimensional subspaces are absolutely continuous with respect to the m-dimensional Lebesgue measure. Of course, for this system the m-dimensional subsystems are atypical and the projections onto them are not absolutely continuous. Our idea is that a small coupling will make these coordinate planes typical ones. However, one has to be careful since in [HK] Hunt and Kaloshin proved that these projection results are not valid in infinite dimensional spaces.

We adopt the very general formulation of projection results due to Peres and Schlag [PS]. We begin by recalling the notation from [PS] which we will use later.

2.1. Definition. *Let (X,d) be a compact metric space, $Q \subset \mathbb{R}^n$ an open connected set, and $\Pi : Q \times X \to \mathbb{R}^m$ a continuous map with $n \geq m$. For any multi-index $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, let $|\eta| = \sum_{i=1}^n \eta_i$ be the length of it, and $\partial^\eta = \frac{\partial^{|\eta|}}{(\partial \varepsilon_1)^{\eta_1} \dots (\partial \varepsilon_n)^{\eta_n}}$ where $\epsilon = (\varepsilon_1, \dots, \varepsilon_n) \in Q$. Let L be a positive integer and $\delta \in [0,1)$. We say that $\Pi \in C^{L,\delta}(Q)$ if for any compact set $Q' \subset Q$ and for any multi-index η with $|\eta| \leq L$*

there exist constants $C_{\eta,Q'}$ *and* $C_{\delta,Q'}$ *such that*

$$|\partial^\eta \Pi(\epsilon,x)| \le C_{\eta,Q'} \text{ and } \sup_{|\eta'|=L} |\partial^{\eta'}\Pi(\epsilon,x) - \partial^{\eta'}\Pi(\epsilon',x)| \le C_{\delta,Q'}|\epsilon-\epsilon'|^\delta$$

for all $\epsilon, \epsilon' \in Q'$ *and* $x \in X$.

Next we will give a definition of a subclass of $C^{L,\delta}(Q)$ from [PS].

2.2. Definition. *Let* $\Pi \in C^{L,\delta}(Q)$ *for some* L *and* δ. *Define for all* $x \neq y \in X$

$$\Phi_{x,y}(\epsilon) = \frac{\Pi(\epsilon,x) - \Pi(\epsilon,y)}{d(x,y)}.$$

Let $\beta \in [0,1)$. *The set* Q *is a region of transversality of order* β *for* Π *if there exists a constant* C_β *such that for all* $\epsilon \in Q$ *and for all* $x \neq y \in X$ *the condition* $|\Phi_{x,y}(\epsilon)| \le C_\beta d(x,y)^\beta$ *implies*

$$\det(D\Phi_{x,y}(\epsilon)(D\Phi_{x,y}(\epsilon))^T) \ge C_\beta^2 d(x,y)^{2\beta}.$$

Here the derivative with respect to ϵ *is denoted by* D *and* A^T *is the transpose of a matrix* A.

Further, Π *is* (L,δ)*-regular on* Q *if there exists a constant* $C_{\beta,L,\delta}$ *and for all multi-indices* η *with* $|\eta| \le L$ *there exists a constant* $C_{\beta,\eta}$ *such that for all* $\epsilon, \epsilon' \in Q$ *and for all distinct* $x, y \in X$

$$|\partial^\eta \Phi_{x,y}(\epsilon)| \le C_{\beta,\eta} d(x,y)^{-\beta|\eta|}$$

and

$$\sup_{|\eta'|=L} |\partial^{\eta'}\Phi_{x,y}(\epsilon) - \partial^{\eta'}\Phi_{x,y}(\epsilon')| \le C_{\beta,L,\delta}|\epsilon-\epsilon'|^\delta d(x,y)^{-\beta(L+\delta)}.$$

2.3. Remark. *Note that if the determinant in Definition 2.2 is bounded away from zero then* Q *is a region of transversality of order* β *for all* $\beta \in [0,1)$.

2.4. Definition. *Let* μ *be a Borel measure on* X *and* $\alpha \in \mathbb{R}$. *The* α*-energy of* μ *is*

$$\mathcal{E}_\alpha(\mu) = \int_X \int_X d(x,y)^{-\alpha} d\mu(x) d\mu(y).$$

We denote the image of a measure μ under a map $f : X \to Y$ by $f_*\mu$, that is, $f_*\mu(A) = \mu(f^{-1}(A))$ for all $A \subset Y$. The following theorem from [PS] gives a relation between Sobolev-norms of images of measures under $C^{L,\delta}(Q)$-mappings and energies of original measures.

2.5. Theorem. *Let* $Q \subset \mathbb{R}^n$ *and* $\Pi \in C^{L,\delta}(Q)$ *such that* $L + \delta > 1$. *Let* $\beta \in [0,1)$. *Assume that* Q *is a region of transversality of order* β *for* Π *and that* Π *is* (L,δ)*-regular on* Q. *Let* μ *be a finite Borel measure on* X *such that* $\mathcal{E}_\alpha(\mu) < \infty$ *for some* $\alpha > 0$. *Then there exist a constant* a_0 *depending only on* m, n, *and* δ *such that for any compact* $Q' \subset Q$

$$\int_{Q'} \|\Pi_*\mu\|_{2,\gamma}^2 d\mathcal{L}^n(\epsilon) \le C_\gamma \mathcal{E}_\alpha(\mu)$$

for some constant C_γ provided that $0 < (m+2\gamma)(1+a_0\beta) \le \alpha$ and $2\gamma < L+\delta-1$. Here $\|\cdot\|_{2,\gamma}$ is the Sobolev norm, that is,

$$\|\nu\|_{2,\gamma}^2 = \int_{\mathbb{R}^m} |\hat{\nu}(\xi)|^2 |\xi|^{2\gamma} d\mathcal{L}^m(\xi)$$

for any finite compactly supported Borel measure on $\mathbb{R}^m$ where

$$\hat{\nu}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} d\nu(x)$$

is the Fourier transform of ν.

Proof. [PS, Theorem 7.3]. $\square$

2.6. Remark. *Let ν be a finite compactly supported Borel measure on $\mathbb{R}^n$. If $\|\nu\|_{2,0} < \infty$ then ν is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$ and its Radon-Nikodym derivative is L^2-integrable, that is, $D(\nu,\mathcal{L}^n) \in L^2(\mathbb{R}^n)$ (see (3.5)). Indeed, if $\hat{\nu} \in L^2(\mathbb{R}^n)$ then by the surjectivity of the Fourier transform* [SW, Theorem 2.3 p. 17] *there exists $f \in L^2(\mathbb{R}^n)$ such that $\hat{f} = \hat{\nu}$. Thus by* [T, Definition 1.7 p. 262] *$f = \nu$ as a distribution meaning that $f = D(\nu,\mathcal{L}^n)$. Note also that $\|\nu\|_{2,\gamma} < \infty$ for $\gamma \ge n+2$ implies that $D(\nu,\mathcal{L}^n)$ has L^2-integrable derivatives of order γ, that is, $D(\nu,\mathcal{L}^n) \in W_2^\gamma(\mathbb{R}^n)$. So by* [SW, Lemma 3.17 p. 26] *$D(\nu,\mathcal{L}^n)$ is continuously differentiable.*

3. Results

Let $\Omega = \prod_{\mathbb{Z}^d} S^1$, where $d \ge 1$ is an integer and $S^1 \subset \mathbb{C}$ is the unit circle. We use the notation $\Omega_\Lambda = \prod_\Lambda S^1$ for all $\Lambda \subset \mathbb{Z}^d$. For $\Lambda \subset \tilde{\Lambda} \subset \mathbb{Z}^d$ let $\pi_\Lambda : \Omega \to \Omega_\Lambda$ and $\pi_{\tilde{\Lambda},\Lambda} : \Omega_{\tilde{\Lambda}} \to \Omega_\Lambda$ be the natural projections. Let $\varepsilon_0 > 0$ and let $A_\epsilon : \Omega \to \Omega$ be such that its lift $\overline{A}_\epsilon : \overline{\Omega} \to \overline{\Omega}$, where $\overline{\Omega} = \prod_{\mathbb{Z}^d} \mathbb{R}$, is

$$\overline{A}_\epsilon(x)_i = x_i + \sum_{l\in\mathbb{Z}^d} \varepsilon_{il} 2^{-|i-l|} \overline{g}(x_l) \tag{3.1}$$

for all $i \in \mathbb{Z}^d$ where $|\cdot|$ is a metric on $\mathbb{Z}^d$, $\varepsilon_{il} \in (-\varepsilon_0,\varepsilon_0)$ for all $i,l \in \mathbb{Z}^d$ and $\overline{g}$ is continuously differentiable and 1-periodic. (We use the covering map $p : \overline{\Omega} \to \Omega$ such that $\prod_{\mathbb{Z}^d}[0,1]$ is a covering domain. Then $A_\epsilon = p \circ \overline{A}_\epsilon \circ p^{-1}$.)

Set $E = \prod_{\mathbb{Z}^d\times\mathbb{Z}^d}(-\varepsilon_0,\varepsilon_0)$ and denote by $\mathcal{L}$ the product over $\mathbb{Z}^d\times\mathbb{Z}^d$ of normalized Lebesgue measures on $(-\varepsilon_0,\varepsilon_0)$. It is not difficult to see that A_ϵ is invertible for all $\epsilon \in E$ provided ε_0 is small enough (depending on $|\overline{g}'|$). We fix such ε_0 and set $T_\epsilon = A_\epsilon \circ F \circ A_\epsilon^{-1}$, where $F : \Omega \to \Omega$ is the product over $\mathbb{Z}^d$ of maps $z \mapsto z^3$ (or $t \mapsto 3t$ mod 1 if S^1 is viewed as $[0,1]$). Let $\mathcal{K} = \prod_{\mathbb{Z}^d} K$ and $\mu = \prod_{\mathbb{Z}^d} \mathcal{H}^s|_K$ where K is the $\frac{1}{3}$-Cantor set on S^1 (or $[0,1]$) and $\mathcal{H}^s|_K$ is the restriction of the s-dimensional Hausdorff measure to K with $s = \frac{\log 2}{\log 3}$. (Note that s is the Hausdorff dimension of K). Now $(A_\epsilon)_*\mu$ is clearly T_ϵ-invariant, that is, $(T_\epsilon)_*(A_\epsilon)_*\mu = (A_\epsilon)_*\mu$. Our aim is to show that for $\mathcal{L}$-almost all ϵ the projection $(\pi_\Lambda)_*(A_\varepsilon)_*\mu$ is absolutely continuous with respect to the Lebesgue measure on Ω_Λ for all finite $\Lambda \subset \mathbb{Z}^d$.

Let $\Lambda \subset \mathbb{Z}^d$. We denote the restriction of A_ϵ to Ω_Λ by $A_{\epsilon,\Lambda}$, that is,

$$\overline{A}_{\epsilon,\Lambda}(x)_i = x_i + \sum_{l\in\Lambda} \varepsilon_{il} 2^{-|i-l|} \overline{g}(x_l)$$

for all $i \in \Lambda$. Set $\mu_\Lambda = \prod_\Lambda \mathcal{H}^s|_K$ and $\mathcal{K}_\Lambda = \prod_\Lambda K$. Let $\Lambda \subset \tilde{\Lambda} \subset \mathbb{Z}^d$ be such that $|\tilde{\Lambda}|s > |\Lambda|$ where the number of elements in Λ is denoted by $|\Lambda|$. Let $E_{\Lambda\times\tilde{\Lambda}} = \prod_{\Lambda\times\tilde{\Lambda}}(-\varepsilon_0, \varepsilon_0)$ and let $\mathcal{L}^{\Lambda\times\tilde{\Lambda}}$ be the restriction of $\mathcal{L}$ to $E_{\Lambda\times\tilde{\Lambda}}$. We will first show that for $\mathcal{L}^{\Lambda\times\tilde{\Lambda}}$-almost all $\epsilon \in E_{\Lambda\times\tilde{\Lambda}}$ the measure $(\pi_{\tilde{\Lambda},\Lambda} \circ A_{\varepsilon,\tilde{\Lambda}})_*\mu_{\tilde{\Lambda}}$ is absolutely continuous with respect to the Lebesgue measure on Ω_Λ. As it will be indicated in the proof of Proposition 3.2 this claim follows from Theorem 2.5.

In order to apply Theorem 2.5 we have to give some conditions on g. Since $\overline{g}$ is 1-periodic and continuously differentiable there necessarily exists $t_0 \in [0,1]$ such that $\overline{g}'(t_0) = 0$. In order to satisfy the transversality assumption in Theorem 2.5, we demand that $\overline{g}' \neq 0$ on K. More precisely, let $b > 0$ and let $\overline{g}$ be increasing on $[0, 1/6]$ such that $\overline{g}(0) = 0$ and $\overline{g}'(t) \geq b$ for all $t \in [0, t_1]$ for some $1/9 < t_1 < 1/6$. Define $\overline{g}(t + 1/6) = \overline{g}(1/6 - t)$ for $t \in [0, 1/6]$ and $\overline{g}(1 - t) = -\overline{g}(t)$ for $t \in [0, 1/3]$. We extend $\overline{g}$ to the interval $[1/3, 2/3]$ such that $\overline{g}$ is continuously differentiable, $\overline{g}([0, 1]) \subset [-1, 1]$, for some $B \geq b$ we have $|\overline{g}'(t)| \leq B$ for all $t \in [0, 1]$, and $|\overline{g}'(t)| \geq b$ for all $t \in [1/3, 1/3 + t_2] \cup [2/3 - t_2, 2/3]$ where $0 < t_2 < 1/9$.

Consider the second step in the construction of the Cantor set K. Call the chosen intervals I_i, $i = 1, \ldots, 4$, that is, $I_1 = [0, 1/9]$, $I_2 = [2/9, 1/3]$, $I_3 = [2/3, 7/9]$, and $I_4 = [8/9, 1]$. Let $x \in \mathcal{K}$ and $\Lambda \subset \mathbb{Z}^d$. Define $\tilde{x} \in \mathcal{K}$ in the following way: For all $i \in \Lambda$ let $\tilde{x}_i = x_i$. For $j \in \Lambda^c = \mathbb{Z}^d \setminus \Lambda$ set $\tilde{x}_j = x_j$ if $x_j \in I_1 \cup I_4$, $\tilde{x}_j = 1/6 - (x_j - 1/6)$ if $x_j \in I_2$, and $\tilde{x}_j = 5/6 + 5/6 - x_j$ if $x_j \in I_3$. Note that with these definitions $\overline{g}(\tilde{x}_j) = \overline{g}(x_j)$ for all $j \in \mathbb{Z}^d$ implying that $\pi_\Lambda \circ A_\epsilon(\tilde{x}) = \pi_\Lambda \circ A_\epsilon(x)$. Further, if $x_j \notin [-t_1, t_1]$ for some $j \in \Lambda^c$ then $\tilde{x}_j \in [-t_1, t_1]$.

Let $x, y \in \mathcal{K}$ such that $x_i \in I_1$ and $y_i \in I_2$ for some $i \in \Lambda$. Then

$$\begin{aligned}
\overline{A}_\epsilon(y)_i - \overline{A}_\epsilon(x)_i &\geq y_i - x_i - \sum_{l\in\mathbb{Z}^d} \varepsilon_{il} 2^{-|i-l|} |\overline{g}(y_l) - \overline{g}(x_l)| \\
&\geq y_i - x_i - \sum_{l\in\mathbb{Z}^d} \varepsilon_{il} 2^{-|i-l|} B|y_l - x_l| \geq y_i - x_i - C\varepsilon_0 \geq \frac{1}{18}
\end{aligned} \tag{3.2}$$

for ε_0 small enough since $y_i - x_i \geq 1/9$. Thus the cubes at the second stage of the construction of $\mathcal{K}$ with i:th side I_1 will not overlap with cubes with i:th side I_2 under the projection $\pi_\Lambda \circ A_\epsilon$ provided that $i \in \Lambda$. (The same argument works in other cases as well, see (3.3) below.) More precisely, there exists a constant $c > 0$ such that

$$|\pi_\Lambda \circ A_\epsilon(x) - \pi_\Lambda \circ A_\epsilon(y)| \geq c \tag{3.3}$$

for all $x, y \in \mathcal{K}$ with $x_i \in I_1 \cup I_4$ and $y_i \in I_2 \cup I_3$ (or $x_i \in I_2$ and $y_i \in I_3$) for some $i \in \Lambda$. Further, as in (3.2) we see that there exists $\tilde{c} > 0$ such that $|\overline{A}_\epsilon(x)_i - 1/6| \geq \tilde{c}$ for all $i \in \Lambda$ and $x \in \mathcal{K}$, giving the existence of $\delta > 0$ such that

$$\pi_{\{i\}} \circ A_\epsilon(\mathcal{K}) \cap [\frac{1}{6} - \delta, \frac{1}{6} + \delta] = \emptyset \tag{3.4}$$

for all $i \in \Lambda$. We fix ε_0 and δ such that the above results hold.

3.1. Lemma. *Let $\Lambda \subset \tilde{\Lambda} \subset \mathbb{Z}^d$ be finite such that $|\tilde{\Lambda}|s > |\Lambda|$. Set $X_{\tilde{\Lambda}} = \prod_{\tilde{\Lambda}}[-t_1, t_1]$. Define $\Pi : E_{\Lambda\times\tilde{\Lambda}} \times X_{\tilde{\Lambda}} \to \Omega_\Lambda$ by $\Pi(\epsilon, x) = \pi_{\tilde{\Lambda},\Lambda} \circ A_{\epsilon,\tilde{\Lambda}}(x)$. Then* *the assumptions of Theorem 2.5 are valid for $\delta = 0$, $\beta = 0$, and for all integers $L > 1$. Further, $\mathcal{E}_\alpha(\mu_{\tilde\Lambda}) < \infty$ for any $|\Lambda| < \alpha < |\tilde\Lambda| s$.*

Proof. We may replace Ω_Λ by $\mathbb{R}^m$ where $m = |\Lambda|$. Let $i_0 \in \Lambda$. Note that $X_{\tilde\Lambda}$ is a compact metric space equipped with the metric

$$d(x,y)^2 = \sum_{l\in\tilde\Lambda} 2^{-2|i_0-l|}|x_l - y_l|^2.$$

Clearly $\Pi \in C^{L,0}(E_{\Lambda\times\tilde\Lambda})$ for all positive integers L since all the first order partial derivatives are constants. Note that Q' in Definition 2.1 will not play any role here since all the estimates are independent of Q'.

To check the transversality assumption in Definition 2.2, define for all $x \neq y \in X_{\tilde\Lambda}$

$$\Phi_{x,y}(\epsilon) = \frac{\Pi(\epsilon,x) - \Pi(\epsilon,y)}{d(x,y)}.$$

Fix $i \in \Lambda$, $k = (k_1,k_2) \in \Lambda \times \tilde\Lambda$, and $x,y \in X_{\tilde\Lambda}$ such that $x \neq y$. Then

$$D\Phi_{x,y}(\epsilon)_{i,k} = \delta_{i,k_1} 2^{-|i-k_2|}\frac{\overline{g}(x_{k_2}) - \overline{g}(y_{k_2})}{d(x,y)}$$

where $\delta_{i,j}$ is the Kronecker's delta. Thus for $i,j \in \Lambda$

$$\begin{aligned}(D\Phi_{x,y}(\epsilon)D\Phi_{x,y}(\epsilon)^T)_{i,j} =& \frac{\delta_{i,j}}{d(x,y)^2}\sum_{l\in\tilde\Lambda} 2^{-|i-l|-|j-l|}(\overline{g}(x_l) - \overline{g}(y_l))^2 \\ &\geq \delta_{i,j} b^2 2^{-|i-i_0|-|j-i_0|}.\end{aligned}$$

By Remark 2.3 the transversality assumption is valid for $\beta = 0$ with the constant $C_0 = b^m 2^{-\sum_{i\in\Lambda}|i-i_0|}$.

Finally, Π is obviously $(L,0)$-regular (in fact (L,δ)-regular for all $\delta \in [0,1)$) on $E_{\Lambda\times\tilde\Lambda}$ for all positive integers L. The last assertion follows from the well-known properties of the Hausdorff measure $\mathcal{H}^s|_K$ (see [M, Chapter 8]). □

The following absolute continuity result follows from Theorem 2.5 and Lemma 3.1.

3.2. Proposition. *Let $\Lambda \subset \tilde\Lambda \subset \mathbb{Z}^d$ be finite such that $|\tilde\Lambda| s > |\Lambda|$. Then for $\mathcal{L}^{\Lambda\times\tilde\Lambda}$-almost all $\epsilon \in E_{\Lambda\times\tilde\Lambda}$ the measure $(\pi_{\tilde\Lambda,\Lambda} \circ A_{\epsilon,\tilde\Lambda})_*\mu_{\tilde\Lambda}$ is absolutely continuous with respect to the Lebesgue measure on Ω_Λ.*

Proof. By the arguments given before stating Lemma 3.1 we may replace $\Omega_{\tilde\Lambda}$ by $X_{\tilde\Lambda} = \prod_{\tilde\Lambda}[-t_1,t_1]$. Lemma 3.1 and Theorem 2.5 give $\|(\pi_{\tilde\Lambda,\Lambda} \circ A_{\epsilon,\tilde\Lambda})_*\mu_{\tilde\Lambda}\|_{2,0} < \infty$ for $\mathcal{L}^{\Lambda\times\tilde\Lambda}$-almost all $\epsilon \in E_{\Lambda\times\tilde\Lambda}$ which by Remark 2.6 implies the claim. □

In Proposition 3.3 we will prove that one may replace $A_{\epsilon,\tilde\Lambda}$ by A_ϵ and $\mu_{\tilde\Lambda}$ by μ in Proposition 3.2. For this purpose we use differentiation theory of measures. Let ν and λ be Radon measures on $\mathbb{R}^n$. Recall that the lower derivative of ν with respect to λ at a point $x \in \mathbb{R}^n$ is defined by

$$\underline{D}(\nu,\lambda,x) = \liminf_{r\to 0}\frac{\nu(B(x,r))}{\lambda(B(x,r))} \tag{3.5}$$

where $B(x,r)$ is the closed ball with centre at x and with radius r. If the limit exists it is called the Radon-Nikodym derivative of ν with respect to λ and is denoted by $D(\nu,\lambda,x)$. Further, ν is absolutely continuous with respect to λ if and only if $\underline{D}(\nu,\lambda,x) < \infty$ for ν-almost all $x \in \mathbb{R}^n$ [M, Theorem 2.12].

3.3. Proposition. *Let $\Lambda \subset \tilde{\Lambda} \subset \mathbb{Z}^d$ be finite such that $|\tilde{\Lambda}|s > |\Lambda|$ and let $\epsilon_1 \in E_{\Lambda\times\tilde{\Lambda}}$ such that the conclusion of Proposition 3.2 is valid. Then for all $\epsilon \in E$ with $\epsilon_{\Lambda\times\tilde{\Lambda}} = \epsilon_1$ we have*

$$\underline{D}((\pi_\Lambda \circ A_\epsilon)_*\mu, \mathcal{L}^\Lambda, x) < \infty$$

for $(\pi_\Lambda \circ A_\epsilon)_*\mu$-almost all $x \in \Omega_\Lambda$. Here $\mathcal{L}^\Lambda$ is the Lebesgue measure on Ω_Λ and $\epsilon_{\Lambda\times\tilde{\Lambda}} = (\varepsilon_{ij})_{(i,j)\in\Lambda\times\tilde{\Lambda}}$.

Proof. Let $\epsilon, \epsilon_0 \in E$ such that $\epsilon_{\Lambda\times\tilde{\Lambda}} = \epsilon_1$, $\epsilon_{\tilde{\Lambda}\times\tilde{\Lambda}} = (\epsilon_0)_{\tilde{\Lambda}\times\tilde{\Lambda}}$, and $(\epsilon_0)_{\mathbb{Z}^d\times\tilde{\Lambda}^c} = (\epsilon_0)_{\tilde{\Lambda}^c\times\mathbb{Z}^d} = 0$. Set $\nu_\epsilon = (\pi_\Lambda \circ A_\epsilon)_*\mu$ and $\nu_0 = (\pi_{\tilde{\Lambda},\Lambda} \circ A_{\epsilon_0,\tilde{\Lambda}})_*\mu_{\tilde{\Lambda}}$. Then ν_ϵ and ν_0 are Radon measures with compact supports [M, Theorem 1.18]. It follows directly from (3.1) that $(A_{\epsilon_0,\tilde{\Lambda}})_*\mu_{\tilde{\Lambda}} = (\pi_{\tilde{\Lambda}} \circ A_{\epsilon_0})_*\mu$, meaning that $\nu_0 = (\pi_\Lambda \circ A_{\epsilon_0})_*\mu$. By Proposition 3.2 the measure ν_0 is absolutely continuous with respect to $\mathcal{L}^\Lambda$. Set $m = |\Lambda|$. We will first show that there exists a constant $C > 0$ such that for all $r > 0$

$$\int_{\Omega_\Lambda} \nu_\epsilon(B(x,r))d\nu_\epsilon(x) \le C\int_{\Omega_\Lambda} \nu_0(B(x,\sqrt{m}r))d\nu_0(x). \tag{3.6}$$

By [FO, Lemma 2.6] it is enough to prove that

$$\sum_{Q\in\mathcal{D}(r,\Lambda)} \nu_\epsilon(Q)^2 \le C \sum_{Q\in\mathcal{D}(r,\Lambda)} \nu_0(Q)^2 \tag{3.7}$$

where $\mathcal{D}(r,\Lambda)$ is the family of r-mesh cubes in $\mathbb{R}^\Lambda$, that is, cubes of the form $[l_1 r, (l_1+1)r) \times \cdots \times [l_m r, (l_m+1)r)$ where $l_i \in \mathbb{Z}$ for all $i = 1,\ldots,m$.

Let $r > 0$. Consider the cubes at the n:th stage of the construction of $\mathcal{K}$ where $3^{-n} < r$. Call this n:th stage approximation $\mathcal{K}(n)$. Setting $V_0 = A_{\epsilon_0,\tilde{\Lambda}}(\mathcal{K}_{\tilde{\Lambda}}(n)) \times \mathcal{K}_{\tilde{\Lambda}^c}(n) = A_{\epsilon,\tilde{\Lambda}}(\mathcal{K}_{\tilde{\Lambda}}(n)) \times \mathcal{K}_{\tilde{\Lambda}^c}(n)$, we get $A_{\epsilon_0}(\operatorname{spt}\mu) \subset V_0$ implying that $\operatorname{spt}\nu_0 \subset \pi_\Lambda(V_0)$. Here the support of a measure λ is denoted by $\operatorname{spt}\lambda$.

If $i \in \tilde{\Lambda}$ and $x, y \in X = \prod_{\mathbb{Z}^d}[-t_1, t_1]$ such that $x_k = y_k$ for all $k \in \tilde{\Lambda}$, then

$$A_\epsilon(x)_i - A_\epsilon(y)_i = \sum_{l\in\tilde{\Lambda}^c} \varepsilon_{il}2^{-|i-l|}(\overline{g}(x_l) - \overline{g}(y_l)). \tag{3.8}$$

(Recall the discussion before Lemma 3.1 according to which we can assume that $x_i \in [-t_1, t_1]$ for all $i \in \mathbb{Z}^d$). Note that the difference in (3.8) depends only on x_j for $j \in \tilde{\Lambda}^c$. Defining $V_\epsilon = A_\epsilon(\mathcal{K}(n))$, we have $\operatorname{spt}\nu_\epsilon \subset \pi_\Lambda(V_\epsilon)$. Further, $A_\epsilon(x)_i = A_{\epsilon,\tilde{\Lambda}}(x)_i$ for all $i \in \tilde{\Lambda}$ if $x_j = 0$ for all $j \in \tilde{\Lambda}^c$ meaning that the restriction of V_ϵ to the subspace $\Omega_{\tilde{\Lambda}} \subset \Omega$ equals $A_{\epsilon,\tilde{\Lambda}}(\mathcal{K}_{\tilde{\Lambda}}(n)) = A_{\epsilon_0,\tilde{\Lambda}}(\mathcal{K}_{\tilde{\Lambda}}(n))$. So by (3.8) V_ϵ is obtained from V_0 by tilting the rows of "cubes" above each "cube" in $A_{\epsilon,\tilde{\Lambda}}(\mathcal{K}_{\tilde{\Lambda}}(n))$ in such a way that the amount of translation does not depend on x_i for $i \in \tilde{\Lambda}$. Thus ν_ϵ is obtained from ν_0 by spreading around the "cubes" defining ν_0.

Let $Q \in \mathcal{D}(r,\Lambda)$. If there is $Q' \in \mathcal{D}(r,\Lambda)$ such that a part of the "cubes" above it in V_0 are tilted above Q then the corresponding "cubes" above Q (in V_0) are removed away by (3.8). Define

$$A_Q = \{Q' \in \mathcal{D}(r,\Lambda) \mid \pi_\Lambda(A_\epsilon(A_{\epsilon,\tilde{\Lambda}}^{-1}(Q' \times X_{\tilde{\Lambda}\setminus\Lambda}) \times X_{\tilde{\Lambda}^c})) \cap Q \neq \emptyset\}.$$

Then for all $Q' \in A_Q$ with $\pi_\Lambda(V_\epsilon) \cap \pi_\Lambda(A_\epsilon(A^{-1}_{\epsilon,\tilde\Lambda}(Q' \times X_{\tilde\Lambda\setminus\Lambda}) \times X_{\tilde\Lambda^c})) \cap Q \neq \emptyset$ we have $V_0 \cap (Q' \times X_{\Lambda^c}) \neq \emptyset$. Further,

$$Q \times X_{\Lambda^c} = \bigcup_{\substack{Q' \in \mathcal{D}(r,\Lambda) \\ Q \in A_{Q'}}} P_Q(Q') \tag{3.9}$$

where

$$P_Q(Q') = \{x \in Q \times X_{\Lambda^c} \mid \pi_\Lambda(A_\epsilon(A^{-1}_{\epsilon,\tilde\Lambda}(x_{\tilde\Lambda}) \times x_{\tilde\Lambda^c})) \in Q'\}.$$

Observe that

$$(A_{\epsilon_0})_*\mu(P_Q(Q')) = (A_\epsilon)_*\mu(A_\epsilon(A^{-1}_{\epsilon_0}(P_Q(Q')))). \tag{3.10}$$

Note that by (3.8) the geometric shape of this partition is independent of Q, that is, if $Q_1 \in \mathcal{D}(r,\Lambda)$ with

$$Q_1 \times X_{\Lambda^c} = \bigcup_{\substack{Q' \in \mathcal{D}(r,\Lambda) \\ Q_1 \in A_{Q'}}} P_{Q_1}(Q'),$$

then for all $Q_2 = \tau(Q_1) \in \mathcal{D}(r,\Lambda)$ (τ is a translation) we have

$$Q_2 \times X_{\Lambda^c} = \bigcup_{\substack{Q' \in \mathcal{D}(r,\Lambda) \\ Q_1 \in A_{Q'}}} \tau(P_{Q_1}(Q')).$$

Naturally, this partition can be restricted to V_0. Hence for all $Q \in \mathcal{D}(r,\Lambda)$ there are non-negative numbers $p_Q(Q') = \frac{1}{\nu_0(Q)}(A_{\epsilon_0})_*\mu(P_Q(Q'))$ adding to 1 such that

$$\begin{aligned} \nu_0(Q) = (A_{\epsilon_0})_*\mu(Q \times X_{\Lambda^c}) &= \sum_{\substack{Q' \in \mathcal{D}(r,\Lambda) \\ Q \in A_{Q'}}} (A_{\epsilon_0})_*\mu(P_Q(Q')) \\ &= \sum_{\substack{Q' \in \mathcal{D}(r,\Lambda) \\ Q \in A_{Q'}}} p_Q(Q')\nu_0(Q). \end{aligned} \tag{3.11}$$

This gives by (3.10) that

$$\nu_\epsilon(Q) = \sum_{Q' \in A_Q} (A_{\epsilon_0})_*\mu(P_{Q'}(Q)) = \sum_{Q' \in A_Q} p_{Q'}(Q)\nu_0(Q'). \tag{3.12}$$

The numbers $p_Q(Q')$ depend on both Q and $P_Q(Q')$. Enumerating the partition of $Q \times X_{\Lambda^c}$ given in (3.9) we get $Q \times X_{\Lambda^c} = \cup_i P_Q(i)$, where the geometric shape of $P_Q(i)$ may vary as i varies. However, for all i and $Q, \widetilde{Q} \in \mathcal{D}(r,\Lambda)$ we have $P_{\widetilde{Q}}(i) = \tau(P_Q(i))$ where τ is the translation with $\widetilde{Q} = \tau(Q)$. Hence the differences in $P_Q(i)$ as Q varies and i is kept fixed are due to the fact that the measure is not evenly distributed inside horizontal $|\Lambda|$-dimensional slices of $Q \times X_{\Lambda^c}$. Note that if such a horizontal slice intersects an element $P_Q(Q')$ of the partition (3.9), then, by

(3.8), it may intersect only the elements $P_Q(Q'')$ where Q'' is a neighbour of Q' in $\mathcal{D}(r,\Lambda)$. Let $N = 3^{|\Lambda|}$ be the number of neighbours. We say that Q' and Q'' are related ($Q' \sim Q''$) if there exists Q such that $Q', Q'' \in A_Q$. Then by (3.11) and (3.12)

$$\begin{aligned}
N &\sum_{Q\in\mathcal{D}(r,\Lambda)} \nu_0(Q)^2 - \sum_{Q\in\mathcal{D}(r,\Lambda)} \nu_\epsilon(Q)^2 \\
&= N \sum_{Q\in\mathcal{D}(r,\Lambda)} \sum_{\substack{Q'\in\mathcal{D}(r,\Lambda)\\ Q\in A_{Q'}}} \sum_{\substack{Q''\in\mathcal{D}(r,\Lambda)\\ Q\in A_{Q''}}} p_Q(Q')p_Q(Q'')\nu_0(Q)^2 \\
&- \sum_{Q\in\mathcal{D}(r,\Lambda)} \sum_{Q'\in A_Q} \sum_{Q''\in A_Q} p_{Q'}(Q)p_{Q''}(Q)\nu_0(Q')\nu_0(Q'') \\
&= \sum_{\substack{Q',Q''\in\mathcal{D}(r,\Lambda)\\ Q'\sim Q''}} \sum_{\substack{Q\in\mathcal{D}(r,\Lambda)\\ Q',Q''\in A_Q}} p_{Q'}(Q)p_{Q''}(Q)(\nu_0(Q')-\nu_0(Q''))^2 + P \geq 0
\end{aligned}$$

since the remainder P (which is due to the occasionally very generous compensation factor N) is non-negative. This concludes the proof of (3.7).

Let α be the $\mathcal{L}^\Lambda$-measure of the m-dimensional unit ball. By [M, Theorem 2.12] $D(\nu_0, \mathcal{L}^\Lambda, x)$ exists and is finite for $\mathcal{L}^\Lambda$-almost all x. By Proposition 3.2 the same is true for ν_0-almost all x. By Remark 2.6 we can choose $D(\nu_0, \mathcal{L}^\Lambda)$ as smooth as we like by increasing $\tilde{\Lambda}$. In particular, it can be chosen to be uniformly continuous so that one can find $r_0 > 0$ such that $\nu_0(B(x,r))\alpha^{-1}r^{-m} \leq \max\{2D(\nu_0,\mathcal{L}^\Lambda,x),1\}$ for all $0 < r < r_0$ and $x \in \Omega_\Lambda$. Thus using Fatou's lemma, (3.6), the theorem of dominated convergence, and Theorem 2.5 together with Plancharel's formula [SW, Theorem 2.1 p. 16], we have

$$\begin{aligned}
\int \underline{D}(\nu_\epsilon, \mathcal{L}^\Lambda, x)d\nu_\epsilon(x) &= \int \liminf_{r\to 0} \nu_\epsilon(B(x,r))\alpha^{-1}r^{-m}d\nu_\epsilon(x) \\
&\leq \liminf_{r\to 0} \int \nu_\epsilon(B(x,r))\alpha^{-1}r^{-m}d\nu_\epsilon(x) \\
&\leq \liminf_{r\to 0} C \int \nu_0(B(x,\sqrt{m}r))\alpha^{-1}r^{-m}d\nu_0(x) \\
&= C(\sqrt{m})^m \int \underline{D}(\nu_0,\mathcal{L}^\Lambda,x)d\nu_0(x) \\
&= C' \int \underline{D}(\nu_0,\mathcal{L}^\Lambda,x)^2 d\mathcal{L}^\Lambda(x) < \infty.
\end{aligned}$$

Thus $\underline{D}(\nu_\epsilon, \mathcal{L}^\Lambda, x)$ is finite for ν_ϵ-almost all x. □

3.4. Theorem. *For $\mathcal{L}$-almost all ϵ the map T_ϵ has infinitely many SRB-measures.*

Proof. For all finite $\Lambda \subset \mathbb{Z}^d$, let

$$E_g(\Lambda) = \{\epsilon \in E \mid (\pi_\Lambda \circ A_\epsilon)_*\mu \text{ is absolutely continuous with respect to } \mathcal{L}^\Lambda\}.$$

By Propositions 3.2 and 3.3 and [M, Theorem 2.12] we get for all finite $\Lambda \subset \mathbb{Z}^d$

$$\mathcal{L}(E_g(\Lambda)) = 1.$$

Defining

$$E_g = \bigcap_{\substack{\Lambda \subset \mathbb{Z}^d \\ |\Lambda| < \infty}} E_g(\Lambda)$$

we have $\mathcal{L}(E_g) = 1$. Further, for all $\epsilon \in E_g$ the measure $(\pi_\Lambda \circ A_\epsilon)_*\mu$ is absolutely continuous with respect to $\mathcal{L}^\Lambda$ for all finite $\Lambda \subset \mathbb{Z}^d$. Since by (3.4) the measure $(A_\epsilon)_*\mu$ is different from the SRB-measure constructed by Bricmont and Kupiainen there are at least two SRB-measures. Instead of considering the standard Cantor set one can study the Cantor set where the first two intervals are chosen and the third one is removed. Defining g properly the above proofs work for both of these sets. Since at each direction one can choose either of these Cantor sets and each choice gives a different measure there are infinite number of SRB-measures. □

3.5. Remarks. *1) Note that if one takes any coupled map lattice which is close to T_0 in the sense that it has an invariant set close to $\mathcal{K}$, one can repeat the above arguments. Thus one can decompose a suitable space of coupled map lattices into leaves such that inside each leaf almost every system has infinitely many SRB-measures. Thus the uniqueness of the SRB-measure is a very atypical situation.*

2) One can use similar methods for coupled axiom A diffeomorphisms and show that typically the projections of a SRB-measure onto finite dimensional subsystems are absolutely continuous with respect to the corresponding Lebesgue measure.

3) Note that by Theorem 2.5 and Remark 2.6 the densities of $(\pi_\Lambda \circ A_\epsilon)_*\mu$ are smooth, in particular, Hölder continuous. The uniqueness proof of Bricmont and Kupiainen fails for these measures because there are regions where the density is zero (see (3.4)) and thus one cannot take the logarithm of the densities.

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