DETERMINISTIC AND RANDOM ASPECTS OF POROSITIES

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Abstract. We study porosities of limit sets of finite conformal iterated function systems and certain random fractals. We characterize systems with positive porosity and prove that porosity is continuous within a special class of one dimensional systems. We also show that for certain typical random fractals related to fractal percolation both 0-porous and 1/2-porous points are dense, that is, porosity obtains its minimum and maximum values in a dense set.

1. Introduction

Since the introduction of Hausdorff dimension at the beginning of the last century the number of different kinds of dimensions used both in fractal theory and applications has exploded. In addition to Hausdorff dimension, the most widely used ones are perhaps packing and box-counting dimensions. Each of these dimensions has its own basic properties and one can find examples of sets which can be distinguished by a given dimension but not by any other one. On the other hand there are sets which cannot be distinguished from each other by any dimension although intuitively they may look quite different. As shown later, in some cases porosity is a parameter that can be used to single out sets that are so alike that they cannot be separated from dimensional point of view. Instead of measuring sizes of sets as most concepts of dimensions do, porosity estimates the holes contained in a given set (see Definition 2.3).

There are several variations of Definition 2.3 appearing in the literature. The definition of porosity goes back at least to the 1920's. In fact, in [De] Denjoy introduced a quantity called index which is a slight modification of the concept we are using – the only difference being the choice of an upper limit as a substitute for the lower one we are considering in (2.6). Dolženko [Do] brought into use the term porosity in connection with these quantities. He proved that σ-porous sets (that is, countable unions of porous sets) with respect to upper porosity form a proper subclass of first category sets with measure zero. Based on this result it seems natural that upper porosity is used for describing properties of exceptional sets, for

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example, for measuring sizes of sets where certain functions are nondifferentiable. For more details of upper porosity see [Z]. On the other hand, lower porosity has been used as a tool to find upper bounds for dimensions. Mattila [M] and Salli [Sa] proved that large lower porosity implies small dimension for sets and J.-P. Eckmann, E. Järvenpää, and M. Järvenpää [EJJ], [JJ] considered corresponding questions for measures. For other results related to this theme, see [KR] and [V].

The purpose of this paper is to study porosities from both deterministic and random viewpoints. We will give examples indicating that within a suitable class of self-similar sets porosity can be used to distinguish different sets with same dimension from each other. On the other hand we will illustrate that for certain random fractals porosity will not make a difference even between sets with different dimension. We begin by considering attractors of finite conformal iterated function systems in section 2. We will prove that local porosity is typically a constant on the unique limit set with respect to the natural measure it carries. The finite conformal iterated function systems having positive porosity are also characterized. Finally, we indicate that porosity is continuous within a suitable class of one dimensional systems and give examples illustrating relations between dimensions and porosities of attractors. In section 3 we study random fractal constructions related to Mandelbrot percolation. We show that for typical random sets both 0-porous and 1/2-porous points are dense, that is, porosity obtains the minimum and maximum values in a dense set. This means that typically all the random fractal sets we are considering look the same as far as porosity is concerned. We also prove that with respect to the natural measure typical random sets cannot be uniformly porous.

2. Finite conformal iterated function systems

The setting we use is as in [MU]. Let $I = \{1, \dots, N\}$ be a finite index set containing at least two elements. For all integers $n \geq 1$ let I^n be the set of all n-term sequences of elements of I, and let I^∞ be the corresponding set of infinite sequences. Set $I^* = \cup_{n\geq 0} I^n$ where $I^0 = \{\emptyset\}$.

A finite iterated function system $S = \{\phi_i : X \to X \mid i \in I\}$ is a collection of injections from a compact metric space (X, ρ) into itself which are contractive meaning that for all $i \in I$ there is a constant $0 < s_i < 1$ such that

$$\rho(\phi_i(x), \phi_i(y)) \leq s_i \rho(x, y)$$

for all $x, y \in X$. For $\tau = (\tau_1, \dots, \tau_n) \in I^n$, let

$$\phi_\tau = \phi_{\tau_1} \circ \cdots \circ \phi_{\tau_n}.$$

Further, we denote by $|\tau|$ the length of τ, that is, $|\tau| = n$ for $\tau \in I^n$. If $\tau \in I^\infty \cup I^*$ and $1 \leq k \leq |\tau|$, we use the notation $\tau|_k = (\tau_1, \dots, \tau_k) \in I^k$. For $\tau = (\tau_1, \dots, \tau_l) \in I^l$ and $\sigma = (\sigma_1, \dots, \sigma_k) \in I^k$ let $\tau * \sigma = (\tau_1, \dots, \tau_l, \sigma_1, \dots, \sigma_k) \in I^{l+k}$.

Let $\pi : I^\infty \to X$ be the natural projection, that is,

$$\pi(\tau) = \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X).$$

The limit set associated to the iterated function system $S = \{\phi_i : X \to X \mid i \in I\}$ is defined by

$$J = \pi(I^\infty) = \bigcup_{\tau \in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\tau|_n}(X).$$

The set J is the unique non-empty compact set which is invariant under S, that is,

$$J = \bigcup_{i=1}^{N} \phi_i(J).$$

2.1. Definition. *A finite iterated function system* $S = \{\phi_i : X \to X \mid i \in I\}$ *is conformal if it has the following three properties:*

1. *The set X is a compact subset of the euclidean space $\mathbb{R}^d$ such that* $\overline{\mathrm{Int}(X)} = X$*. Here* $\overline{\mathrm{Int}(X)}$ *is the closure of the interior of* X*.*
2. **The open set condition.** *For all $i \in I$ we have* $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$ *and for $j \neq i$ we have* $\phi_i(\mathrm{Int}(X)) \cap \phi_j(\mathrm{Int}(X)) = \emptyset$*.*
3. *There is an open connected set $V \subset \mathbb{R}^d$ containing X such that for every $i \in I$ the map ϕ_i can be extended to conformal contractive $C^{1+\alpha}$-diffeomorphism $\phi_i : V \to \phi_i(V)$. ($C^{1+\alpha}$ is the family of continuously differentiable functions which have α-Hölder continuous derivatives.)*

2.2. Remarks. *1) Since I is finite Definition 2.1 (3) implies the* **bounded distortion property***: there exists a constant $K \geq 1$ such that for all $\tau \in I^*$ and for all $x, y \in W$ we have $|\phi'_\tau(y)| \leq K|\phi'_\tau(x)|$ (see for example* [MU, Remark 2.3]*). Here W is an open connected set such that $X \subset W \subset \overline{W} \subset V$.*

2) The invariance of domain [ES, Theorem 3.11 p. 303] *implies that in Definition 2.1 (2) we have always* $\phi_i(\mathrm{Int}(X)) \subset \mathrm{Int}(X)$*.*

3) As pointed out to us by P. Mattila the smoothness assumption in Definition 2.1 (3) is needed only in the case $d = 1$. In the plane a conformal map is always analytic or anti-analytic and in higher dimensions the smoothness follows from Liouville's theorem [Re, Theorem 5.10].

The bounded distortion property and conformality imply that there is $R = \mathrm{dist}(X, \partial W) > 0$ such that for all $\tau \in I^*$, $x \in X$, and $0 < r \leq R$

$$B(\phi_\tau(x), \frac{1}{K}\|\phi'_\tau\| r) \subset \phi_\tau(B(x,r)) \subset B(\phi_\tau(x), \|\phi'_\tau\| r). \tag{2.1}$$

Here dist is the distance between two sets, ∂ is the boundary of a set, $B(x,r)$ is the closed ball with centre at x and radius r, and $\|\phi'_\tau\| = \sup_{x \in W} |\phi'_\tau(x)|$.

Next we state some well-known results for finite conformal iterated function systems. The initial ideas for using the thermodynamic formalism in this context are due to Sinai [Si], Bowen [Bo], and Ruelle [Ru1], [Ru2]. For early contributions to this subject see [Be]. The viewpoint we are following here is somewhat different. Its development, particularly within the context of infinite iterated function systems, and all proofs of the following facts can be found in [MU, Lemma 3.6– Lemma 3.14].

Let s be the zero of the topological pressure function

$$P : t \mapsto \lim_{n \to \infty} \frac{1}{n} \log \sum_{\tau \in I^n} \|\phi'_\tau\|^t$$

and let m be the unique s-conformal measure on J. The measure m has the following scaling property: there is a constant $C \geq 1$ such that

$$C^{-1} r^s \leq m(B(x,r)) \leq C r^s \tag{2.2}$$

for all $x \in J$ and for all $0 < r < \frac{1}{2}\operatorname{diam}(X)$ where $\operatorname{diam}(X)$ is the diameter of X. Further, there exists J_0 with $\cup_{i=1}^n \partial(\phi_i(X)) \subset J_0 \subset J$ such that $m(J_0) = 0$ and for every point $x \in J \setminus J_0$ there is a unique $\tau \in I^\infty$ with $\pi(\tau) = x$. In particular, for all $x \in J \setminus J_0$ there exists a unique $i \in \{1, \dots, N\}$ with $x \in \phi_i(X)$. Define $T : J \to J$ for all $x \in J$ by

$$T(x) = \phi_{i_x}^{-1}(x)$$

where $i_x = \min\{1 \le i \le N \mid x \in \phi_i(X)\}$. Let $\sigma : I^\infty \to I^\infty$ be the left shift, that is, $\sigma(\tau)_i = \tau_{i+1}$. Then

$$\pi \circ \sigma(\tau) = T \circ \pi(\tau)$$

for all $\tau \in I^\infty$ with $\pi(\tau) \in J \setminus J_0$. It is well-known that there exists a unique σ-invariant ergodic probability measure μ^* on I^∞ which is equivalent to μ, where $\pi_\sharp \mu = m$. (We denote the image of a measure μ under the map π by $\pi_\sharp \mu$.) Hence the measure $m^* = \pi_\sharp \mu^*$ is T-invariant ergodic probability measure on J. Further, m^* is equivalent to m and there exists $C > 0$ such that

$$C^{-1} < \frac{dm^*}{dm} < C \tag{2.3}$$

where dm^*/dm is the Radon-Nikodym derivative of m^* with respect to m.

Next we will give definitions of porosity and uniform porosity for sets in terms of local porosities.

2.3. Definition. *Let $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $r > 0$. Define*

$$\operatorname{por}(A, x, r) = \sup\{p \ge 0 \mid \text{ there is } z \in \mathbb{R}^d \text{ such that } B(z, pr) \subset B(x, r) \setminus A\}. \tag{2.4}$$

For $q > 0$ and $R > 0$, the set A is uniformly (q, R)-porous if

$$\operatorname{por}(A, x, r) \ge q \tag{2.5}$$

for all $x \in A$ and for all $0 < r \le R$. We say that A is uniformly porous if it is uniformly (q, R)-porous for some $q > 0$ and $R > 0$. The porosity of A is defined by

$$\operatorname{por}(A) = \inf_{x \in A} \operatorname{por}(A, x)$$

where $\operatorname{por}(A, x)$ *is the porosity of A at a point $x \in \mathbb{R}^d$ given by*

$$\operatorname{por}(A, x) = \liminf_{r \to 0} \operatorname{por}(A, x, r). \tag{2.6}$$

2.4. Remark. *Using open balls $U(z, pr)$ and $U(x, r)$ instead of closed ones in Definition 2.3 one can replace supremum with maximum in (2.4). To see this, consider $x \in \mathbb{R}^d$ and $r > 0$ with* $\operatorname{por}(A, x, r) > 0$ *and define*

$$g(z) = \sup\{p \ge 0 \mid U(z, pr) \subset U(x, r) \setminus A\}.$$

Since the continuous function g equals zero in $\partial B(x, r)$ it obtains its maximum at some point $z_0 \in U(x, r)$. Clearly $U(z_0, g(z_0)r) \subset U(x, r) \setminus A$, and so

$$\begin{aligned} \operatorname{por}(A, x, r) &= g(z_0) \\ &= \max\{p \ge 0 \mid \text{ there is } z \in \mathbb{R}^d \text{ such that } U(z, pr) \subset U(x, r) \setminus A\}. \end{aligned}$$

According to the following result porosity of any conformal iterated function system is typically a constant.

2.5. Proposition. *If $S = \{\phi_i : X \to X \mid i \in I\}$ is a finite conformal iterated function system, then the function $x \mapsto \operatorname{por}(J, x)$ is a constant for m-almost all $x \in J$.*

Proof. It suffices to show that

$$\operatorname{por}(J, x) = \operatorname{por}(T(J), T(x)) \tag{2.7}$$

for all $x \in J \setminus J_0$. Using the fact that J is invariant under T this implies that the function $x \mapsto \operatorname{por}(J, x)$ defined on $J \setminus J_0$ is invariant under T, that is, $\operatorname{por}(J, x) = \operatorname{por}(J, T(x))$ for all $x \in J \setminus J_0$, and hence a constant for m-almost all $x \in J$ because m^* is ergodic, m^* and m are equivalent, and $m(J_0) = 0$.

For (2.7), let $x \in J \setminus J_0$ and $\varepsilon > 0$. Since T is a continuously differentiable function on the open set $\cup_{i=1}^N \operatorname{Int}(\phi_i(X)) \supset J \setminus J_0$ we have for all sufficiently small $r > 0$ and for all $y \in B(x, r)$ that $|T'(x)| - \varepsilon \le |T'(y)| \le |T'(x)| + \varepsilon$. Consider $p \ge 0$ and $z \in \mathbb{R}^d$ with $B(z, pr) \subset B(x, r) \setminus J$. Then by conformality and by the fact that for all $i = 1, \ldots, N$ the restriction of T to $\operatorname{Int}(X_i)$ is an injection we get

$$\begin{aligned} B(T(z), (|T'(x)| - \varepsilon)pr) &\subset T(B(z, pr)) \\ &\subset T(B(x, r)) \setminus T(J) \subset B(T(x), (|T'(x)| + \varepsilon)r) \setminus T(J), \end{aligned}$$

and so

$$\operatorname{por}(T(J), T(x), (|T'(x)| + \varepsilon)r) \ge \frac{|T'(x)| - \varepsilon}{|T'(x)| + \varepsilon} \operatorname{por}(J, x, r).$$

Letting $r \to 0$ and finally $\varepsilon \to 0$ gives $\operatorname{por}(T(J), T(x)) \ge \operatorname{por}(J, x)$. The opposite inequality follows similarly. □

In the following proposition we give a sufficient condition for the limit set to be uniformly porous. The same result has been proved earlier by Urbański [U] using different methods. We will give a shorter direct proof which gives an explicit lower bound for porosity.

2.6. Theorem. *If $S = \{\phi_i : X \to X \mid i \in I\}$ is a finite conformal iterated function system with $\operatorname{Int}(X) \setminus \bigcup_{i=1}^N \phi_i(X) \ne \emptyset$, then J is uniformly porous.*

Proof. Let R be as in (2.1). Then there exist $0 < \delta \le R/\operatorname{diam}(X)$ and $x_0 \in X$ such that $B(x_0, \delta \operatorname{diam}(X)) \subset \operatorname{Int}(X)$ and

$$B(x_0, \delta \operatorname{diam}(X)) \cap \bigcup_{i=1}^N \phi_i(X) = \emptyset. \tag{2.8}$$

Noting that there exists D such that for all $x, y \in X$ there is a curve $\mathcal{C}_{xy}$ connecting x to y with $\operatorname{length}(\mathcal{C}_{xy}) \le D \operatorname{dist}(x, y)$, we obtain

$$\operatorname{dist}(\phi_\tau(x), \phi_\tau(y)) \le \|\phi_\tau'\| \operatorname{length}(\mathcal{C}_{xy}) \le D \|\phi_\tau'\| \operatorname{dist}(x, y) \tag{2.9}$$

for all $\tau \in I^*$ and $x, y \in X$. (Note that X is not necessarily convex or connected, but we did use the assumption that V is connected.)

Let $x \in J$ and $r > 0$. Fix $\tau_x \in \pi^{-1}(x)$. Consider a positive integer n such that $D\operatorname{diam}(X)\|\phi'_{\tau_x|n}\| \le r < D\operatorname{diam}(X)\|\phi'_{\tau_x|n-1}\|$. From (2.1) and (2.9) we obtain

$$\begin{aligned} B(\phi_{\tau_x|n}(x_0), \frac{1}{K}\|\phi'_{\tau_x|n}\|\delta\operatorname{diam}(X)) \subset \phi_{\tau_x|n}(B(x_0,\delta\operatorname{diam}(X))) \subset \phi_{\tau_x|n}(X) \\ \subset B(x, D\operatorname{diam}(X)\|\phi'_{\tau_x|n}\|) \subset B(x,r). \end{aligned}$$

By (2.8) the set $\phi_{\tau_x|n}(B(x_0,\delta\operatorname{diam}(X)))$ does not intersect J, and so

$$\operatorname{por}(J,x,r) \ge \frac{\delta\|\phi'_{\tau_x|n}\|}{DK\|\phi'_{\tau_x|n-1}\|} \ge \frac{\delta}{DK^2}\min_{i=1,\dots,N}\|\phi'_i\|$$

which implies the claim. $\square$

2.7. Remark. *Let $S = \{\phi_i : X \to X \mid i = 1,\dots,N\}$ be a finite conformal iterated function system. Clearly the condition* $\operatorname{Int}(X)\setminus\bigcup_{i=1}^N \phi_i(X) \neq \emptyset$ *is necessary for J to have positive porosity. Using Theorem 2.6 and* [MU, Proposition 4.4 and Theorem 4.5] *we have the following characterizations for positive uniform porosity:*

$$\begin{aligned} &J \text{ is uniformly porous} \Leftrightarrow J \text{ has positive porosity} \\ &\Leftrightarrow \operatorname{Int}(X)\setminus\bigcup_{i=1}^N \phi_i(X) \neq \emptyset \Leftrightarrow \mathcal{L}^d(J) = 0 \Leftrightarrow \dim_H(J) < d. \end{aligned}$$

Here $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\dim_H$ is the Hausdorff dimension.

For comparing porosities of conformal attractors to those of conformal measures we need the following definition.

2.8. Definition. *The porosity of a finite Borel measure ν on $\mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is defined by*

$$\operatorname{por}(\nu,x) = \lim_{\varepsilon\to 0}\liminf_{r\to 0}\operatorname{por}(\nu,x,r,\varepsilon)$$

where for all $r, \varepsilon > 0$

$$\begin{aligned} \operatorname{por}(\nu,x,r,\varepsilon) = \sup\{p \ge 0 \mid \text{there is } z \in \mathbb{R}^d \text{ such that } B(z,pr) \subset B(x,r) \\ \text{and } \nu(B(z,pr)) \le \varepsilon\nu(B(x,r))\}. \end{aligned}$$

2.9. Remark. *It is not difficult to see that the scaling property (2.2) of the conformal measure m implies that*

$$\operatorname{por}(J,x) = \operatorname{por}(m,x) = \operatorname{por}(m^*,x) \tag{2.10}$$

for all $x \in J$. In fact, if there is $x \in J$ with $\operatorname{por}(J,x) < t < \operatorname{por}(m,x)$, *then by (2.2) for all sufficiently small $\varepsilon, r > 0$ there is $z \in \mathbb{R}^d$ such that $B(z,tr) \subset B(x,r)$ and $m(B(z,tr)) \le \varepsilon m(B(x,r)) \le C\varepsilon r^s$. For all $y \in J \cap B(z,tr)$ let* $\operatorname{dist}(y,\partial(B(z,tr)) = q_y r$. *The inequality*

$$C^{-1}(q_y r)^s \le m(B(y,q_y r)) \le m(B(z,tr)) \le C\varepsilon r^s$$

gives $q_y \le C^{2/s}\varepsilon^{1/s}$ *implying that*

$$B(z,(t-C^{2/s}\varepsilon^{1/s})r) \subset B(x,r) \setminus J.$$

This gives the contradiction $\operatorname{por}(J,x) \ge t$. *The second equality in (2.10) follows from (2.3).*

As the last result of this section we will prove that porosity is continuous on a special class of conformal iterated function systems on the real line. After that we will give some examples. We begin by defining a metric on the space of conformal iterated function systems consisting of N maps. Let $S = \{\phi_i : [0,1] \to [0,1] \mid i = 1,\dots,N\}$ be a conformal iterated function system. For simplicity we assume that $\phi_i'(x) > 0$ for all $i = 1,\dots,N$ (and for all $x \in [0,1]$). Assume that $\phi_1(0) = 0$, $\phi_N(1) = 1$, and $\gamma_i = \phi_{i+1}(0) - \phi_i(1) > 0$ for $i = 1,\dots,N-1$. Let H be the maximum of the Hölder constants of ϕ_i', that is, $|\phi_i'(x) - \phi_i'(y)| \le H|x-y|^\alpha$ for all $i = 1,\dots,N$ and for all $x,y \in [0,1]$. Denote the space of these systems by $\mathcal{K}^N$ and define a metric d on $\mathcal{K}^N$ by

$$d(S,\tilde{S}) = \sum_{i=1}^{N} \max_x |\phi_i'(x) - \tilde{\phi}_i'(x)| + \sum_{i=1}^{N-1} |\gamma_i - \tilde{\gamma}_i| + |H - \tilde{H}|,$$

where $S = \{\phi_1,\dots,\phi_N\}$, $\tilde{S} = \{\tilde{\phi}_1,\dots,\tilde{\phi}_N\}$, and $\tilde{\gamma}_i$ and $\tilde{H}$ are the gap lengths and Hölder constant of $\tilde{S}$, respectively. Let $J(S)$ be the limit set of the system S.

For all $1 \le i \le N-1$ let g_i be the open interval $(\phi_i(1), \phi_{i+1}(0))$. For all $(\eta_1,\dots,\eta_p) \in I^*$, the intervals $\phi_{\eta_1} \circ \cdots \circ \phi_{\eta_p}(g_i)$ are called gaps. (We use the interpretation $\phi_{\eta_1} \circ \cdots \circ \phi_{\eta_p}(g_i) = g_i$ for $p = 0$.) Note that $g \cap J(S) = \emptyset$ and $\overline{g} \cap J(S) \ne \emptyset$ for all gaps g by the assumptions $\phi_1(0) = 0$ and $\phi_N(1) = 1$.

2.10. Lemma. *Let* $x \in J(S)$. *Assume that the continuous map* $r \mapsto \operatorname{por}(J(S),x,r)$ *has a local minimum at the point* r_0. *Then there is a gap* $G \subset B(x,r_0) \setminus J(S)$ *such that* $\operatorname{length}(g \cap B(x,r_0)) \le \operatorname{length}(G)$ *for all gaps* g *with* $g \cap B(x,r_0) \ne \emptyset$.

Proof. There are a gap G' with $G' \cap B(x,r_0) \ne \emptyset$ and $\delta_0 > 0$ such that for all gaps g with $g \cap B(x,r_0) \ne \emptyset$ and $\operatorname{length}(g \cap B(x,r_0)) \ne \operatorname{length}(G' \cap B(x,r_0))$ we have $\operatorname{length}(g \cap B(x,r_0)) \le \operatorname{length}(G' \cap B(x,r_0)) - \delta_0$. Let $\lambda = \operatorname{length}(G' \cap B(x,r_0))$. If there does not exist a gap $G \subset B(x,r_0)$ with $\operatorname{length}(G) = \lambda$ then for all $0 < \delta < \delta_0$

$$\operatorname{por}(J(S),x,r_0-\delta) = \frac{\lambda-\delta}{2(r_0-\delta)} < \frac{\lambda}{2r_0} = \operatorname{por}(J(S),x,r_0)$$

giving a contradiction since r_0 is a local minimum point of the porosity function $r \mapsto \operatorname{por}(J(S),x,r)$. $\square$

2.11. Theorem. *The function* $S \mapsto \operatorname{por}(J(S))$ *is continuous on* $\mathcal{K}^N$.

Proof. Let $S \in \mathcal{K}^N$ with gap lengths $\gamma_i = \operatorname{length}(g_i)$, $i = 1,\dots,N-1$, and Hölder constant H. Set $\lambda_{\max} = \max_i\{\|\phi_i'\|\}$, $\lambda_{\min} = \min_{x,i}\{\phi_i'(x)\}$, and $\gamma_{\min} = \min_i\{\gamma_i\}$. Let $\varepsilon > 0$ be sufficiently small. Let $x \in J(S)$, $\tau = (\tau_1,\tau_2,\dots) \in I^\infty$ such that $\pi(\tau) = x$, and $r > 0$. Set

$$\lambda_{\tau_j} = \frac{\operatorname{length}(\phi_{\tau_1} \circ \cdots \circ \phi_{\tau_j}([0,1]))}{\operatorname{length}(\phi_{\tau_1} \circ \cdots \circ \phi_{\tau_{j-1}}([0,1]))}.$$

Note that $\lambda_{\tau_j} \leq \lambda_{\max}$. Fix a positive integer k with

$$\prod_{j=1}^{k} \lambda_{\tau_j} \leq r < \prod_{j=1}^{k-1} \lambda_{\tau_j}. \tag{2.11}$$

Let $a \in [\lambda_{\tau_k}, 1]$ be such that $r = a \prod_{j=1}^{k-1} \lambda_{\tau_j}$.

Since $\mathrm{por}(J(S), x) = \liminf_{r \to 0} \mathrm{por}(J(S), x, r)$ we may assume that r is a local minimum point of the map $r \mapsto \mathrm{por}(J(S), x, r)$ when determining the value of $\mathrm{por}(J(S), x, r)$. Let $G = \phi_{\eta_1} \circ \cdots \circ \phi_{\eta_P}(g_j) \subset B(x, r) \setminus J(S)$ be as in Lemma 2.10 and let $M = \max\{m \geq 0 \mid \lambda_{\max}^m \geq K^{-2}\gamma_{\min}\}$. Let $L \geq 0$ be the positive integer such that $\eta_i = \tau_i$ for all $i = 1, \ldots, L$ and $\eta_{L+1} \neq \tau_{L+1}$. Since $|\phi'_{\tau|_L}(x_0)| = \lambda_{\tau_1} \ldots \lambda_{\tau_L}$ for some $x_0 \in [0, 1]$, the bounded distortion property implies that for all $y \in [0, 1]$

$$\frac{1}{K}\lambda_{\tau_1} \ldots \lambda_{\tau_L} \leq |\phi'_{\tau|_L}(y)| \leq K\lambda_{\tau_1} \ldots \lambda_{\tau_L}. \tag{2.12}$$

By (2.12) the ball $B(x, r)$ contains a gap of length at least $\frac{1}{K}\lambda_{\tau_1} \ldots \lambda_{\tau_L}\gamma_{\min}$ giving

$$\frac{1}{K}\lambda_{\tau_1} \ldots \lambda_{\tau_L}\gamma_{\min} \leq r < \lambda_{\tau_1} \ldots \lambda_{\tau_{k-1}}.$$

If $L < k - 1$ this implies $\lambda_{\max}^{M+1} < \frac{1}{K}\gamma_{\min} < \lambda_{\max}^{k-1-L}$. Hence $L \geq k - 1 - M$. Using again the fact that $B(x, r)$ contains a gap of length at least $\frac{1}{K}\lambda_{\tau_1} \ldots \lambda_{\tau_L}\gamma_{\min}$ and assuming that $P \geq L + M + 1$ we get by (2.12) and Lemma 2.10 the contradiction

$$\mathrm{length}(G) \leq K\lambda_{\tau_1} \ldots \lambda_{\tau_L}\lambda_{\max}^{M+1}\gamma_j < \frac{1}{K}\lambda_{\tau_1} \ldots \lambda_{\tau_L}\gamma_{\min}.$$

Thus $P \leq L + M$. Let Q be the smallest integer such that $\lambda_{\max}^Q < \varepsilon^{\frac{1}{\alpha}}$. Then

$$\mathrm{diam}(\phi_{\eta_{L-Q+1}} \circ \cdots \circ \phi_{\eta_P}(g_j) \cup \phi_{\tau_{L-Q+1}} \circ \cdots \circ \phi_{\tau_{k-1}}([0, 1])) < \varepsilon^{\frac{1}{\alpha}}. \tag{2.13}$$

Further, there exists a constant H_1 depending on H, α, $\lambda_{\max}$, $\lambda_{\min}$, and K such that for all $\omega = (\omega_1, \ldots, \omega_n) \in I^*$ and for all $y, z, w \in [0, 1]$

$$|\phi'_\omega(y) - \phi'_\omega(z)| \leq H_1\phi'_{(\omega_2,\ldots,\omega_n)}(w)|y - z|^\alpha. \tag{2.14}$$

To see this, write

$$|\phi'_\omega(y) - \phi'_\omega(z)| \leq \sum_{k=0}^{n-1} \phi'_{\omega_1}(\phi_{(\omega_2,\ldots,\omega_n)}(z))\phi'_{(\omega_2,\ldots,\omega_k)}(\phi_{(\omega_{k+1},\ldots,\omega_n)}(z))\phi'_{(\omega_{k+2},\ldots,\omega_n)}(y)$$
$$H|\phi_{(\omega_{k+2},\ldots,\omega_n)}(y) - \phi_{(\omega_{k+2},\ldots,\omega_n)}(z)|^\alpha.$$

Since $\phi'_k(a) \leq \frac{\lambda_{\max}}{\lambda_{\min}}\phi'_l(b)$ for all $a, b \in [0, 1]$ and k, l, the existence of H_1 follows by the bounded distortion property.

Let $\tilde{S} \in \mathcal{K}^N$ such that $d(S, \tilde{S}) \leq \varepsilon$. Then $|\phi'_i(x) - \tilde{\phi}'_i(x)| \leq \varepsilon$, $|\gamma_i - \tilde{\gamma}_i| \leq \varepsilon$, and $|H - \tilde{H}| \leq \varepsilon$ for all i and x. Thus all the constants defined above can be chosen such that they work for all $\tilde{S}$ with $d(S, \tilde{S}) \leq \varepsilon$. This will be used in the forth inequality in (2.15). There is a natural bijection between $J(S)$ and $J(\tilde{S})$ via the coding space. For $x \in J(S)$ we denote the corresponding point in $J(\tilde{S})$ by $\tilde{x}$. Let $\tilde{\lambda}_{\tau_j}$ be the quantities of $\tilde{S}$ that correspond to λ_{τ_j}. Define $\tilde{a}$ as the image of a under the affine bijection between $[\lambda_{\tau_k}, 1]$ and $[\tilde{\lambda}_{\tau_k}, 1]$ that fixes 1. Setting $\tilde{r} = \tilde{a} \prod_{j=1}^{k-1} \tilde{\lambda}_{\tau_j}$ we obtain a bijection between radii in $J(S)$ and $J(\tilde{S})$. Let $w \in \phi_{\eta_{L-Q+1}} \circ \cdots \circ \phi_{\eta_P}(\overline{g}_j) \cup \phi_{\tau_{L-Q+1}} \circ \cdots \circ \phi_{\tau_{k-1}}([0,1])$ be the minimum point of $\phi'_{\tau|_{L-E}}$. Then by (2.13) and (2.14) there are constants C_1, C_2, and C_3 independent of ε such that

$$
\begin{aligned}
\operatorname{por}(J,x,r) &= \frac{\int_{g_j} \phi'_{\eta|_P}(t)\, d\mathcal{L}(t)}{a \int_{[0,1]} \phi'_{\tau|_{k-1}}(t)\, d\mathcal{L}(t)} = \frac{\int_{\phi_{\eta_{L-Q+1}} \circ \cdots \circ \phi_{\eta_P}(g_j)} \phi'_{\eta|_{L-Q}}(t)\, d\mathcal{L}(t)}{a \int_{\phi_{\tau_{L-Q+1}} \circ \cdots \circ \phi_{\tau_{k-1}}([0,1])} \phi'_{\tau|_{L-Q}}(t)\, d\mathcal{L}(t)} \\
&\le \frac{\int_{\phi_{\eta_{L-Q+1}} \circ \cdots \circ \phi_{\eta_P}(g_j)} \big(\phi'_{\eta|_{L-Q}}(w) + H_1 \phi'_{(\eta_2,\dots,\eta_{L-Q})}(w)\varepsilon\big)\, d\mathcal{L}(t)}{a \int_{\phi_{\tau_{L-Q+1}} \circ \cdots \circ \phi_{\tau_{k-1}}([0,1])} \phi'_{\tau|_{L-Q}}(w)\, d\mathcal{L}(t)} \\
&\le \frac{\operatorname{length}(\phi_{\eta_{L-Q+1}} \circ \cdots \circ \phi_{\eta_P}(g_j))}{a \operatorname{length}(\phi_{\tau_{L-Q+1}} \circ \cdots \circ \phi_{\tau_{k-1}}([0,1]))} + C_1 \varepsilon \\
&\le \frac{\operatorname{length}(\tilde{\phi}_{\eta_{L-Q+1}} \circ \cdots \circ \tilde{\phi}_{\eta_P}(\tilde{g}_j))}{\tilde{a} \operatorname{length}(\tilde{\phi}_{\tau_{L-Q+1}} \circ \cdots \circ \tilde{\phi}_{\tau_{k-1}}([0,1]))} + C_2 \varepsilon^{\alpha} |\log \varepsilon| \\
&\le \frac{\int_{\tilde{g}_j} \tilde{\phi}'_{\eta|_P}(t)\, d\mathcal{L}(t)}{\tilde{a} \int_{[0,1]} \tilde{\phi}'_{\tau|_{k-1}}(t)\, d\mathcal{L}(t)} + C_3 \varepsilon^{\alpha} |\log \varepsilon| \le \operatorname{por}(\tilde{J}, \tilde{x}, \tilde{r}) + C_3 \varepsilon^{\alpha} |\log \varepsilon|
\end{aligned} \tag{2.15}
$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$. Here in the third inequality we use the Hölder continuity of $\tilde{\phi}'$ and the fact that for a constant C independent of ε we have $|\phi_\omega(t) - \tilde{\phi}_\omega(t)| \le C\varepsilon$ for all $\omega \in I^*$ and $t \in [0,1]$. The factor $|\log \varepsilon|$ is due to the definition of Q. Hence

$$
\operatorname{por}(J,x) \le \operatorname{por}(\tilde{J}, \tilde{x}) + C_3 \varepsilon^{\alpha} |\log \varepsilon|. \tag{2.16}
$$

Changing the role of $J(S)$ and $J(\tilde{S})$ we get the opposite inequality. Since (2.16) is valid for all $x \in J(S)$ and $\tilde{x} \in J(\tilde{S})$ the claim follows. □

2.12. Remarks. *1) The above proof can be extended to the space* $\cup_{n=1}^{\infty} \mathcal{K}^n$ *equipped with a metric defined in a way that all the systems which are close to a fixed system have the same number of maps.*

2) For similarities we can ease the assumptions $\phi_1(0) = 0$ *and* $\phi_N(1) = 1$ *since they do not affect porosity (see Example 1 below).*

We conclude this section by studying examples of self-similar iterated function systems. The first example gives a class of self-similar attractors which can be distinguished neither by dimension nor by porosity. The second one indicates that in some cases porosity can be used to sort out sets that have same dimension.

Examples. 1) Let $S(a_1, a_2) = \{\phi_i : [0,1] \to [0,1] \mid i = 1,2\}$ where $\phi_1(x) = \lambda x + a_1$ and $\phi_2(x) = \lambda x + a_2$ for some $0 < \lambda < 1/2$ and $a_1, a_2 \in [0,1]$ with $a_1 + \lambda \le a_2 \le 1 - \lambda$. Then

$$
\operatorname{por}(J(S)) = \frac{1 - 2\lambda}{2(1-\lambda)} \tag{2.17}
$$

independently of the values of a_i. It is not difficult to see that (2.17) gives the right value for $a_1 = 0$ and $a_2 = 1 - \lambda$. The critical points are the inner endpoints of intervals. The claim follows by noting that the affine bijection between $J(S(0, 1-\lambda))$ and $J(S(a_1, a_2))$, $x \mapsto \frac{a_2 - a_1}{1-\lambda} x + \frac{a_1}{1-\lambda}$, preserves porosity.

2) Let $S = \{\phi_i : [0,1] \to [0,1] \mid i = 1,2,3\}$ where $\phi_i(x) = \lambda x + a_i$ such that $\phi_1(0) = 0$, $\phi_3(1) = 1$, and $0 < \gamma_1 = \phi_2(0) - \phi_1(1) < \gamma_2 = \phi_3(0) - \phi_2(1)$. Let $d_1 = \operatorname{dist}(J(S), 1/2)$ and $d_2 = \operatorname{dist}(J(S), \lambda - \lambda^2 + (3\lambda^2 + \gamma_1 + \lambda\gamma_1)/2)$. Then

$$\operatorname{por}(J(S)) = \begin{cases} \frac{\gamma_2}{\frac{1}{2} - d_1 + \gamma_2} & \text{for } \frac{\lambda\gamma_2}{\frac{1}{2} - d_1} < \gamma_1, \\ \frac{\gamma_1}{\lambda + \gamma_1} & \text{for } \lambda\gamma_2 < \gamma_1 \leq \frac{\lambda\gamma_2}{\frac{1}{2} - d_1}, \\ \frac{\gamma_2}{1 + \gamma_2} & \text{for } \frac{\frac{3}{2}\lambda^2 - d_2}{1 - \gamma_2 - \lambda\gamma_2}\gamma_2 < \gamma_1 \leq \lambda\gamma_2, \\ \frac{\gamma_1}{\gamma_1 + \frac{1}{2}(3\lambda^2 + \gamma_1 + \lambda\gamma_1) - d_2} & \text{for } \lambda^2\gamma_2 < \gamma_1 \leq \frac{\frac{3}{2}\lambda^2 - d_2}{1 - \gamma_2 - \lambda\gamma_2}\gamma_2, \\ \frac{\lambda^2\gamma_2}{\lambda^2\gamma_2 + \frac{1}{2}(3\lambda^2 + \gamma_1 + \lambda\gamma_1) - d_2} & \text{for } \gamma_1 \leq \lambda^2\gamma_2. \end{cases}$$

3. Uniform porosity and random fractals

The results of this section centre around special random fractals in the real line. The higher dimensional cases can be treated similarly. Our setup is as in [MW], [GMW].

Let $I = \{0, 1\}$. We denote by Ω the set of functions $\omega : I^* \to \{c, n\}$. Each $\omega \in \Omega$ can be thought of as a code that tells us which intervals we choose (c) and which we neglect (n). More precisely, let $\omega \in \Omega$. For all positive integers k we divide the unit interval $[0,1]$ into 2^k closed dyadic intervals of length 2^{-k}. For all $\sigma \in I^k$ we use the notation J_σ for the closed dyadic subinterval of $[0,1]$ of length 2^{-k} containing those points whose base-2 expansion begins with σ. If $\omega(\sigma) = n$ for $\sigma \in I^k$, then $J_\sigma(\omega) = \emptyset$, and if $\omega(\sigma) = c$, then $J_\sigma(\omega) = J_\sigma$. In the case $\omega(\emptyset) = c$ we set $J_\emptyset(\omega) = [0,1]$. Define

$$K_\omega = \bigcap_{k=0}^{\infty} \bigcup_{\sigma \in I^k} J_\sigma(\omega).$$

Fix $0 \leq p \leq 1$. We make the above construction random by demanding that if J_σ is chosen then $J_{\sigma*0}$ and $J_{\sigma*1}$ are chosen independently with probability p. Let P be the natural probability measure on Ω, that is, for all $\sigma \in I^*$:

$$\begin{aligned} &P(\{\omega \in \Omega \mid \omega(\emptyset) = c\}) = 1, \\ &P(\{\omega \in \Omega \mid \omega(\sigma * 0) = c \text{ and } \omega(\sigma * 1) = c\} \mid \{\omega \in \Omega \mid \omega(\sigma) = c\}) = p^2, \\ &P(\{\omega \in \Omega \mid \omega(\sigma * 0) = n \text{ and } \omega(\sigma * 1) = n\} \mid \{\omega \in \Omega \mid \omega(\sigma) = c\}) = (1-p)^2, \\ &P(\{\omega \in \Omega \mid \omega(\sigma * 0) = c \text{ and } \omega(\sigma * 1) = n\} \mid \{\omega \in \Omega \mid \omega(\sigma) = c\}) = p(1-p) \\ &P(\{\omega \in \Omega \mid \omega(\sigma * 1) = c \text{ and } \omega(\sigma * 0) = n\} \mid \{\omega \in \Omega \mid \omega(\sigma) = c\}) = p(1-p) \\ &P(\{\omega \in \Omega \mid \omega(\sigma * 0) = n \text{ and } \omega(\sigma * 1) = n\} \mid \{\omega \in \Omega \mid \omega(\sigma) = n\}) = 1 \end{aligned}$$

where the notation $P(A \mid B)$ means the conditional probability of A given B.

It is a well-known result in the theory of branching processes that if the expectation of the number of chosen intervals of length $1/2$ is bigger than one then the limit set K_ω is non-empty with positive probability (see [AN, Theorem 1, p.7]). In our case this expectation equals $2p$ since both left and right intervals are chosen independently with probability p. So

$$Q = P(\{\omega \in \Omega \mid K_\omega \neq \emptyset\}) > 0 \quad \text{if} \quad p > \frac{1}{2}.$$

The value of Q is easy to calculate. In fact, if K_ω is non-empty, then either $K_\omega \cap [0,1/2] \neq \emptyset$ or $K_\omega \cap [1/2,1] \neq \emptyset$. Thus

$$Q = 2pQ - p^2Q^2, \tag{3.1}$$

that is, $Q = 0$ or $Q = (2p-1)/p^2$. Since $Q > 0$ for $p > 1/2$ the only possible solution is $(2p-1)/p^2$. From now on we assume that $1/2 < p < 1$. Note that for $p = 1$ we have $K_\omega = [0,1]$ for P-almost all $\omega \in \Omega$.

For $0 \leq t \leq 1/2$ define $A_\omega(t) = \{x \in K_\omega \mid \operatorname{por}(K_\omega, x) = t\}$. Let $B = \{\omega \in \Omega \mid K_\omega \neq \emptyset\}$. We use the notation $\mathcal{D}$ for the family of all closed dyadic subintervals of the unit interval $[0,1]$.

3.1. Theorem. *For P-almost all $\omega \in B$ both $A_\omega(0)$ and $A_\omega(1/2)$ are dense in K_ω.*

Proof. Let $D \in \mathcal{D}$ and let $\sigma \in I^*$ be such that $J_\sigma = D$. We say that D is chosen by $\omega \in \Omega$ if $\omega(\sigma) = c$ and D is neglected by ω if $\omega(\sigma) = n$.

We first prove that for P-almost all $\omega \in B$ the set $A_\omega(1/2)$ is dense in K_ω. Since $p < 1$ we have $P(E_D) = 0$ for all $D \in \mathcal{D}$ where

$$E_D = \{\omega \in \Omega \mid D' \text{ is chosen by } \omega \text{ for all dyadic intervals } D' \subset D\}.$$

Define

$$E = \bigcup_{D \in \mathcal{D}} E_D.$$

Then $P(E) = 0$. Let $\omega \in B \setminus E$. For any $x \in K_\omega$ and $r > 0$ there is a dyadic interval $D \subset B(x,r)$ containing x and a dyadic interval $D' \subset D$ which is not chosen by ω. Since K_ω is closed one of the endpoints of the connected component of $[0,1] \setminus K_\omega$ including D', say y, belongs to the set $K_\omega \cap B(x,r)$. Clearly $\operatorname{por}(K_\omega, y) = 1/2$ implying the first claim.

Now we show that for P-almost all $\omega \in B$ the set $A_\omega(0)$ is dense in K_ω. A sequence $\tau = (\tau_1, \tau_2, \ldots) \in I^\infty$ is said to *carry a (k,ω)-block* if $\pi(\tau) \in K_\omega$ and if there exists a positive integer l such that for all $\sigma \in I^k$ we have $J_{\tau|_l * \sigma} \cap K_\omega \neq \emptyset$, $J_{\tau|_{l-1} * \tau_l^c * \sigma} \cap K_\omega \neq \emptyset$, and $J_{\tau|_{l-2} * \tau_{l-1}^c * \tau_l^c * \sigma)} \cap K_\omega \neq \emptyset$ where $\tau_l^c = 0$ if $\tau_l = 1$ and $\tau_l^c = 1$ if $\tau_l = 0$. The dyadic interval $J_{\tau|_l}$ is the *seed* of the (k,ω)-block and $J_{\tau|_{l-2}}$ is the *root* of it.

We begin by proving that for all positive integers k for P-almost all $\omega \in B$ there exists $\tau \in I^\infty$ which carries a (k,ω)-block. Setting for fixed k

$$Q = P(\{\omega \in \Omega \mid \text{there exists } \tau \in I^\infty \text{ which carries a } (k,\omega)\text{-block}\})$$

we have

$$Q = 2pQ - p^2Q^2 + f(p,Q)$$

where the term $f(p,Q) \geq 0$ stems from the case that a (k,ω)-block starts right from the second level and there are no (k,ω)-blocks starting from higher levels. More precisely, $f(p,Q)$ is the P-measure of the functions ω that satisfy the following property: there is $\tau \in I^\infty$ that carries a (k,ω)-block having a dyadic interval of length $1/4$ as the seed, and there are no points $\tau' \in I^\infty$ carrying (k,ω)-blocks having seeds with length shorter than $1/4$. Since the parabola $-p^2x^2 + (2p-1)x$ is positive for $x \in (0,(2p-1)/p^2)$, the only possible solutions for Q are 0 and $(2p-1)/p^2$ (recall that $P(B) = (2p-1)/p^2$). Clearly, the set of those functions ω for which there is $\tau \in I^\infty$ carrying a (k,ω)-block that has a dyadic interval of length $1/4$ as a seed has positive P-measure, and so the only solution for Q is $(2p-1)/p^2$.

What we have proved is the following: for all positive integers k and for all dyadic intervals D

$$\begin{aligned} &P(\{\omega \in \Omega \mid \text{there is } \tau \in I^\infty \text{ with } \pi(\tau) \in D \text{ such that } \tau \text{ carries a } (k,\omega)\text{-block} \\ &\qquad \text{having the root in } D\} \mid \{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset\}) = 1. \end{aligned} \tag{3.2}$$

Let

$$B_1 = \{\omega \in B \mid \text{there is } \tau \in I^\infty \text{ carrying a } (1,\omega)\text{-block}\}.$$

Then $B_1 = \cup_{D\in\mathcal{D}} C_1^D$ for

$$C_1^D = \{\omega \in B_1 \mid \text{there is } \tau \in I^\infty \text{ carrying a } (1,\omega)\text{-block having the seed } D\},$$

and according to (3.2) we have $P(B_1) = P(B)$. For $j = 2,3,\dots$ set $B_j = \cup_{D\in\mathcal{D}} B_j^D$, where

$$B_j^D = \{\omega \in C_{j-1}^D \mid \text{there is } \tau \in I^\infty \text{ carrying a } (j,\omega)\text{-block having the seed in } D\}$$

and

$$\begin{aligned} C_k^D = \{\omega \in B_k \mid &\text{for all } l = 1,\dots,k \text{ there is } \tau_l \in I^\infty \text{ carrying an } (l,\omega)\text{-block} \\ &\text{having the seed } S_l \text{ with } S_1 \supset \cdots \supset S_{k-1} \supset S_k = D\}. \end{aligned}$$

From (3.2) we get $P(B_j^D) = P(C_{j-1}^D)$ giving $P(B_j) = P(B)$ since $B_{j-1} = \cup_{D\in\mathcal{D}} C_{j-1}^D$.

If $\omega \in B_\infty = \cap_{i=1}^\infty B_i$, then for all i there is $\tau_i \in I^\infty$ with $\pi(\tau_i) \in K_\omega$ such that for all $l = 1,\dots,i$ the sequence τ_i carries an (l,ω)-block having the seed S_l such that $S_1 \supset \cdots \supset S_i$. Since K_ω is closed $x_\infty = \lim_{i\to\infty} \pi(\tau_i) \in K_\omega$. It is not difficult to see that $\operatorname{por}(K_\omega, x_\infty) = 0$ (for details see Lemma 3.7). Note that $P(B_\infty) = P(B)$. Hence for any $D \in \mathcal{D}$ for P-almost all $\omega \in \Omega$ with $K_\omega \cap D \neq \emptyset$ there is $y \in K_\omega \cap D$ such that $\operatorname{por}(K_\omega, y) = 0$. This in turn implies the claim. $\square$

In Section 2 we saw that porosity may vary for self-similar sets having same dimension. Theorem 3.1 in turn indicates that almost surely the random fractal sets we are considering look the same as far as porosity is concerned. It is well-known that the dimension of a typical set depends on p [MW, Theorem 1.1]. Denoting by ν_ω the natural measure carried by a typical set [MW, Theorem 3.1], it is natural to ask what is the ν_ω-measure of a typical set $A_\omega(t)$. In particular one may ask what is $\operatorname{ess\,inf}_x \operatorname{por}(K_\omega, x)$ with respect to ν_ω. We give the following conjecture.

3.2. Conjecture.

$$\nu_\omega - \operatorname*{ess\,inf}_x \operatorname{por}(K_\omega, x) = 0.$$

We finish this section by taking a step into this direction by showing that K_ω is not ν_ω-uniformly porous.

3.3. Definition. *Let μ be a finite Borel measure on $\mathbb{R}^d$. A set $A \subset \mathbb{R}^d$ is μ-uniformly (q, R)-porous if (2.5) is true for μ-almost all $x \in A$ and for all $0 < r \leq R$. The set A is called μ-uniformly porous if it is μ-uniformly (q, R)-porous for some q and R.*

3.4. Proposition. *For all $q > 0$ and $R > 0$*

$$P(\{\omega \in \Omega \mid K_\omega \textit{ is non-empty and } \nu_\omega\textit{-uniformly } (q, R)\textit{-porous}\}) = 0.$$

As an immediate consequence we have:

3.5. Corollary.

$$P(\{\omega \in \Omega \mid K_\omega \textit{ is non-empty and } \nu_\omega\textit{-uniformly porous}\}) = 0.$$

3.6. Remark. *Using* [MW, Theorem 3.4] *it is easy to see that the support of ν_ω equals K_ω. Thus Proposition 3.4 follows from Theorem 3.1 using the fact that the closure of any uniformly (q, R)-porous set is uniformly (q, R)-porous. However, we give an alternative proof which, although being longer, is much more informative and gives some intuition what is going on.*

For the proof of Proposition 3.4 we need the following lemmas.

3.7. Lemma. *Let $q > 0$ and $R > 0$. There are positive integers $n = n(q)$ and $L = L(q, R)$ such that the following property holds for P-almost all $\omega \in B$ for which K_ω is ν_ω-uniformly (q, R)-porous: for all $l \geq L$ and for all $\tau \in I^\infty$ with $\pi(\tau) \in K_\omega$ there exists a finite word σ with $|\sigma| = n$ such that $J_{\tau|_{ln} * \sigma} \cap K_\omega = \emptyset$.*

Proof. Let

$$A = \{\omega \in B \mid K_\omega \text{ is } \nu_\omega\text{-uniformly } (q, R)\text{-porous}\}.$$

Let n and L be such that $q \geq 2^{-n+2}$ and $R \geq 2^{-Ln-1}$. Setting, for all dyadic subintervals D of $[0, 1]$, $C_D = \{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset\}$ and $A_D = \{\omega \in C_D \mid \nu_\omega(D) > 0\}$, [MW, Theorem 3.4] implies that $P(C_D \setminus A_D) = 0$. (Note that in [MW, Theorem 3.4] it is assumed that the expectation of certain random variable needed in the construction of ν_ω is positive. In our case this expectation equals 1 [MW, pp. 327–328]). Hence $P(E) = 0$ for $E = \cup_D(C_D \setminus A_D)$. Consider $\omega \in A \setminus E$. Let $l \geq L$ and $\tau \in I^\infty$ such that $\pi(\tau) \in K_\omega$. Assuming that $J_{\tau|_{ln} * \sigma} \cap K_\omega \neq \emptyset$ for all σ with $|\sigma| = n$, we have $\nu_\omega(J_{\tau|_{ln} * \sigma_0} \cap K_\omega) > 0$ for $\sigma_0 = (1, 0, 0, \ldots, 0) \in I^n$. Then for a set of points $x \in J_{\tau|_{ln} * \sigma_0}$ with positive ν_ω-measure we have

$$2^{-n+2} \leq \operatorname{por}(K_\omega, x, 2^{-ln-1}) \leq \frac{2^{-ln-n}}{2^{-ln-1}} = 2^{-n+1}$$

which is a contradiction. $\square$

For all positive integers k, let $\mathcal{D}_k$ be the family of all closed dyadic subintervals of the unit interval of length 2^{-k}. Let n be an integer and let $D \in \mathcal{D}_k$. The set $K_\omega \cap D$ is *n-notched* if there is $D' \in \mathcal{D}_{k+n}$ with $D' \subset D$ and $K_\omega \cap D' = \emptyset$.

3.8. Lemma. *Let n be a positive integer. There exists a real number $\gamma < 1$ depending on p and n such that for all positive integers k and for all $D \in \mathcal{D}_k$*

$$P(\{\omega \in \Omega \mid K_\omega \cap D \textit{ is non-empty and } n\textit{-notched}\}) = \gamma P(\{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset\}).$$

Proof. Let k be a positive integer and $D \in \mathcal{D}_k$. Let $\sigma \in I^k$ such that $J_\sigma = D$. Set $Q = (2p-1)/p^2 = P(B)$. We use for all positive integers i and j and for all $D \in \mathcal{D}_i$ the notation $\mathcal{D}^D_{i+j}$ for the family of all $\widetilde{D} \in \mathcal{D}_{i+j}$ with $\widetilde{D} \subset D$. Setting

$$F = \{\omega \in \Omega \mid K_\omega \cap D \text{ is non-empty and } n\text{-notched}\},$$

we have

$$\begin{aligned} &P(\{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset\}) \\ &= P(F) + P(\{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset \text{ and } K_\omega \cap D' \neq \emptyset \text{ for all } D' \in \mathcal{D}^D_{k+n}\}) \\ &= P(F) + P(\{\omega \in \Omega \mid \omega(\sigma) = c\}) Q^{2^n} \prod_{i=1}^n p^{2^i}. \end{aligned}$$

Since $P(\{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset\}) = QP(\{\omega \in \Omega \mid \omega(\sigma) = c\})$ the claim follows by setting $\gamma = 1 - Q^{2^n-1} \prod_{i=1}^n p^{2^i}$. $\square$

Proof of Theorem 3.4. Let

$$F = \{\omega \in \Omega \mid K_\omega \text{ is non-empty and } \nu_\omega\text{-uniformly } (q,R)\text{-porous}\}.$$

Let n and L be as in Lemma 3.7. Defining for all positive integers $l \geq L$

$$F_l = \bigcup_{\substack{A \subset \mathcal{D}_{ln} \\ A \neq \emptyset}} \Big(\{\omega \in \Omega \mid K_\omega \cap D \neq \emptyset \text{ if and only if } D \in A\} \cap \bigcap_{D \in A} \{\omega \in \Omega \mid K_\omega \cap D \text{ is } n\text{-notched}\} \Big),$$

Lemma 3.7 implies that for all positive integers k

$$P(F) \leq P\Big(\bigcap_{l=L}^{L+k} F_l\Big).$$

Hence, it suffices to prove that

$$P\Big(\bigcap_{l=L}^{L+k} F_l\Big) \leq \gamma^{k+1} P(B), \tag{3.3}$$

where $\gamma < 1$ is as in Lemma 3.8. This implies the claim since γ does not depend on k. Using the fact that for every $D_{k-1} \in \mathcal{D}_{(L+k-1)n}$,

$$\begin{aligned} &\{\omega \in \Omega \mid K_\omega \cap D_{k-1} \text{ is non-empty and } n\text{-notched}\} \\ &= \bigcup_{\substack{A_k \subset \mathcal{D}^{D_{k-1}}_{(L+k)n} \\ A_k \neq \emptyset \\ \bigcup_{D_k \in A_k} D_k \neq D_{k-1}}} \{\omega \in \Omega \mid K_\omega \cap D_k \cap D_{k-1} \neq \emptyset \text{ if and only if } D_k \in A_k\}, \end{aligned} \tag{3.4}$$

we get

$$P\Big(\bigcap_{l=L}^{L+k} F_l\Big) = \sum_{\substack{A_0\subset\mathcal{D}_{Ln}\\ A_0\neq\emptyset}} \Big[\prod_{D_0\in A_0}\Big(\sum_{\substack{A_1\subset\mathcal{D}^{D_0}_{(L+1)n}\\ A_1\neq\emptyset\\ \bigcup_{D_1\in A_1} D_1\neq D_0}} \Big[\prod_{D_1\in A_1}\cdots\Big(\sum_{\substack{A_k\subset\mathcal{D}^{D_{k-1}}_{(L+k)n}\\ A_k\neq\emptyset\\ \bigcup_{D_k\in A_k} D_k\neq D_{k-1}}}$$

$$\Big[\prod_{D_k\in A_k} P(\{\omega\in\Omega\mid K_\omega\cap D_k \text{ is non-empty and } n\text{-notched}\})\Big]$$

$$\times\Big[\prod_{E_k\in\mathcal{D}^{D_{k-1}}_{(L+k)n}\setminus A_k} P(\{\omega\in\Omega\mid K_\omega\cap E_k=\emptyset\})\Big]\Big)$$

$$\times\Big[\prod_{E_{k-1}\in\mathcal{D}^{D_{k-2}}_{(L+(k-1))n}\setminus A_{k-1}} P(\{\omega\in\Omega\mid K_\omega\cap E_{k-1}=\emptyset\})\Big]\Big)$$

$$\cdots\Big]\times\Big[\prod_{E_0\in\mathcal{D}_{Ln}\setminus A_0} P(\{\omega\in\Omega\mid K_\omega\cap E_0=\emptyset\})\Big]. \tag{3.5}$$

Note that by Lemma 3.8 and (3.4) we have

$$\sum_{\substack{A_k\subset\mathcal{D}^{D_{k-1}}_{(L+k)n}\\ A_k\neq\emptyset\\ \bigcup_{D_k\in A_k} D_k\neq D_{k-1}}} \Big[\prod_{D_k\in A_k} P(\{\omega\in\Omega\mid K_\omega\cap D_k \text{ is non-empty and } n\text{-notched}\})\Big]$$

$$\times\Big[\prod_{E_k\in\mathcal{D}^{D_{k-1}}_{(L+k)n}\setminus A_k} P(\{\omega\in\Omega\mid K_\omega\cap E_k=\emptyset\})\Big]\Big)$$

$$= \sum_{\substack{A_k\subset\mathcal{D}^{D_{k-1}}_{(L+k)n}\\ A_k\neq\emptyset\\ \bigcup_{D_k\in A_k} D_k\neq D_{k-1}}} \gamma^{|A_k|}\Big[\prod_{D_k\in A_k} P(\{\omega\in\Omega\mid K_\omega\cap D_k\neq\emptyset\})\Big]$$

$$\times\Big[\prod_{E_k\in\mathcal{D}^{D_{k-1}}_{(L+k)n}\setminus A_k} P(\{\omega\in\Omega\mid K_\omega\cap E_k=\emptyset\})\Big]\Big)$$

$$\leq \gamma \sum_{\substack{A_k\subset\mathcal{D}^{D_{k-1}}_{(L+k)n}\\ A_k\neq\emptyset\\ \bigcup_{D_k\in A_k} D_k\neq D_{k-1}}} \Big[\prod_{D_k\in A_k} P(\{\omega\in\Omega\mid K_\omega\cap D_k\neq\emptyset\})\Big]$$

$$\times\Big[\prod_{E_k\in\mathcal{D}^{D_{k-1}}_{(L+k)n}\setminus A_k} P(\{\omega\in\Omega\mid K_\omega\cap E_k=\emptyset\})\Big]\Big)$$

$$= \gamma P(\{\omega\in\Omega\mid K_\omega\cap D_{k-1} \text{ is non-empty and } n\text{-notched}\})$$

where the number of elements in A_k is denoted by $|A_k|$. Iterating this k times in (3.5) implies (3.3). $\square$

3.9. Remarks. *1) The inequality (2.2) is not valid for* ν_ω. *So it is possible that* $\mathrm{por}(\nu_\omega, x) > \mathrm{por}(K_\omega, x)$ *for some* $x \in K_\omega$ *which makes the evaluation of* $\mathrm{por}(\nu_\omega)$ *an interesting (open) problem.*

2) As mentioned at the beginning of this section the proofs go through for much more general systems. One can divide a cube in $\mathbb{R}^d$ *into a finite number of subcubes of different side lengths. Also each subcube can be chosen with different probability. Of course, in this general setting for example the equation (3.1) is much more complicated.*

3) One natural question for further investigations is how generally the results of this section are valid. For the denseness of $1/2$*-porous points one needs only that the probability for choosing everything is zero. For proving the denseness of* 0*-porous points at least some geometrical constrains must be implied.*

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