QUASIHYPERBOLIC BOUNDARY CONDITIONS AND CAPACITY: POINCARÉ DOMAINS

PEKKA KOSKELA, JANI ONNINEN, AND JEREMY T. TYSON

ABSTRACT. We prove that a domain in $\mathbb{R}^n$ whose quasihyperbolic metric satisfies a logarithmic growth condition with coefficient $\beta \le 1$ is a (q,p)-Poincaré domain for all p and q satisfying $p \in [1,\infty) \cap (n - n\beta, n)$ and $q \in [p, \beta p^*)$, where $p^* = np/(n-p)$ denotes the Sobolev conjugate exponent. An elementary example shows that the given ranges for p and q are sharp. The proof makes use of estimates for a variational capacity and Frostman's theorem. When $p = 2$ we give an application to the solvability of the Neumann problem on domains with irregular boundaries. We also discuss the relationship between this growth condition on the quasihyperbolic metric and the s-John condition.

1. Introduction

Let Ω be a domain of finite volume in $\mathbb{R}^n$, $n \ge 2$, and let $1 \le p \le q < \infty$. We say that Ω is a (q,p)-*Poincaré domain* if there exists a constant $M_{q,p} = M_{q,p}(\Omega)$ so that

$$(1.1) \qquad \left(\int_\Omega |u(x) - u_\Omega|^q \, dx \right)^{1/q} \le M_{q,p} \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}$$

for all $u \in C^\infty(\Omega)$. Here $u_\Omega = |\Omega|^{-1} \int_\Omega u(x)\, dx$. When $q = p$ we say that Ω is a p-*Poincaré domain*.

It is a problem of some interest to determine geometric conditions on a domain Ω (possibly with a very irregular boundary) sufficient to guarantee the satisfaction of the Poincaré inequality (1.1). A number of geometric assumptions (cone/cusp conditions, John conditions, etc.) have been considered in this context. Our purpose in this paper is to study sufficient conditions for the Poincaré inequality in terms of another geometric assumption, namely, a growth condition on the quasihyperbolic metric in Ω.

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Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$. The *quasihyperbolic distance* between a pair of points $x, y \in \Omega$ is defined to be

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds}{\operatorname{dist}(z,\partial\Omega)},$$

where the infimum is taken over all curves γ in Ω joining x to y. This metric arises naturally in the theory of conformal geometry where, for example, it plays an important rôle in the study of the boundary behavior of quasiconformal maps. As another application, we mention the result of Jones [11] characterizing BMO-extension domains in terms of a growth condition on the quasihyperbolic metric. See the survey article [14] for further applications.

A connection between the quasihyperbolic metric and the global Poincaré inequality (1.1) was first demonstrated by Jerison [9], who proved that a planar domain Ω of finite area is a 2-Poincaré domain provided

$$\int_\Omega k_\Omega(x_0, x)\, dx < \infty$$

for some (each) $x_0 \in \Omega$. The analogous result in higher dimensions, due to Hurri [7] and Smith and Stegenga [24], states that a domain $\Omega \subset \mathbb{R}^n$ of finite volume is an n-Poincaré domain if

$$\int_\Omega k_\Omega(x_0, x)^{n-1}\, dx < \infty \tag{1.2}$$

for some (each) $x_0 \in \Omega$. A natural question then arises: can the integrability condition (1.2) be verified under some simpler geometric restriction on the quasihyperbolic metric?

Let $\beta > 0$. We say that Ω satisfies a *β-quasihyperbolic boundary condition* if the growth condition

$$k_\Omega(x_0, x) \leq \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} + C_0 \tag{1.3}$$

is satisfied for all $x \in \Omega$, where $x_0 \in \Omega$ is a fixed basepoint and $C_0 = C_0(x_0) < \infty$. Gehring and Martio [3] demonstrated a close connection between condition (1.3) and global regularity (specifically, Hölder continuity) of quasiconformal maps; their results generalize previous work of Becker and Pommerenke [1] for conformal maps in simply connected plane domains. By results from [24], $\int_\Omega k_\Omega(x_0, x)^p\, dx < \infty$ for all $p \geq 1$ whenever Ω satisfies (1.3) for some $\beta > 0$; it follows that such a domain is necessarily an n-Poincaré domain.

In fact, this result can be improved. Smith and Stegenga [25] (see also Hurri [7]) prove that (1.3) implies (1.1) for some $p < n$; this conclusion is stronger as can be verified using

Hölder's inequality. The exact infimum of the values of p for which (1.1) holds with $q = p$ has been unknown until now (see Theorem 1.4).

The results of [25] provide an estimate for p in terms of the Minkowski dimension $\dim_M \partial\Omega$ of the boundary of Ω, which is known to be strictly less than n in domains which satisfy (1.3). In fact, by an observation of Edwards and Hurri-Syrjänen [2], Ω is a p-Poincaré domain for all $p > \dim_M \partial\Omega$. An estimate for $\dim_M \partial\Omega$ in terms of β can be derived from the work of Jones and Makarov [12] when $n = 2$ and an explicit statement valid in all dimensions was given by Koskela and Rohde [16]. Specifically, one has $\dim_M \partial\Omega \geq n - c(n)\beta^{n-1}$, where $c(n) > 0$ is a constant depending only on n. In the case $n = 2$, Jones and Makarov further show that the estimate $\dim_M \partial\Omega \geq 2 - c\beta$ is essentially sharp in the following sense: there exist constants c_1 and c_2 with $0 < c_1 < c_2 < \infty$ so that $\dim_M \partial\Omega \geq 2 - c_2\beta$ whenever Ω satisfies (1.3) but there exist domains Ω satisfying (1.3) for which $\dim_M \partial\Omega < 2 - c_1\beta$.

The preceding paragraphs set the stage for our main result:

Theorem 1.4. *Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, satisfy the quasihyperbolic boundary condition (1.3) for some $\beta \leq 1$. Then Ω is a p-Poincaré domain provided $p \in [1, \infty) \cap (n - n\beta, n)$.*

For each $1 \leq p < n - n\beta$, there exist domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying (1.3) which are not p-Poincaré domains.

Thus the infimum of the values $p \geq 1$ for which (1.1) holds with $q = p$ is equal to $\max\{1, n - n\beta\}$. The boundary case $p = n - n\beta$ is still open, but we conjecture that Ω is also an $(n - n\beta)$-Poincaré domain in this setting.

In contrast with previous proofs of the Poincaré inequality (1.1) under the assumption of the quasihyperbolic boundary condition, we make no use of the Minkowski dimension of the boundary. We rely instead on an estimate for a variational capacity which is known to imply the Poincaré inequality in quite general situations, see [6]. The use of capacity estimates in the context of Sobolev-Poincaré inequalities is a major theme in work of Maz´ya [19], [20], [21], [22]. Our proof of these capacity estimates as a consequence of the quasihyperbolic boundary condition, however, is new. We use a chaining argument involving the classical Poincaré inequality on Whitney cubes together with Frostman's theorem. In a companion paper [15], we use this technique to solve an open problem on the global regularity of quasiconformal mappings.

More generally, we prove the following:

Theorem 1.5. *Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, satisfy the quasihyperbolic boundary condition (1.3) for some $\beta \leq 1$. Then Ω is a (q,p)-Poincaré domain provided $p \in [1,\infty) \cap (n - n\beta, n)$ and $q \in [p, \beta np/(n-p))$.*

For each $q > \beta np/(n-p)$, there exist domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfying (1.3) which are not (q,p)-Poincaré domains.

Recall that $p^* = np/(n-p)$ is the classical Sobolev conjugate exponent for $p < n$; the (p^*, p)-Poincaré inequality is the sharpest Sobolev embedding theorem which can possibly hold (and does in "sufficiently nice" domains).

The structure of this paper is as follows. In section 2 we review basic results on the quasihyperbolic metric and prove a number of technical lemmas relating to the geometry of Whitney cubes and quasihyperbolic geodesics. Section 3 contains the proof of the aforementioned capacity estimate which directly implies Theorem 1.5 by a result of Hajłasz and Koskela [6]. Here, as mentioned above, we make use of Frostman's theorem to construct measures in the domain Ω with prescribed growth behaviors. In fact, this ingredient in the proof is only necessary if we are interested in the (q,p)-Poincaré inequality for $q > p$; to prove the (p,p)-Poincaré inequality we may use the Lebesgue measure in Ω in place of the Frostman measure. Our technique thus yields a completely elementary proof of the p-Poincaré inequality in domains satisfying the quasihyperbolic boundary condition for the sharp range of exponents $p \in (n - n\beta, n)$.

In section 4 we consider two applications of our results. We discuss the relevance of Theorem 1.5 for the Neumann problem on general domains and we establish Poincaré inequalities for simply connected planar domains for which the Riemann map from the unit disc is globally Hölder continuous.

In section 5 we consider a related geometric condition on domains: the s-John ("twisted cusp") condition. Hajłasz and Koskela [6] have given sharp results for the problem of determining when s-John domains are (q,p)-Poincaré domains. We observe that domains which satisfy the quasihyperbolic boundary condition (1.3) for some $\beta \leq 1$ are $\frac{1}{\beta}$-John domains. In spite of this fact, Theorem 1.5 is not merely a special case of the results in [6]. Indeed, the domains which satisfy (1.3) for some $\beta \leq 1$ form a strict subclass of the $\frac{1}{\beta}$-John domains and Theorem 1.5 provides the sharp result for the question of when such domains are (q,p)-Poincaré domains — the range of allowable exponents q in Theorem 1.5 is larger than the corresponding range for $\frac{1}{\beta}$-John domains in [6]. We discuss all of this in detail in section 5, where we also generalize an example in [6] to prove the sharpness in Theorem 1.5 and also to provide examples of $\frac{1}{\beta}$-John domains which do not satisfy (1.3).

1.6. **Notations and definitions.** We denote by $\mathbb{R}^n$, $n \geq 1$, the Euclidean space of dimension n. For a cube $Q \subset \mathbb{R}^n$ with center x and side length $s(Q)$ and for a factor $\lambda > 0$, we denote by λQ the dilated cube which is again centered at x but has side length $\lambda s(Q)$. We denote the Lebesgue measure in $\mathbb{R}^n$ by m, although we usually abbreviate $dm(x) = dx$ and write $|A|$ for the Lebesgue measure of A.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Set $s(\Omega) = n^{-1/2} \operatorname{diam} \Omega$. We denote by $\mathcal{W} = \mathcal{W}(\Omega)$ a *Whitney decomposition* of the domain Ω into *Whitney cubes* Q, i.e., the cubes in $\mathcal{W}$ have pairwise disjoint interiors and vertices in the set

$$2^{-\mathbb{N}} s(\Omega) \cdot \mathbb{Z}^n := \left\{ \left(2^{-j} s(\Omega) l_1, \ldots, 2^{-j} s(\Omega) l_n\right) : j \in \mathbb{N}, l_1, \ldots, l_n \in \mathbb{Z} \right\}$$

and satisfy $\operatorname{diam} Q \leq \operatorname{dist}(Q, \partial\Omega) \leq 4 \operatorname{diam} Q$ for each $Q \in \mathcal{W}$. For the existence of such a decomposition, we refer to Stein's book [26, VI.1]. For any λ, $1 < \lambda < 5/4$, the expanded collection of cubes $\{\lambda Q : Q \in \mathcal{W}\}$ has bounded overlap, specifically,

$$\sup_{x \in \Omega} \sum_{Q \in \mathcal{W}} \chi_{\lambda Q}(x) \leq 12^n < \infty.$$

See, e.g., [26, VI.1.3, Proposition 3]. For $j \in \mathbb{N}$, we let $\mathcal{W}_j$ denote the collection of cubes $Q \in \mathcal{W}$ for which $\operatorname{diam} Q = 2^{-j} \operatorname{diam} \Omega$.

2. Preliminary results on the quasihyperbolic metric

Throughout this section, Ω will denote a proper subdomain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Recall that the *quasihyperbolic metric* k_Ω in Ω is defined to be

$$k_\Omega(x,y) = \inf_\gamma k_\Omega - \operatorname{length}(\gamma),$$

where the infimum is taken over all rectifiable curves γ in D which join x to y and

$$k_\Omega - \operatorname{length}(\gamma) = \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial\Omega)}$$

denotes the *quasihyperbolic length* of γ in D. This metric was introduced by Gehring and Palka in [5]. A curve γ joining x to y for which $k_\Omega - \text{length}(\gamma) = k_\Omega(x,y)$ is called a *quasihyperbolic geodesic*. Quasihyperbolic geodesics joining any two points of a proper subdomain of $\mathbb{R}^n$ always exist, see [4, Lemma 1]. If γ is a quasihyperbolic geodesic in Ω and $x', y' \in \gamma$, we denote by $\gamma(x',y')$ the portion of γ which joins x' to y'.

When x and y are sufficiently far apart, $k_\Omega(x,y)$ is roughly equal to the number $N(x,y)$ of Whitney cubes Q that intersect a quasihyperbolic geodesic γ joining x to y. More precisely,

$$N(x,y)/C \le k_\Omega(x,y) \le CN(x,y)$$

for all $x, y \in \Omega$ with $|x-y| \ge \text{dist}(x,\partial\Omega)/2$, where $C = C(n)$.

Let $\beta \in (0,1]$ and fix a basepoint $x_0 \in \Omega$. Following Gehring and Martio [3], we say that Ω satisfies a *β-quasihyperbolic boundary condition* if for some (each) $x_0 \in \Omega$ there exists a constant $C_0 = C_0(x_0) < \infty$ so that

$$k_\Omega(x_0,x) \le \frac{1}{\beta}\log\frac{\text{dist}(x_0,\partial\Omega)}{\text{dist}(x,\partial\Omega)} + C_0 \tag{2.1}$$

for all $x \in \Omega$. Then Ω is bounded, in fact $\operatorname{diam}\Omega \le (2/\beta)e^{C_0\beta}\,\text{dist}(x_0,\partial\Omega)$ by [3, Lemma 3.9]. The value of β is necessarily less than or equal to one as a consequence of the following simple estimate (c.f. [5]):

$$k_\Omega(x_0,x) \ge \log\frac{\text{dist}(x_0,\partial\Omega)}{\text{dist}(x,\partial\Omega)} \tag{2.2}$$

for all $x \in \Omega$.

The following result of Smith and Stegenga [24, Theorem 3] is fundamental to our work.

Lemma 2.3. *Let $\Omega \subsetneq \mathbb{R}^n$ satisfy the quasihyperbolic boundary condition (2.1). Then there exists a finite constant $C_1 = C_1(\beta, C_0)$ so that for all $x_1 \in \Omega$, we have*

$$k_\Omega(x_0,x) \le \frac{1}{\beta}\log\frac{\text{dist}(x_0,\partial\Omega)}{\text{length}(\gamma(x,x_1))} + C_1$$

whenever γ is a quasihyperbolic geodesic joining x_0 to x_1 and $x \in \gamma$.

For the remainder of this section, we assume that Ω satisfies the quasihyperbolic boundary condition (2.1) for some $\beta \le 1$. Our first lemma controls the number of Whitney cubes of a given size or larger which can intersect a given quasihyperbolic geodesic.

Lemma 2.4. *Let γ be a quasihyperbolic geodesic in Ω starting at the basepoint x_0. Then there exists a constant $C = C(n, \beta, C_0)$ so that*

$$\operatorname{card}\{Q \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j : Q \cap \gamma \neq \emptyset\} \leq Cj$$

for all $j \geq 1$. Here $\operatorname{card} S$ *denotes the cardinality of the set S.*

Proof. Assume that we have N Whitney cubes $Q_1, \ldots, Q_N$ satisfying $s(Q_i) \geq 2^{-j} \operatorname{diam} \Omega$ and $Q_i \cap \gamma \neq \emptyset$, $i = 1, \ldots, N$. Fix $\lambda = \frac{9}{8}$ so that the dilated cubes λQ_i have bounded overlap. If we let γ_i denote the part of the curve γ which lies in the cube λQ_i, then the quasihyperbolic lengths of the curves γ_i are uniformly bounded from below:

$$k_\Omega - \operatorname{length}(\gamma_i) \geq \frac{\operatorname{length}(\gamma \cap \lambda Q_i)}{\sup\{\operatorname{dist}(x, \partial\Omega) : x \in \lambda Q_i\}} \geq \frac{1}{C(n)} > 0$$

for $i = 1, \ldots, N$.

In order to apply Lemma 2.3, let $x_1 \in Q_N \cap \gamma$. If N is chosen sufficiently large relative to n, then one of the cubes λQ_i, $N/2 \leq i \leq N$, will be disjoint from λQ_N and hence will satisfy $\operatorname{dist}(Q_i, Q_N) \geq c2^{-j} \operatorname{diam} \Omega$ for some $c > 0$. Let x denote the terminal point of exit of γ from the cube Q_i. By Lemma 2.3,

$$\begin{aligned}
\frac{1}{C(n)} \frac{N}{2} &\leq \sum_{i=1}^{N/2} k_\Omega - \operatorname{length}(\gamma_i) \leq k_\Omega(x_0, x) \\
&\leq \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{length}(\gamma(x, x_1))} + C_1 \\
&\leq \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(Q_i, Q_N)} + C_0 \\
&\leq C(n, \beta, C_0) j.
\end{aligned}$$

The lemma follows. □

We now fix a Whitney cube Q_0 and assume that x_0 is the center of Q_0. For each cube $Q \in \mathcal{W}$, we choose a quasihyperbolic geodesic γ joining x_0 to the center of Q and we let $P(Q)$ denote the collection of all of the Whitney cubes $Q' \in \mathcal{W}$ which intersect γ. Then we define the *shadow* $S(Q)$ of the cube Q to be

$$S(Q) = \bigcup_{\substack{Q_1 \in \mathcal{W} \\ Q \in P(Q_1)}} Q_1.$$

Informally speaking, our next lemma says that the amount of overlap of the shadows of Whitney cubes of a fixed size is bounded.

Lemma 2.5. *There exists a finite constant $C = C(n, \beta, C_0)$ so that*

$$\sum_{Q \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j} \chi_{S(Q)}(x) \le Cj$$

for every $j \ge 1$ and $x \in \Omega$.

Proof. Since the Whitney collection $\mathcal{W}$ has bounded overlap, we may without loss of generality work with the (disjoint) interiors of the Whitney cubes. If $Q_1, \ldots, Q_N \in \mathcal{W}_1 \cup \cdots \cup \mathcal{W}_j$ are such that $F := S(Q_1) \cap \cdots \cap S(Q_N)$ is nonempty, then F contains an entire Whitney cube; in particular, it contains its center point x. But then the chosen quasihyperbolic geodesic joining x_0 to x intersects each of the cubes Q_i, $i = 1, \ldots, N$. Then the result follows from Lemma 2.4. □

We now estimate the size of the shadow of a Whitney cube Q in terms of the size of Q.

Lemma 2.6. *There exists $C = C(n, \beta, C_0)$ so that*

$$\operatorname{diam} S(Q) \le C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^\beta$$

for all $Q \in \mathcal{W}$.

Proof. We first show that $\operatorname{diam} Q_1 \le C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^\beta$ for each cube $Q_1 \subset S(Q)$. If $Q_1 = Q$ this is obvious so assume $Q_1 \ne Q$. Let x_1 denote the center of Q_1, let γ be a quasihyperbolic geodesic joining x_0 to x_1, and let x be any point in $Q \cap \gamma$. It is clear that the (Euclidean) length of that portion of γ which lies in Q_1 is at least $c \operatorname{diam} Q_1$ for some constant $c = c(n) > 0$. We apply Lemma 2.3 together with (2.2) to deduce that

$$\log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{dist}(x, \partial\Omega)} \le k_\Omega(x_0, x) \le \frac{1}{\beta} \log \frac{\operatorname{dist}(x_0, \partial\Omega)}{\operatorname{diam} Q_1} + C_1.$$

The desired result follows since $\operatorname{dist}(x, \partial\Omega) \approx \operatorname{diam} Q$.

It thus suffices to show that the set Z consisting of all of the centers of cubes contained in $S(Q)$ satisfies $\operatorname{diam} Z \le C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^\beta$. To this end, let $x_1, x_2 \in Z$. Choose points x_1' and x_2' in $\gamma_{x_1} \cap Q$ and $\gamma_{x_2} \cap Q$, respectively, where γ_x denotes the chosen quasihyperbolic geodesic joining x to x_0. Then

$$
\begin{aligned}
|x_1 - x_2| &\le \operatorname{length}(\gamma_{x_1}(x_1', x_1)) + \operatorname{diam} Q + \operatorname{length}(\gamma_{x_2}(x_2', x_2)) \\
&\le \operatorname{diam} Q + C \operatorname{dist}(x_0, \partial\Omega) e^{-\beta k_\Omega(x_0, x_1')} + C \operatorname{dist}(x_0, \partial\Omega) e^{-\beta k_\Omega(x_0, x_2')} \\
&\le \operatorname{diam} Q + C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} \operatorname{dist}(x_1', \partial\Omega)^\beta + C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} \operatorname{dist}(x_2', \partial\Omega)^\beta \\
&\le (\operatorname{diam} \Omega)^{1-\beta} (\operatorname{diam} Q)^\beta + C \operatorname{dist}(x_0, \partial\Omega)^{1-\beta} (\operatorname{diam} Q)^\beta
\end{aligned}
$$

by Lemma 2.3 and (2.2). Since $\operatorname{diam} \Omega \le C(\beta, C_0) \operatorname{dist}(x_0, \partial\Omega)$, the result follows. □

3. Proof of Theorem 1.5

The following general result characterizing Poincaré domains in terms of a capacity-type estimate is due to Maz´ya [22]; the formulation here is from [6, Theorem 1].

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^n$ and let $1 \le p \le q < \infty$. Then Ω is a (q,p)-Poincaré domain if and only if the following holds: for an arbitrary cube Q_0 compactly contained in Ω there exists a constant $M = M(\Omega, Q_0, p, q)$ so that*

$$
\int_\Omega |\nabla u(x)|^p \, dx \ge \frac{1}{M} |A|^{p/q} \tag{3.2}
$$

whenever A is an admissible subset of Ω which is disjoint from Q_0 and $u \in C^\infty(\Omega)$ satisfies $u|_A \ge 1$ and $u|_{Q_0} = 0$.

Here, we say that a subset $A \subset \Omega$ is *admissible* if A is open and if $\partial A \cap \Omega$ is a smooth submanifold.

We will use Theorem 3.1 to prove Theorem 1.5 in the case $p > 1$; for the case $p = 1$ see section 5. In fact, we will prove the following significantly more general statement.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with diameter one and let $0 < \beta \le 1$. Let $\max\{1, n - n\beta\} < p < n$ and $p \le q < \beta np/(n-p)$. Fix a Whitney cube $Q_0 \subset \Omega$ and assume that there exists a finite constant $\tilde{C}$ and a constant $\lambda < \frac{\beta}{p-1}\left(\frac{np}{q} - \frac{n-p}{\beta}\right)$ so that*

$$
\sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(x) \le \tilde{C} 2^{\lambda j} \tag{3.4}
$$

for all $j \geq 1$ and $x \in \Omega$ and that

$$\operatorname{diam} S(Q) \leq \tilde{C}(\operatorname{diam} Q)^\beta \tag{3.5}$$

for all $Q \in \mathcal{W}$, where the shadow $S(Q)$ of the Whitney cube Q is defined as before relative to the fixed cube Q_0. Then Ω is a (q,p)-Poincaré domain.

Recall that $\operatorname{diam}\Omega \leq C(\beta, C_0)\operatorname{dist}(x_0, \partial\Omega)$ if Ω satisfies (2.1); thus in order to derive Theorem 1.5 from Theorem 3.3 we may scale the domain Ω to have diameter one.

Note also that if Ω satisfies the quasihyperbolic boundary condition (2.1), then Lemmas 2.5 and 2.6 show that the hypotheses of Theorem 3.3 are satisfied. Indeed, the conclusion of Lemma 2.5 (the maximal overlap among Whitney cubes down to the level j grows at most linearly in j) is significantly stronger than what is needed in (3.4) (the overlap among the Whitney cubes at the level j grows at most exponentially in j). Thus Theorem 3.3 potentially applies to a much greater class of domains. In fact, it is not difficult to construct examples of domains for which conditions (3.4) and (3.5) are satisfied but for which the quasihyperbolic boundary condition (2.1) is not satisfied. We include one such example following the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $\Omega \subset \mathbb{R}^n$ be a domain with diameter one which satisfies conditions (3.4) and (3.5) for some $\beta \in (0,1]$. Fix p and q satisfying $\max\{1, n - n\beta\} < p < n$ and $p \leq q < \beta np/(n-p)$. We will verify that the conclusion of Theorem 3.1 is satisfied for each admissible subset $A \subset \Omega$ which is disjoint from Q_0 and any $u \in C^\infty(\Omega)$ satisfying $u|_A \geq 1$ and $u|_{Q_0} = 0$. Let u be such a function. We divide the set A into a "good set"

$$A_g := \{x \in A : u_Q \leq \tfrac{1}{2} \text{ for some Whitney cube } Q \ni x\}$$

and a "bad set"

$$A_b := \{x \in A : u_Q \geq \tfrac{1}{2} \text{ for some Whitney cube } Q \ni x\}.$$

(These sets may overlap slightly along the boundaries of Whitney cubes but this is immaterial to the ensuing discussion.)

For points x in the good set A_g the standard Poincaré inequality on cubes provides a trivial estimate

$$\frac{1}{2}|A \cap Q|^{\frac{1}{p^*}} \leq \left(\int_Q |u - u_Q|^{p^*}\right)^{\frac{1}{p^*}} \leq C\left(\int_Q |\nabla u|^p\right)^{\frac{1}{p}},$$

where $p^* = \frac{np}{n-p}$ and $Q \ni x$ is a Whitney cube. Since $q < p^*$ this yields

$$\int_Q |\nabla u|^p \geq \frac{1}{C}|A \cap Q|^{\frac{p}{q}}$$

and by summing over all such Whitney cubes we deduce that

$$\int_\Omega |\nabla u|^p \geq \frac{1}{C}|A_g|^{\frac{p}{q}}. \tag{3.6}$$

The bad set A_b requires more work and it is here where we will make use of the various assumptions on Ω. We will prove that

$$\int_\Omega |\nabla u|^p \geq \frac{1}{C}|A_b|^{\frac{p}{q}} \tag{3.7}$$

and then (3.2) follows by adding together (3.6) and (3.7).

To see why (3.7) holds, choose a Frostman measure μ on A_b relative to the growth function $\varphi(r) = r^{np/q}$, i.e., μ is a Borel measure supported on A_b satisfying the conditions

$$\mu(A_b \cap B(x,r)) \leq r^{np/q} \tag{3.8}$$

for all balls $B(x,r)$ and

$$\mu(A_b) \geq \frac{1}{C(n)}\mathcal{H}^\infty_{np/q}(A_b) \geq \frac{1}{C(n)}|A_b|^{p/q} \tag{3.9}$$

where $\mathcal{H}^\infty_s$ denotes s-dimensional Hausdorff content; $\mathcal{H}^\infty_s(A_b) = \inf \sum_i r_i^s$, where the infimum is taken over all coverings of A_b with balls $B(x_i, r_i)$, $i = 1, 2, \ldots$. See, for example, [18, Theorem 8.8]. Note that in the case when $q = p$ the measure μ can be taken to be Lebesgue measure restricted to the set A_b and the use of Frostman's theorem can be avoided.

For each $x \in A_b$, let $Q(x)$ denote a Whitney cube containing x for which $⨍_{Q(x)} u(y)\,dy \geq \frac{1}{2}$. Then a straightforward chaining argument involving the Poincaré inequality (c.f. [9, pp. 519-520] or [25, Lemma 8]) shows that

$$1 \leq C \sum_{Q \in P(Q(x))} \operatorname{diam} Q ⨍_Q |\nabla u(y)|\,dy; \tag{3.10}$$

recall that $P(Q(x))$ consists of the collection of all of the Whitney cubes which intersect the quasihyperbolic geodesic joining x_0 to the center of $Q(x)$.

Integrating (3.10) over the set A_b with respect to the Frostman measure μ and applying Hölder's inequality yields

$$\mu(A_b) \le C \int_{A_b} \sum_{Q \in P(Q(x))} (\operatorname{diam} Q)^{1-n/p} \left(\int_Q |\nabla u(y)|^p \, dy \right)^{1/p} d\mu(x).$$

We now interchange the order of summation and integration to deduce that

$$\mu(A_b) \le C \sum_{Q \in \mathcal{W}} \mu(S(Q) \cap A_b) (\operatorname{diam} Q)^{1-n/p} \left(\int_Q |\nabla u(y)|^p \, dy \right)^{1/p}$$

Applying Hölder's inequality again leads to

$$\begin{aligned}
\mu(A_b) &\le C \left(\sum_{Q \in \mathcal{W}} \mu(S(Q) \cap A_b)^{\frac{p}{p-1}} (\operatorname{diam} Q)^{(1-\frac{n}{p})(\frac{p}{p-1})} \right)^{1-1/p} \\
&\qquad \times \left(\sum_{Q \in \mathcal{W}} \int_Q |\nabla u(y)|^p \, dy \right)^{1/p} \\
&\le C \left(\sum_{Q \in \mathcal{W}} \mu(S(Q) \cap A_b)^{1+1/(p-1)} (\operatorname{diam} Q)^{-\frac{n-p}{p-1}} \right)^{1-1/p} \\
&\qquad \times \left(\int_\Omega |\nabla u(y)|^p \, dy \right)^{1/p}.
\end{aligned} \tag{3.11}$$

Set $\delta = \frac{1}{p-1}$ and $\epsilon = \frac{n-p}{\beta}$. We require an estimate for terms of the form

$$\sum_{Q \in \mathcal{W}} \mu(S(Q) \cap A_b)^{1+\delta} (\operatorname{diam} Q)^{-\beta\delta\epsilon},$$

which we give in the following lemma:

Lemma 3.12. *Let Ω be a domain in $\mathbb{R}^n$ with diameter one and let $0 < \delta < \infty$ and $0 < \epsilon < \nu \le n$. Assume that Ω satisfies (3.5) for some $0 < \beta \le 1$ and that Ω satisfies condition (3.4) for some $\lambda < \beta\delta(\nu - \epsilon)$. Let μ be a Borel measure on $\mathbb{R}^n$ which satisfies the growth condition $\mu(B(x,r)) \le r^\nu$. Then there exists a constant C so that*

$$\sum_{Q \in \mathcal{W}} \mu(S(Q) \cap E)^{1+\delta} (\operatorname{diam} Q)^{-\beta\delta\epsilon} \le C \mu(E)$$

whenever $E \subset \Omega$.

We defer the proof of this lemma momentarily. To complete the proof of Theorem 1.5, we apply Lemma 3.12 in (3.11) with $\delta = \frac{1}{p-1}$, $\epsilon = \frac{n-p}{\beta}$, $\nu = \frac{np}{q}$, and $E = A_b$ to see that

$$\mu(A_b) \leq C\mu(A_b)^{1-1/p} \left(\int_\Omega |\nabla u(y)|^p \, dy \right)^{1/p}$$

for some $C = C(n, p, q, \beta, \lambda, \tilde{C})$. By (3.9),

$$\int_\Omega |\nabla u(y)|^p \, dy \geq \frac{1}{C}\mu(A_b) \geq \frac{1}{M}|A_b|^{p/q}$$

for some finite constant $M = M(n, p, q, \beta, \lambda, \tilde{C})$. This proves (3.7) and hence also (3.2). The proof of the theorem is now complete as a consequence of Theorem 3.1. □

Proof of Lemma 3.12. The growth condition on μ together with (3.5) and (3.4) yield

$$\begin{aligned}
\sum_{Q \in \mathcal{W}} \mu(S(Q) \cap E)^{1+\delta} (\operatorname{diam} Q)^{-\beta\delta\epsilon} &\leq \sum_{j=1}^{\infty} \sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E)(\operatorname{diam} S(Q))^{\nu\delta} (\operatorname{diam} Q)^{-\beta\delta\epsilon} \\
&\leq C \sum_{j=1}^{\infty} \sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E)(\operatorname{diam} Q)^{\beta\delta(\nu-\epsilon)} \\
&\leq C \sum_{j=1}^{\infty} 2^{-j\beta\delta(\nu-\epsilon)} \sum_{Q \in \mathcal{W}_j} \mu(S(Q) \cap E) \\
&\leq C\mu(E) \sum_{j=1}^{\infty} 2^{-j(\beta\delta(\nu-\epsilon)-\lambda)} \\
&\leq C\mu(E).
\end{aligned}$$

□

Example 3.13. We construct a planar domain Ω which satisfies conditions (3.4) and (3.5) but does not satisfy the quasihyperbolic boundary condition (2.1). Fix $\frac{1}{2} < \beta < 1$ and $2 - 2\beta < 1 \leq p < 2$. For simplicity we only consider the case when $q = p$. We will construct a domain $\Omega \subset \mathbb{C}$ for which

$$\sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(x) \leq \tilde{C} 2^{\lambda j} \tag{3.14}$$

for all j and for some

$$\lambda < \frac{\beta}{p-1}\left(2 - \frac{2-p}{\beta}\right) = \frac{p-2+2\beta}{p-1}. \tag{3.15}$$

and

$$\operatorname{diam} S(Q) \le \tilde{C}(\operatorname{diam} Q)^{\beta} \tag{3.16}$$

for all $Q \in \mathcal{W}$, but Ω fails to satisfy the consequence of the quasihyperbolic boundary condition given in Lemma 2.3, namely,

$$k_{\Omega}(z_0, z) \le \frac{1}{\beta} \log \frac{\operatorname{dist}(z_0, \partial\Omega)}{\operatorname{length}(\gamma(z, z_1))} + C_1 \tag{3.17}$$

for all quasihyperbolic geodesics γ joining z_0 to z_1 in Ω and all points $z \in \gamma$. The domain Ω will consist of the union of an infinite sequence of concentric annular regions. We open small "gates" in the boundaries of these regions in order to make Ω connected but we also insert "walls" within each annular region to force certain quasihyperbolic geodesics in Ω to stay close to the boundary over a very long (Euclidean) length. This will guarantee that (3.17) is violated. The final domain Ω will (essentially) approximate a spiral domain centered at the origin.

For each $i = 0, 1, 2, \ldots$, let $A_i = \{z \in \mathbb{C} : 2^{-i-1} < |z| < 2^{-i}\}$. Next, for each $i = 1, 2, \ldots$ and each $k = 1, 2, \ldots, N_i$, let $C_{ik} \subset A_i$ be concentric annular regions with width $\approx 2^{-\frac{i}{\beta}}$. Thus the number N_i of these regions is approximately equal to $2^{i(\frac{1}{\beta}-1)}$. For each of the annular regions C_{ik}, open a "gate" G_{ik} of size $\approx 2^{-\frac{i}{\beta}}$ in the outer wall of C_{ik} and insert a wall W_{ik} inside C_{ik} joining the outer and inner boundaries of C_{ik}. Let Ω be the resulting domain:

$$\Omega = A_0 \cup \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{N_i} (C_{ik} \cup G_{ik} \setminus W_{ik}).$$

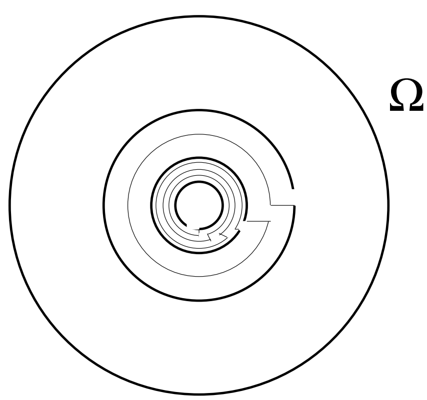

FIGURE 1. A "maze-like" domain Ω

We may arrange the gates and walls so that any curve in Ω which passes from $C_{i,k-1}$ to $C_{i,k+1}$ must travel almost all of the way around the annular region A_i. The construction is easiest to comprehend by looking at the picture in Figure 1.

Fix a basepoint $z_0 \in A_0$. We claim first that (3.16) holds for each Whitney cube Q. It suffices to verify this for Whitney cubes $Q \subset A_i$ which satisfy $\operatorname{diam} Q \approx 2^{-\frac{i}{\beta}}$. But it is immediately clear from the picture that in this case $S(Q) \subset B(0, 2^{-i})$ and so $\operatorname{diam} S(Q) \leq \operatorname{diam} A_i \leq 2^{-i+1} \leq \tilde{C}(\operatorname{diam} Q)^{\beta}$.

Next, we show that the overlap condition (3.14) holds for some $\lambda > 0$. The maximal overlap among the shadows of Whitney cubes $Q \in \mathcal{W}_j$ is obtained when the cubes line up along the center of the regions C_{ik} as in Figure 2(a). Note that such a Whitney cube in C_{ik} must have diameter $\approx 2^{-\frac{i}{\beta}}$. Since also $\operatorname{diam} Q \approx 2^{-j}$, we have $i \approx \beta j$. It is clear that these cubes can only appear in a bounded number of regions $A_{i_0}, A_{i_0+1}, \ldots, A_{i_0+N}$. Within any one of these regions A_i, the total number of Whitney cubes $Q \in \mathcal{W}_j$ is at most N_i times the number of cubes in any of the regions C_{ik}, which is again approximately equal to $2^{i(\frac{1}{\beta}-1)}$. Thus the total number of Whitney cubes of a given size is at most

$$C \times 2^{i(\frac{1}{\beta}-1)} \times 2^{i(\frac{1}{\beta}-1)} = C2^{2i(\frac{1}{\beta}-1)}$$

and this yields the desired upper estimate for the overlap of the shadows $S(Q)$, $Q \in \mathcal{W}_j$,

$$\sum_{Q \in \mathcal{W}_j} \chi_{S(Q)}(z) \leq C2^{\lambda j},$$

where $\lambda = 2(1-\beta)$.

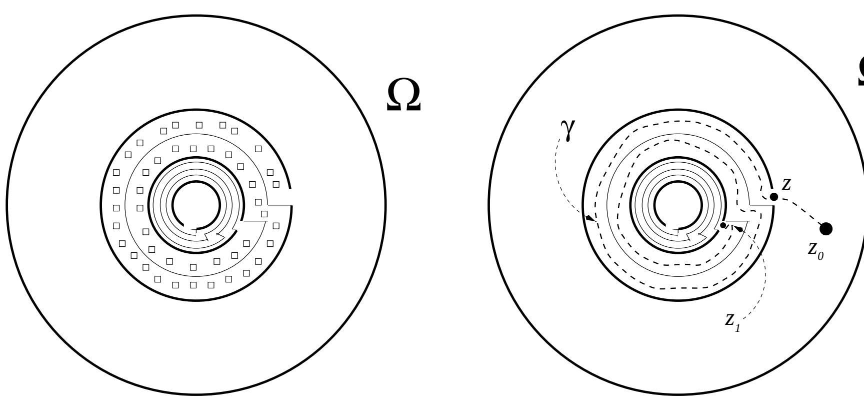

FIGURE 2. (a) Maximal collection of Whitney cubes of a fixed size with overlapping shadows; (b) A quasihyperbolic geodesic γ which threads the maze Ω

We next verify that Ω does not satisfy the quasihyperbolic boundary condition (2.1). Fix some $i \in \mathbb{N}$. Let z_1 be some point contained in the final gate $G_{i+1,i}$ exiting from the region A_i and let γ be the quasihyperbolic geodesic joining z_1 back to the basepoint z_0. Let $z \in \gamma$ be a point contained in the first gate $G_{i,1}$ entering the region A_i. See Figure 2(b). Then the length of the portion of γ joining z to z_1 is at least N_i times the length of γ within any of the regions C_{ik} and the latter quantity is comparable to 2^{-i}. Thus

$$\mathrm{length}(\gamma(z,z_1)) \geq \frac{1}{C}2^{i(\frac{1}{\beta}-1)}2^{-i} = \frac{1}{C}2^{i(\frac{1}{\beta}-2)}.$$

On the other hand,

$$\mathrm{dist}(z,\partial\Omega) \leq 2^{-\frac{i}{\beta}}.$$

If Ω did satisfy the quasihyperbolic boundary condition (2.1), then (3.17) would hold, i.e.,

$$\log\frac{\mathrm{dist}(z_0,\partial\Omega)}{\mathrm{dist}(z,\partial\Omega)} \leq k_\Omega(z_0,z) \leq \frac{1}{\beta}\log\frac{\log(\mathrm{dist}(z_0,\partial\Omega)}{\mathrm{length}(\gamma(z,z_1))} + C_1$$

which would imply that

$$\frac{1}{\beta}\log 2\cdot i - C \leq -\frac{1}{\beta}\left(\frac{1}{\beta}-2\right)\log 2\cdot i + C_1,$$

i.e.,

$$\frac{1}{\beta}\left(\frac{1}{\beta}-1\right)\log 2\cdot i \leq C$$

for all i. Our assumption $\beta < 1$ guarantees that this cannot occur.

The final thing which we must check is that our choice of $\lambda = 2(1-\beta)$ satisfies the required assumption (3.15), i.e.,

$$2(1-\beta) < \frac{p-2+2\beta}{p-1},$$

but this follows from our assumption $\beta > \frac{1}{2}$.

4. Applications

We discuss two applications of our main result, first, for simply connected planar domains for which the Riemann map from the unit disc is uniformly Hölder continuous, and second, for the theory of the Neumann problem in domains satisfying (2.1).

4.1. **Conformal maps of simply connected plane domains.** Let $\Omega' \subsetneq \mathbb{C}$ be simply connected and let $f : \mathbb{D} \to \Omega'$ be a conformal mapping with $f(0) = z_0$. Recall that the *hyperbolic distance* from z_0 to a point $z \in \Omega'$ is defined to be

$$\rho_{\Omega'}(z_0, z) = \rho_{\mathbb{D}}(0, f^{-1}(z)) = \log \frac{1 + |f^{-1}(z)|}{1 - |f^{-1}(z)|}.$$

The Koebe Distortion Theorem implies that

$$\frac{1}{1-|\zeta|^2} \leq \frac{|f'(\zeta)|}{\operatorname{dist}(f(\zeta), \partial\Omega')} \leq \frac{4}{1-|\zeta|^2}, \qquad \zeta \in \mathbb{D},$$

(see [23, 4.6(5)]) which in turn implies that

$$\frac{1}{2}\rho_{\Omega'}(z_0, z) \leq k_{\Omega'}(z_0, z) \leq 2\rho_{\Omega'}(z_0, z) \tag{4.2}$$

for $z \in \Omega'$ (see [23, 4.6(5)]). (Note that our definition of the hyperbolic metric differs from that of [23, Section 4.6] by a factor of two.)

Assume now that $f : \mathbb{D} \to \Omega'$ is uniformly Hölder continuous with exponent $\alpha \leq 1$. By Becker and Pommerenke [1], this is equivalent to the requirement that the hyperbolic metric in Ω' satisfy the growth condition

$$\rho_{\Omega'}(z_0, z) \leq \frac{1}{\alpha} \log \frac{\operatorname{dist}(z_0, \partial\Omega')}{\operatorname{dist}(z, \partial\Omega')} + C_0.$$

By (4.2), it follows that the quasihyperbolic metric satisfies

$$k_{\Omega'}(z_0, z) \leq \frac{2}{\alpha} \log \frac{\operatorname{dist}(z_0, \partial\Omega')}{\operatorname{dist}(z, \partial\Omega')} + C_0$$

and so Ω' is a p-Poincaré domain for all $2 - \alpha < p < 2$ by Theorem 1.4.

On the other hand, for each $\alpha \leq 1$ the "room-and-corridor" domain considered in section 5) provides an example of a simply connected domain $\Omega' \subset \mathbb{C}$ which is not a p-Poincaré domain for any $p < 2 - 2\alpha$ but for which a conformal map $f : \mathbb{D} \to \Omega'$ is uniformly α-Hölder continuous. Note the gap between the positive result ($p > 2 - \alpha$) and the negative examples ($p < 2 - 2\alpha$) which arises because we switch between the hyperbolic and quasihyperbolic metrics using the Koebe Distortion Theorem. We do not know what is the exact range of p's for which such a domain Ω' is a p-Poincaré domain, but we conjecture that a version of our argument in section 3 can be carried out for the hyperbolic metric and that the sharp value is again $p = 2 - 2\alpha$ in this case.

4.3. **The Neumann problem.** Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, be as before and let $A : \Omega \to \mathbb{R}^{n\times n}$ be a measurable function taking values in the space of $n \times n$ matrices satisfying the following two conditions:

(i) $A(x)^T = A(x)$ for a.e. $x \in \Omega$;
(ii) there exists $C \geq 1$ so that $|\xi|^2/C \leq \xi^T A(x)\xi \leq C|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$.

Fix $q \in [1,\infty)$. The *Neumann problem* associated with A is the elliptic PDE generated by the operator $L_A : L^q(\Omega) \cap W^{1,2}(\Omega) \to L^{q'}(\Omega)$ given by

$$u \mapsto \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right), \tag{4.4}$$

where $A(x) = (a_{ij}(x))_{i,j}$. We interpret (4.4) in the weak sense: $L_A u$ is the function in $L^{q'}(\Omega)$, $q' = q/(q-1)$, which satisfies

$$\int_\Omega v(x) L_A u(x)\, dx = \int_\Omega \nabla u(x)^T A(x) \nabla v(x)\, dx$$

for all $v \in L^q(\Omega) \cap W^{1,2}(\Omega)$.

We say that the Neumann problem associated with A is *q-solvable* if the following holds: for each $w \in L^{q'}(\Omega)$ with $\int_\Omega w(x)\, dx = 0$ there exists $u \in L^q(\Omega) \cap W^{1,2}(\Omega)$ so that

$$L_A u = w. \tag{4.5}$$

By [22, Lemma 4.10.1], the Neumann problem for A is q-solvable for some $2 \leq q < \infty$ if and only if Ω is a $(q,2)$-Poincaré domain. Furthermore, the spectrum of the operator $L_A : L^2(\Omega) \cap W^{1,2}(\Omega) \to L^2(\Omega)$ is discrete if and only if the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. By a general version of the Rellich-Kondrachov compactness theorem (see e.g. [22, Corollary 4.8.3.3] or [6, Theorem 5]), this holds provided $W^{1,2}(\Omega)$ embeds boundedly into $L^q(\Omega)$ for some $q > 2$, that is, provided Ω is a $(q,2)$-Poincaré domain for some $q > 2$. Putting all of this together yields the following corollary to Theorem 1.5.

Corollary 4.6. *Let $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, satisfy the quasihyperbolic boundary condition (2.1) for some $1 - \frac{2}{n} < \beta \leq 1$. Then the Neumann problem (4.5) on Ω is q-solvable for each $q \in [2, \frac{2\beta n}{n-2})$ and the operator $L_A : L^2(\Omega) \cap W^{1,2}(\Omega) \to L^2(\Omega)$ has a discrete spectrum.*

Conversely, for each $q > \frac{2\beta n}{n-2}$ there exist domains $\Omega \subsetneq \mathbb{R}^n$ which satisfy (2.1) with exponent β but for which the Neumann problem (4.5) is not q-solvable.

5. Comparison between the quasihyperbolic boundary condition and the s-John condition

For the sake of brevity, we say that a domain $\Omega \subsetneq \mathbb{R}^n$, $n \geq 2$, is a *β-QHBC domain* if it satisfies the quasihyperbolic boundary condition (2.1) for $\beta \leq 1$. In this section, we compare our results for β-QHBC domains with some known results for domains satisfying a weaker geometric assumption: the s-John (or "twisted cusp") condition.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. We say that Ω is an *s-John domain*, $s \geq 1$, if for some (each) $x_0 \in \Omega$ there is a constant $C_0' = C_0'(x_0) < \infty$ so that each point $x \in \Omega$ can be joined to x_0 (within Ω) by a rectifiable curve (called a *John curve*) $\gamma : [0, l] \to \Omega$, $\gamma(0) = x$, $\gamma(l) = x_0$, parameterized by arc length, and such that the distance to the boundary satisfies

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \frac{1}{C_0'} t^s \tag{5.1}$$

for all $t \in [0, l]$. If $s = 1$ then we say, for simplicity, that Ω is a John domain. John domains were introduced by Martio and Sarvas [17]. F. John [10] had earlier considered a similar class of domains.

It is well-known that a John domain is a p-Poincaré domain for all $p \geq 1$ [8, Theorem 3]. Furthermore, every John domain is a β-QHBC domain for an appropriate choice of $\beta > 0$. The class of s-John domains for a fixed $s > 1$ is in general much larger than the class of John domains. Smith and Stegenga [25, Thereom 10] proved that each s-John domain is a p-Poincaré domain provided $p \in [1, \infty) \cap ((n-1)(s-1), n)$ and Hajłasz and Koskela [6, Corollary 6] extended this to show that each s-John domain is a (q,p)-Poincaré domain provided $p \in [1, \infty) \cap ((n-1)(s-1), n)$ and

$$1 \leq p \leq q \leq \frac{np}{(n-1)s - (p-1)}. \tag{5.2}$$

Note that [6, Corollary 6] only covers the case $q < \frac{np}{(n-1)s-(p-1)}$; the extension to the borderline case is due to Kilpeläinen and Malý [13].

Note that a β-QHBC domain $\Omega \subsetneq \mathbb{R}^n$ is a $\frac{1}{\beta}$-John domain. This is easy to prove, indeed, the John curve may be taken to be the quasihyperbolic geodesic joining x_0 to x and then (5.1) follows from Lemma 2.3 with $C_0' = e^{C_1} \operatorname{dist}(x_0, \partial\Omega)^{(1/\beta)-1}$. Thus the fact that β-QHBC domains are (q,p)-Poincaré domains for some values of p and q can already be derived from the results of [6]. However, the sharp result (modulo the endpoint estimate) for such domains is Theorem 1.5: a β-QHBC domain is a (q,p)-Poincaré domain for all p and q satisfying

$$1 \le p \le q < \beta \frac{np}{n-p}. \tag{5.3}$$

Indeed, note that

$$\frac{np}{(n-1)s-(p-1)} \le \beta \frac{np}{n-p} \tag{5.4}$$

and that strict inequality holds in (5.4) unless $p=1$ or $\beta=1$. (Incidentally, this explains a point made earlier in section 3, namely, that Theorem 1.5 holds even when $p=1$ although the proof given in section 3 works only for $p>1$. Note that when $p=1$ the ranges (5.2) and (5.3) coincide and so Theorem 1.5 for $p=1$ is just a special case of [6, Corollary 6].) Thus, stronger Poincaré inequalities hold for the class of β-QHBC domains than hold for the (possibly larger) class of $\frac{1}{\beta}$-John domains. We conclude this paper with a "room-and-corridor"-type example which demonstrates both the sharpness in Theorems 1.4 and 1.5 and also the strict inclusion of the class of β-QHBC domains in the class of $\frac{1}{\beta}$-John domains.

Example 5.5. Fix two parameters $\sigma, \tau \ge 1$ and let

$$\Omega = Q_0 \cup C_{01} \cup Q_1 \cup C_{02} \cup Q_2 \cup C_{03} \cup Q_3 \cup \cdots \subset \mathbb{R}^n,$$

where, for each $m=0,1,2,\ldots$, Q_m is an open cube with center x_m and side length $2r_m$ ($r_0=1$) and for each $m=1,2,\ldots$, C_{0m} is a cylindrical domain with height ${r_m}^\tau$ and radius ${r_m}^\sigma$. We arrange these pieces so that the top face of C_{0m} is contained in the boundary of Q_0 and the bottom face of C_{0m} is contained in the boundary of Q_m. Thus Ω consists of a central "room" Q_0 connected by cylindrical "corridors" C_{0m} to smaller adjacent "rooms" Q_m as in Figure 3. We can ensure that the rooms Q_m, $m=1,2,\ldots$, are pairwise disjoint if we require the sequence (r_m) to decrease to zero sufficiently rapidly.

Our goal is to determine for which choices of the parameters σ and τ the domain Ω is or is not

(i) a β-QHBC domain, $\beta \le 1$;
(ii) an s-John domain, $s \ge 1$;
(iii) a (q,p)-Poincaré domain, $1 \le p \le q$.

Specifically, we will show that

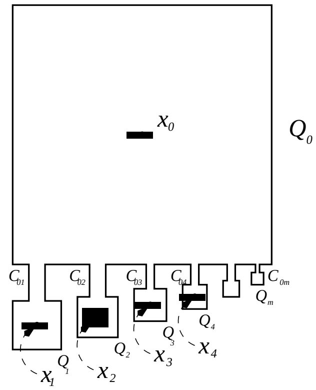

FIGURE 3. A "room-and-corridor type domain Ω

(i) Ω is a β-QHBC domain if $\frac{1}{\beta} = \sigma \leq \tau$ and Ω is not a β-QHBC domain for any $\beta > 0$ if $1 \leq \tau < \sigma$;

(ii) Ω is an s-John domain if $s = \sigma$ and Ω is not an s-John domain if $s < \sigma$ (independent of τ);

(iii) Ω is not a (q,p)-Poincaré domain if

$$q > \frac{np}{(n-1)\sigma - (p-1)\tau}.$$

We will say more about why these facts are true shortly. First, let us discuss their significance.

When $\tau = 1$ and $s = \sigma$, Ω is an s-John domain which is not a (q,p)-Poincaré domain for any

$$q > \frac{np}{(n-1)s - (p-1)}.$$

This shows the sharpness in [6, Corollary 6]. (This special case of the example was already given in [6, p. 437].)

When $\tau = \sigma = \frac{1}{\beta}$, Ω is a β-QHBC domain which is not a (q,p)-Poincaré domain for any

$$q > \frac{np}{(n-1)\frac{1}{\beta} - (p-1)\frac{1}{\beta}} = \beta\frac{np}{n-p}.$$

This shows the sharpness in Theorem 1.5 and hence also Theorem 1.4 of this paper.

Finally, when $1 \leq \tau < \sigma = \frac{1}{\beta}$, Ω is a $\frac{1}{\beta}$-John domain which is not a β-QHBC domain.

To see that Ω is not a QHBC domain if $1 \le \tau < \sigma$, note that the quasihyperbolic distance from the center of Q_0 to the center of Q_m is

$$k_\Omega(x_0, x) \ge r_m^{\tau-\sigma} = \left(\frac{1}{\operatorname{dist}(x_m, \partial\Omega)}\right)^{\sigma-\tau}$$

which grows algebraically (rather than logarithmically) as a function of $\operatorname{dist}(x_m, \partial\Omega)$. We leave it to the reader to verify that Ω is a β-QHBC domain if $\frac{1}{\beta} = \sigma \le \tau$.

We next show that Ω is not an s-John domain if $s < \sigma$. If Ω were s-John, then each point x_m could be joined to x_0 by a rectifiable curve $\gamma_m : [0, l_m] \to \Omega$, $\gamma_m(0) = x_m$, $\gamma_m(l_m) = x_0$, parameterized by arc length and such that $\operatorname{dist}(\gamma_m(t), \partial\Omega) \ge t^s/C_0'$ for all $0 \le t \le l_m$. Let t_m be any parameter value for which $\gamma_m(t_m) \in C_{0m}$. Then $\operatorname{dist}(\gamma_m(t_m), \partial\Omega) = {r_m}^\sigma$ and $t_m \ge \operatorname{dist}(x_m, C_{0m}) = r_m$ so ${r_m}^\sigma \ge {r_m}^s/C_0'$ for all m, which contradicts the assumption that $s < \sigma$. Again, we leave to the reader the verification of the fact that Ω is s-John if $s = \sigma$.

Finally, we show that Ω is not a (q,p)-Poincaré domain if

$$q > \frac{np}{(n-1)\sigma - (p-1)\tau}. \tag{5.6}$$

Define a sequence of piecewise linear functions (u_m) on Ω as follows: $u_m \equiv 1$ on Q_m, $u_m \equiv 0$ on $\Omega \setminus (Q_m \cup C_{0m})$, and u_m is linear on C_{0m}. Then $|\nabla u_m| = {r_m}^{-\tau}$ on C_{0m} and $|\nabla u_m| = 0$ elsewhere.

By [6, Lemma 1], if Ω is a (q,p)-Poincaré domain then

$$\left(\int_\Omega |u_m(x)|^q \, dx\right)^{1/q} \le C \left(\int_\Omega |\nabla u_m(x)|^p \, dx\right)^{1/p} \tag{5.7}$$

for all m. Thus it suffices to show that (5.7) does not hold. We estimate

$$\left(\int_\Omega |u_m(x)|^q \, dx\right)^{1/q} \ge |Q_m|^{1/q} = (2r_m)^{n/q}$$

and

$$\left(\int_\Omega |\nabla u_m(x)|^p \, dx\right)^{1/p} = {r_m}^{-\tau} |C_{0m}|^{1/p} = C(n) {r_m}^{-\tau + (\tau + (n-1)\sigma)/p}.$$

Thus (5.7) would lead to an inequality of the form

$${r_m}^{\frac{n}{q}} \le C {r_m}^{\frac{(n-1)\sigma - (p-1)\tau}{p}},$$

which does not hold for all m by the choice of q in (5.6).

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P. Koskela, Department of Mathematics, University of Jyväskylä, P.O. Box 35 (MaD), FIN-40351, Jyväskylä, Finland

E-mail address: pkoskela@math.jyu.fi

J. Onninen, Department of Mathematics, University of Jyväskylä, P.O. Box 35 (MaD), FIN-40351, Jyväskylä, Finland

E-mail address: jaonnine@math.jyu.fi

J. T. Tyson, Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794-3651

E-mail address: tyson@math.sunysb.edu