POROUS MEASURES ON $\mathbb{R}^n$: LOCAL STRUCTURE AND DIMENSIONAL PROPERTIES

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Abstract. We study dimensional properties of porous measures on $\mathbb{R}^n$. As a corollary of a theorem describing the local structure of nearly uniformly porous measures we prove that the packing dimension of any Radon measure on $\mathbb{R}^n$ has an upper bound depending on porosity. This upper bound tends to $n-1$ as porosity tends to its maximum value.

1. Introduction and preliminaries

Porosity is a quantity that describes irregularities of fractals. The study of dimensional properties featured by porous sets was pioneered by P. Mattila. In [M] he verified the existence of a non-increasing function which gives an upper bound for Hausdorff dimension of any set in $\mathbb{R}^n$ as a function of porosity. Furthermore, he showed that this upper bound tends to $n-1$ as porosity tends to its maximum value. In [S] A. Salli generalized the corresponding results for packing dimension and established the correct asymptotic behaviour for the upper bound.

For measures the following definition of porosity was introduced in [EJJ].

1.1. Definition. *The porosity of a Radon measure μ on $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by*

$$\operatorname{por}(\mu,x) = \lim_{\epsilon\downarrow 0}\liminf_{r\downarrow 0}\operatorname{por}(\mu,x,r,\varepsilon) \tag{1.1}$$

where for all $r,\varepsilon > 0$

$$\begin{aligned}\operatorname{por}(\mu,x,r,\varepsilon) = \sup\{p \ge 0 \mid {}& \textit{there is } z \in \mathbb{R}^n \textit{ such that } B(z,pr) \subset B(x,r)\\ & \textit{and } \mu(B(z,pr)) \le \varepsilon\mu(B(x,r))\}.\end{aligned} \tag{1.2}$$

The porosity of μ is

$$\begin{aligned}\operatorname{por}(\mu) &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}\operatorname{por}(\mu,x)\\ &= \inf\{s \ge 0 \mid \operatorname{por}(\mu,x) \le s \textit{ for } \mu\textit{-almost all } x \in \mathbb{R}^n\}.\end{aligned} \tag{1.3}$$

We will relate porosity of measures to packing dimension defined as follows.

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1.2. Definition. *Let μ be a Radon measure on $\mathbb{R}^n$. The packing dimension of μ is defined in terms of upper local dimensions*

$$(1.4)\quad \dim_{\mathrm{p}}(\mu)=\sup\{s\geq 0 \mid \limsup_{i\to\infty}\frac{\log\mu(D_i(x))}{\log 2^{-i}}\geq s \text{ for } \mu\text{-almost all } x\in\mathbb{R}^n\}$$

where $D_i(x)$ is the closed dyadic cube of side-length 2^{-i} containing x. Equivalently this definition can be given using packing dimensions of Borel sets with positive μ-measure

$$(1.5)\qquad \dim_{\mathrm{p}}(\mu)=\inf\{\dim_{\mathrm{p}}(A)\mid A \text{ is a Borel set with } \mu(A)>0\}.$$

Replacing "liminf" by "limsup" in (1.1) gives the upper porosity of a measure, which was studied by M. E. Mera and M. Morán in [MM]. They showed that if μ satisfies the doubling condition, that is,

$$\limsup_{r\to 0}\frac{\mu(B(x,2r))}{\mu(B(x,r))}<\infty$$

for μ-almost all $x\in\mathbb{R}^n$, then the upper porosity of μ is either 0 or 1/2. (Above $B(x,r)$ is the closed ball with radius r and centre x.) Furthermore, for any non-doubling measure the upper porosity equals 1. Note that the (lower) porosity may obtain any value between 0 and 1/2 for both doubling and non-doubling measures. The upper porosity is too weak for the purpose of obtaining a non-trivial upper bound for dimension; for any $p=0,1/2,1$ and $0\leq d\leq n$ there exists a Radon measure μ with the upper porosity equal to p and with both Hausdorff and packing dimension equal to d.

In this paper we will establish a connection between porosity and packing dimension for all Radon measures on $\mathbb{R}^n$. The case $n=1$ was studied in [JJ]. In [EJJ] the emphasis was given to doubling measures on $\mathbb{R}^n$. For such measures the porosity can be given in terms of porosities of Borel sets with positive measure:

$$\mathrm{por}(\mu)=\sup\{\mathrm{por}(A)\mid A \text{ is a Borel set with } \mu(A)>0\}.$$

(The doubling condition is necessary here, see [EJJ] for details.)

We will generalize the results of [JJ] to higher dimensions by verifying that the packing dimension of any Radon measure on $\mathbb{R}^n$ is bounded above by a function that depends on porosity and by showing that this upper bound goes to $n-1$ as porosity tends to its maximum value 1/2. In particular, the packing dimension of any Radon measure on $\mathbb{R}^n$ having porosity close to 1/2 cannot be much larger than $n-1$.

Our main tools are a dimension estimate obtained from the strong law of large numbers and a description of the local structure of nearly uniformly porous measures. The latter one states that for a given nearly uniformly porous measure any sufficiently small dyadic cube can be divided into three parts two of which having small measure and the remaining one being a narrow boundary of a convex set (see Theorem (2.1)).

We conclude this section by recalling the following lemma from [JJ] according to which we may replace any measure by a nearly uniformly porous measure when estimating packing dimension from above.

1.3. Lemma. *Assume that μ is a Radon measure on $\mathbb{R}^n$ such that* $\mathrm{por}(\mu) \geq p$. *Let $0 < \delta < 1$. Then there is a Radon measure μ_δ with compact support* $\mathrm{spt}(\mu_\delta) \subset \mathrm{spt}(\mu)$ *and with* $\dim_{\mathrm{p}}(\mu_\delta) \geq \dim_{\mathrm{p}}(\mu)$ *such that the following property holds: there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ there are a Borel set $B_{\delta,\varepsilon}$ and $r_{\delta,\varepsilon} > 0$ with $\mu_\delta(\mathbb{R}^n \setminus B_{\delta,\varepsilon}) \leq \delta \mu_\delta(\mathbb{R}^n)$ and*

$$\mathrm{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2}$$

for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$.

Proof. See [JJ, Lemma 2.2]. □

2. Local structure and dimensional properties

For all positive integers i we use the notation $\mathcal{D}_i$ for the family of dyadic cubes in $\mathbb{R}^n$ with side-length 2^{-i}. For all $Q \in \mathcal{D}_i$ and for all positive integers k, let $\mathcal{N}^k(Q) \subset \mathcal{D}_i$ be the family of the $(2k+1)^n$ neighbouring dyadic cubes of Q with side-length 2^{-i} being located symmetrically around Q (including Q itself).

For all $\delta > 0$, *a δ-plate* is a $\frac{\delta}{2}$-neighbourhood of an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. *An affine δ-boundary* of a polyhedron P is the union of parts of δ-plates clued on all faces of P such that the union of P and its affine δ-boundary is a polyhedron obtained by magnifying P.

The following theorem describes the local structure of porous measures by stating that in all sufficiently small dyadic cubes such measures are essentially concentrated on a narrow boundary of some convex set.

2.1. Theorem. *Assume that μ is a Radon measure on $[0,1]^n$ such that* $\mathrm{por}(\mu) \geq \frac{1}{2}(1-\beta)$ *for $0 \leq \beta \leq \frac{1}{18}$. Let K be a positive integer. For all $0 < \delta < \frac{1}{18}$ let μ_δ and ε_δ be as in Lemma 1.3. Let $0 < \varepsilon < \varepsilon_\delta$. Then there is a positive integer i_0 depending on K, δ, and ε such that for all $i \geq i_0$ any cube $Q \in \mathcal{D}_i$ can be divided into three disjoint (not necessarily non-empty) parts*

$$Q = E \cup P \cup I \tag{2.1}$$

where

$$\mu_\delta(E) \leq C_Q^K N \varepsilon$$

for an integer N depending on K, δ, and β and for $C_Q^K = \max_{D \in \mathcal{N}^K(Q)} \mu_\delta(D)$, P is an affine $C_{\beta,\delta} 2^{-i}$-boundary of a convex polyhedron with $C_{\beta,\delta} = 6K(\beta+\delta) + \frac{2n}{K(1-\beta-\delta)}$, and $I \subset Q \setminus B_{\delta,\varepsilon}$. Here $B_{\delta,\varepsilon}$ is as in Lemma 1.3.

Proof. Let $r_{\delta,\varepsilon}$ be as in Lemma 1.3 and let i_0 be the smallest integer such that $(1+K)2^{-i_0} < r_{\delta,\varepsilon}$. Consider an integer $i \geq i_0$. Note that for those $Q \in \mathcal{D}_i$ which do not intersect $B_{\delta,\varepsilon}$ equality (2.1) is trivial. Let $Q \in \mathcal{D}_i$ such that $Q \cap B_{\delta,\varepsilon} \neq \emptyset$. Setting $r_x = \mathrm{dist}(x, \partial Q) + K2^{-i}$ for all $x \in Q \cap B_{\delta,\varepsilon}$, we have $B(x, r_x) \subset \cup_{D \in \mathcal{N}^K(Q)} D$. Lemma 1.3 implies that for all $x \in Q \cap B_{\delta,\varepsilon}$ there is a ball B_x with radius $qr_x = \frac{1}{2}(1-\beta-\delta)r_x$ such that $B_x \subset B(x, r_x)$ and

$$\mu_\delta(B_x) \leq (2K+1)^n C_Q^K \varepsilon. \tag{2.2}$$

Set $a^{1/n} = 2K(1-2q)\alpha^{1/n}$ where $\alpha = \mathcal{L}^n(B(0,1))$ is the Lebesgue measure of the unit ball. Then there are integers $N(a) \geq N(a,Q) \geq 0$ for which there are $x_1, \dots, x_{N(a,Q)} \in Q \cap B_{\delta,\varepsilon}$ such that

$$\mathcal{L}^n\Big((B_{x_j} \cap Q) \setminus \bigcup_{k=1}^{j-1} B_{x_k}\Big) \geq a\mathcal{L}^n(Q)$$

for all $j = 1, \dots, N(a,Q)$ and

$$\mathcal{L}^n\Big((B_x \cap Q) \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big) < a\mathcal{L}^n(Q) \tag{2.3}$$

for all $x \in Q \cap B_{\delta,\varepsilon}$ such that $x \neq x_j$ for all $j = 1, \dots, N(a,Q)$. (We use the interpretation $\cup_{k=1}^{0} B_{x_k} = \emptyset$. In the case $N(a,Q) = 0$ we have $\mathcal{L}^n(B_x \cap Q) < a\mathcal{L}^n(Q)$ for all $x \in Q \cap B_{\delta,\varepsilon}$.)

Define

$$I_1 = \Big\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \operatorname{dist}\Big(y, \partial\Big(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big)\Big) > 2(\frac{a}{\alpha})^{1/n}2^{-i}\Big\}.$$

Then

$$B_x \cap I_1 = \emptyset \tag{2.4}$$

for all $x \in I_1 \cap B_{\delta,\varepsilon}$. In fact, assuming that $B_x = B_{x_j}$ for some $j = 1, \dots, N(a,Q)$, equality (2.4) holds. If $B_x \cap I_1 \neq \emptyset$ for some $x \in I_1 \cap B_{\delta,\varepsilon}$ with $B_x \neq B_{x_j}$ for all $j = 1, \dots, N(a,Q)$, then, as it will be indicated shortly, the set $(B_x \cap Q) \setminus \cup_{k=1}^{N(a,Q)} B_{x_k}$ contains a ball with radius $(a/\alpha)^{1/n}2^{-i}$ contradicting (2.3). To find such a ball, take $z \in B_x \cap I_1$. Then $B(z, 2(a/\alpha)^{1/n}2^{-i}) \subset Q \setminus \cup_{k=1}^{N(a,Q)} B_{x_k}$. Since $(a/\alpha)^{1/n}2^{-i} \leq \frac{1}{2}qr_x$, the ball B_x contains a ball with radius $(a/\alpha)^{1/n}2^{-i}$ having z on its boundary such that the centre of the ball belongs to the line going through z and the centre of B_x. Clearly this ball is a subset of $Q \setminus \cup_{k=1}^{N(a,Q)} B_{x_k}$. This completes the proof of (2.4).

Set

$$I_2 = \Big\{y \in Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k} \mid \operatorname{dist}\Big(y, \partial\Big(Q \setminus \bigcup_{k=1}^{N(a,Q)} B_{x_k}\Big)\Big) > 3(\frac{a}{\alpha})^{1/n}2^{-i}\Big\}.$$

Then $I_2 \subset Q \setminus B_{\delta,\varepsilon}$. To see this, assume that there exists $x \in I_2 \cap B_{\delta,\varepsilon}$. From (2.4) we get

$$\operatorname{dist}(x, \partial B_x) > (\frac{a}{\alpha})^{1/n}2^{-i} = 2K(1-2q)2^{-i}.$$

On the other hand $\operatorname{dist}(x, \partial B_x) \leq r_x(1-2q) \leq (K+1/2)(1-2q)2^{-i}$. Hence $I_2 \subset Q \setminus B_{\delta,\varepsilon}$.

Let $B_{x_k} = B(z_k, qr_{x_k})$. Since $\frac{n}{qK}2^{-i}$ is an upper bound for the height of a segment of any ball with radius qr_x having chord with length at most $\sqrt{n}2^{-i}$, the intersections of each of the annuli $B(z_k, qr_x + 3(a/\alpha)^{1/n}2^{-i}) \setminus B_{x_k}$ and Q can be included in a $C_{\beta,\delta}2^{-i}$-plate. Adding parts of the affine $C_{\beta,\delta}$-boundary of Q if necessary concludes the construction of P. Setting

$$I = I_2 \setminus P$$

and

$$E = Q \setminus (P \cup I) \subset Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k},$$

equality (2.1) follows since

$$\mu_\delta(Q \cap \bigcup_{k=1}^{N(a,Q)} B_{x_k}) \le N(a)(2K+1)^n C_Q^K \varepsilon$$

by (2.2). □

Let μ be a Radon probability measure on $[0,1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Letting k be a positive integer set $I = \{1, \dots, 2^{kn}\}$. For all positive integers j, denote by $\mathbf{I}^j$ the set of all j-term sequences of integers belonging to I and by $\mathbf{I}^\infty$ the corresponding set of infinite sequences, that is,

$$\mathbf{I}^j = \{(i_1, \dots, i_j) \mid i_l \in I \text{ for all } l = 1, \dots, j\}$$

and

$$\mathbf{I}^\infty = \{(i_1, i_2, \dots) \mid i_l \in I \text{ for all } l = 1, 2, \dots\}.$$

Divide $[0,1]^n$ into 2^{kn} dyadic subcubes, enumarate them and denote them by D_j, $j = 1, \dots, 2^{kn}$. Define $f : [0,1]^n \to [0,1]^n$ by setting $f(x) = 2^k x \bmod 1$. For each $x \in [0,1]^n$ we define a sequence $\mathbf{i}^x = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ such that $f^{l-1}(x) \in D_{i_l}$ for all $l = 1, 2, \dots$. Note that for all $x = (x_1, \dots, x_n)$ the sequence $\mathbf{i}^x$ is unique unless x_j is a dyadic rational for some $j = 1, \dots, n$. For a positive integer l and $\mathbf{i} = (i_1, i_2, \dots) \in \mathbf{I}^\infty$ let $\mathbf{i}|_l = (i_1, \dots, i_l) \in \mathbf{I}^l$ be the sequence of the first l digits of $\mathbf{i}$ and for all $j = 1, \dots, 2^{kn}$ let $\mathrm{n}_j(\mathbf{i}|_l)$ be the number of j's in $\mathbf{i}|_l$.

We can attach a sequence (P_l^μ) of probability measures on I to μ such that for all $j = 1, \dots, 2^{kn}$ $\mathrm{P}_l^\mu(\{j\})$ gives the probability that the l:th digit (in the above representation) of a random number (with respect to μ) in $[0,1]^n$ equals j, that is,

$$\mathrm{P}_l^\mu(\{j\}) = \sum_{\substack{(i_1,\dots,i_l)\in\mathbf{I}^l \\ i_l = j}} \mu(D_{i_1,\dots,i_l})$$

where $D_{i_1,\dots,i_l}$ is the closed dyadic subcube of $[0,1]^n$ of side-length 2^{-kl} consisting of points whose expansion begins with $(i_1, \dots, i_l)$. The measures P_l^μ are well-defined since $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. We use the notation P^μ for the product measure $\prod_{l=1}^\infty \mathrm{P}_l^\mu$ on the code space $\mathbf{I}^\infty$.

2.2. Proposition. *Let μ be a Radon probability measure on $[0,1]^n$ such that $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. Let $p \le 2^{-kn}$ and $L \in I$. Assume that $\limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^l \mathrm{P}_i^\mu(\{j\}) \le p$ for all $j = 1, \dots, L$. Then*

$$\dim_{\mathrm{p}} \mu \le -\frac{1}{\log 2^k}(Lp\log p + (1-Lp)\log(\frac{1-Lp}{2^{kn}-L})) =: \alpha(p, L).$$

Proof. The strong law of large numbers [Fe, X.7.1] gives for all $j = 1, \dots, L$ that

$$\limsup_{l\to\infty} \frac{1}{l}\, \mathrm{n}_j(\mathbf{i}|_l) \le p$$

for P^μ-almost all $\mathbf{i} \in \mathbf{I}^\infty$. Defining

$$E_{p,L} = \{x \in [0,1]^n \mid \limsup_{l\to\infty} \frac{1}{l}\, \mathrm{n}_j(\mathbf{i}^x|_l) \le p \text{ for all } j = 1, \dots, L\},$$

this implies that $\mu(E_{p,L}) = 1$. Since $E_{p,L}$ is a Borel set it is enough to prove that $\dim_{\mathrm{p}}(E_{p,L}) \le \alpha(p,L)$.

Let ρ be a probability measure on I such that $\rho(\{j\}) = p$ for all $j = 1, \dots, L$ and $\rho(\{j\}) = (1 - Lp)/(2^{kn} - L)$ for all $j = L+1, \dots, 2^{kn}$. Let ν be the image of the infinite product of the measures ρ under the map $\pi : \mathbf{I}^\infty \to [0,1]^n$. Note that since $p \le 2^{-kn}$ we have

$$-u \log p - (1-u) \log\big(\frac{1 - Lp}{2^{kn} - L}\big) \le \alpha(p,L) \log 2^k$$

for all $u \le Lp$. Let $x \in E_{p,L}$. The equality

$$\log \nu(D_{kl}(x)) = \log p \sum_{j=1}^{L} \mathrm{n}_j(\mathbf{i}^x|_l) + \log\big(\frac{1 - Lp}{2^{kn} - L}\big) \sum_{j=L+1}^{2^{kn}} \mathrm{n}_j(\mathbf{i}^x|_l)$$

gives

$$\liminf_{l\to\infty} \frac{1}{l} \log\big(\frac{\nu(D_{kl}(x))}{2^{-klt}}\big) \ge -\log 2^k \alpha(p,L) + t \log 2^k$$

where $D_{kl}(x)$ is the dyadic cube of side-length 2^{-kl} containing x. Thus if $t > \alpha(p,L)$ then $\liminf_{l\to\infty} \frac{\nu(D_{kl}(x))}{2^{-klt}} = \infty$, implying

$$\limsup_{l\to\infty} \frac{\log(\nu(D_{kl}(x)))}{\log 2^{-kl}} \le t.$$

By [Fa, Proposition 2.3 (d)] we get $\dim_{\mathrm{p}}(E_{p,L}) \le \alpha(p,L)$. □

Let k and i be positive integers. Dyadic cubes in $\mathcal{D}_{ki}$ form *a brood* if they belong to the same dyadic cube belonging to $\mathcal{D}_{k(i-1)}$. Note that each brood consists of 2^{kn} dyadic cubes. Given a measure μ on $\mathbb{R}^n$, order the cubes of every brood such that $\mu(D_j) \le \mu(D_{j+1})$ for all $j = 1, \dots, 2^{kn}$. Let $\mathcal{D}^j_{ki}(\mu)$ be the set of the j:th cubes of all broods.

2.3. Theorem. *Let μ be a Radon probability measure on $[0,1]^n$ such that* $\mathrm{por}(\mu) \ge \frac{1}{2}(1-\beta)$ *for $0 \le \beta \le \frac{1}{18}$ and $\mu(V) = 0$ for all affine hyperplanes $V \subset \mathbb{R}^n$. For all $0 < \delta < \frac{1}{18}$ let μ_δ be as in Lemma 1.3. Then there is an integer L with $2^{k(\beta,\delta)n} \ge L \ge 2^{k(\beta,\delta)n} - c2^{k(\beta,\delta)(n-1)}$ where c is a constant depending only on n and $k(\beta,\delta) \to \infty$ as $\delta \to 0$ and $\beta \to 0$ such that the following inequality is valid: for all $j = 1, \dots, L$ we have*

$$\limsup_{l\to\infty} \frac{1}{l} \sum_{i=1}^{l} \sum_{D \in \mathcal{D}^j_{ki}(\mu_\delta)} \mu_\delta(D) \le \delta \mu_\delta([0,1]^n).$$

Proof. Let k_0 be the largest integer such that $\beta+\delta \le 2^{-2k_0}$. Set $K=2^{k_0}$. Let $0<\varepsilon<\varepsilon_\delta$ and let $i \ge i_0$ where ε_δ and i_0 are as in Theorem 2.1. Let k be the largest integer such that $2^{-k} > 4(6+4n)2^{-k_0}$. Then $2^{-k} > 4C_{\beta,\delta}$ where $C_{\beta,\delta}$ is as in Theorem 2.1. Consider $Q \in \mathcal{D}_{ki}$. Let $Q = E_Q \cup P_Q \cup I_Q$ be the partition of Q given in Theorem 2.1. Take any $x \in P_Q$. Then $D_{k(i+1)}(x)$ and its neighbouring cubes in $\mathcal{D}_{k(i+1)}$ cover a part of P_Q such that the $\mathcal{L}^{n-1}$-measure of the covered part of both inner and outer boundary of P_Q is at least $2^{-(n-1)}2^{-k(i+1)(n-1)}$. By the convexity of P_Q the $\mathcal{L}^{n-1}$-measure of the outer boundary of P_Q is less than $2n2^{-ki(n-1)}$. Hence we need at most $3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes from $\mathcal{D}_{k(i+1)}$ to cover P_Q. Thus there are $L \ge 2^{kn} - 3^{2n}2n2^{n-1}2^{k(n-1)}$ cubes in $\mathcal{D}_{k(i+1)}$ which belong to $E_Q \cup I_Q$. Clearly $\mu_\delta(D_j) \le \mu_\delta(E_Q \cup I_Q)$ for all $j=1,\dots,L$. Theorem 2.1 and Lemma 1.3 give

$$\limsup_{l\to\infty} \frac{1}{l}\sum_{i=1}^{l} \sum_{D\in\mathcal{D}_{ki}^j(\mu_\delta)} \mu_\delta(D) \le \limsup_{l\to\infty}\frac{1}{l}\sum_{i=1}^{l}\sum_{Q\in\mathcal{D}_{ki}} \mu_\delta(E_Q\cup I_Q)$$

$$\le \limsup_{l\to\infty}\frac{1}{l}\sum_{i=1}^{l}((2K+1)^n N\varepsilon+\delta)\mu_\delta([0,1]^n) \xrightarrow[\varepsilon\to 0]{} \delta\mu_\delta([0,1]^n).$$

Since $k_0 \to \infty$ as δ and β tend to zero we may let k tend to infinity when $\delta \to 0$ and $\beta \to 0$. □

2.4. Corollary. *Assume that μ is a Radon measure on $\mathbb{R}^n$. If $0<\beta\le 1$ such that* $\operatorname{por}(\nu) \ge \frac{1}{2}(1-\beta)$, *then* $\dim_{\mathrm{p}}(\nu) \le d(\beta)$ *where $d(\beta)\to n-1$ as $\beta \to 0$.*

Proof. The claim follows from Theorem 2.3 and from the obvious generalization of [JJ, Lemma 3.3] (see [JJ, Corollary 3.4]) with $d(\beta)=\lim_{\delta\to 0}\alpha(\delta,L)$ where L is as in Theorem 2.3. Note that by the choices of k and k_0 in Theorem 2.3 we have $C_n(\beta+\delta)^{-1/2} \le 2^k \le \widetilde{C}_n(\beta+\delta)^{-1/2}$ for constants C_n and $\widetilde{C}_n$ depending only on n. By the lower and upper bounds given in Theorem 2.3 for L we obtain that $d(\beta)\to n-1$ when $\beta\to 0$. □

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2.5. Remark. *After finishing this paper we obtained the preprint* [BS] *from D. B. Beliaev and S. K. Smirnov where similar dimension results have been proved using different methods.*

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