ACL homeomorphisms and linear dilatation

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1 Introduction

Let D be a domain in $\mathbf{R}^n$, $n \geq 1$, and $f\colon D \to \mathbf{R}^n$ a homeomorphism. For $x \in D$ and $0 < r < d(x, \partial D)$ we set

$$L(x,f,r) = \sup\{|f(x) - f(y)| : y \in \partial B(x,r)\},$$

$$l(x,f,r) = \inf\{|f(x) - f(y)| : y \in \partial B(x,r)|\}.$$

where $B(x,r)$ stands for the open ball centered at x and radius r and $\partial B(x,r)$ for its boundary. The linear dilatation of f at x is defined as

$$H(x,f) = \limsup_{r \to 0} H(x,f,r)$$

where $H(x,f,r) = L(x,f,r)/l(x,f,r)$. At every point $x \in D$, $H(x,f) \in [1,\infty]$ and $H(x,f) = \|f'(x)\|/l\,(f'(x))$ provided that f is differentiable at x with $l\,(f'(x)) > 0$. Here the norm $\|f'(x)\|$ of the derivative $f'(x)$ of f at x is defined as

$$\|f'(x)\| = \sup_{|h|=1} |f'(x)h|$$

and the minimum norm $l\,(f'(x))$ as

$$l\,(f'(x)) = \inf_{|h|=1} |f'(x)h|.$$

A well known result of Gehring [G1] says that if a homeomorphism f has the linear dilatation $H(x,f)$ uniformly bounded in D, then f is a quasiconformal mapping.

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In particular f is ACL. The ACL property means that f is absolutely continuous on almost every line segment parallel to the coordinate axis in D. In [T2] Tukia conjectured that the condition

$$m(\{x \in D : H(x,f) > t\}) < ct^{-\alpha} \tag{1.1}$$

for some $\alpha > 3$ is sufficient for the ACL property of a plane homeomorphism f. Indeed, this was proved in [FA] together with a space analogue. In this paper we show that $\alpha > 2$ in (1.1) implies the ACL property in the plane case with a corresponding improvement in space. Our proof is based on the Gehring method in [G1].

Theorem 1.2 *Suppose that a homeomorphism $f: D \to \mathbf{R}^n$, $D \subset \mathbf{R}^n$ a domain, a subset S of D and $s \in (1,\infty]$ satisfy the conditions*

(1.3) $s > n/(n-1)$,

(1.4) $H(x,f) < \infty$ *for each* $x \in D \setminus S$,

(1.5) $H(x,f) \in L^s_{loc}(D)$,

(1.6) S *has* σ*-finite* $(n-1)$*-Hausdorff measure.*

Then f is ACL.

Remarks 1.7 (a) The asumption (1.3) rules out the case $n = 1$, see Section 2.
(b) The asumption (1.6) means that the set S is of the form $S = \bigcup S_i$ where $H^{n-1}(S_i) < \infty$ and H^{n-1} refers to the $(n-1)$-dimensional Hausdorff measure. For the definition of the Hausdorff measure see e.g. [G1] or [V].

In Section 2 we consider some properties of homeomorphisms $f: D \to \mathbf{R}^n$ satisfying $H(x,f) < \infty$ a.e. in D. In Corollary 2.4 we show that $f' \in L^p_{loc}(D)$, $p = sn/(n-1+s)$, under the conditions of Theorem 1.2. In particular this implies that f is ACL^p. The section also contains some examples. Section 3 is devoted to the proof of Theorem 1.2.

2 Mappings with $H(x,f)<\infty$ a.e.

If a homeomorphism $f\colon D \to \mathbf{R}^n$ satisfies $H(x,f) < \infty$ a.e. $x \in D$ or even $\operatorname{ess\,sup}_{x\in D} H(x,f) < \infty$, then f need not be ACL. The well known example is constructed from the Cantor staircase function $g\colon [0,1] \to [0,1]$, i.e. g is an increasing function with the property $g'(x) = 0$ for a.e. $x \in [0,1]$. Now $f\colon (0,1)\times(0,1) \to (0,2)\times(0,1)$ defined as $f(x,y) = (g(x)+x, y)$ is a homeomorphism with $H(z,f) = 1$ a.e. but f is not ACL. Moreover, no boundedness condition, except $H(x,f) = 1$ for all x, in the case $n = 1$ implies absolute continuity. Indeed, an increasing homeomorphism $f\colon \mathbf{R} \to \mathbf{R}$ is called K-quasisymmetric if it satisfies

$$\frac{1}{K} \le \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \le K$$

for all $x \in \mathbf{R}$ and $t > 0$. If f is K-quasisymmetric, then $H(x,f) \le K$ for all $x \in \mathbf{R}$. Now Beurling and Ahlfors [BA] constructed for each $K > 1$ a K-quasisymmetric mapping f which is not absolutely continuous. For more striking examples of such mappings see [T1]. Hence no integrability condition for $H(x,f)$ like (1.5) implies absolute continuity for $n = 1$.

However, homeomorphisms which satisfy $H(x,f) < \infty$ a.e. have some nice properties.

Theorem 2.1 *Suppose that a homeomorphism $f\colon D \to \mathbf{R}^n$ satisfies $H(x,f) < \infty$ a.e. in D. Then f is a.e. differentiable.*

Proof. Fix an open set $G \subset\subset D$ and let $\Phi(E) = |f(E)|$ for each Borel set $E \subset G$. Then Φ is a finite Borel measure on G and hence it has a finite derivative

$$\Phi'(x) = \lim_{r\to 0} \frac{\Phi(B(x,r))}{|B(x,r)|}$$

at a.e. $x \in G$. Here and in the following $|A|$ means the Lebesgue measure of a set $A \subset \mathbf{R}^n$.

Now at an almost every point x of G, $\Phi'(x)$ exists and $H(x,f) < \infty$. Fix such a point x. Let $y \in G$ with $0 < |x-y| < d(x, \partial G)$. Now

$$\begin{aligned}\left(\frac{|f(x)-f(y)|}{|y-x|}\right)^n &\le \left(\frac{L(x,f,|y-x|)}{l(x,f,|y-x|)}\right)^n \left(\frac{l(x,f,|y-x|)}{|y-x|}\right)^n \\ &\le H(x,f,|y-x|)^n \frac{\Phi(B(x,|y-x|))}{|B(x,|y-x|)|}\end{aligned}$$

and letting $y \to x$ we see that

$$\limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} \le H(x,f)\Phi'(x)^{\frac{1}{n}} \le \infty.$$

By the Rademacher-Stepanov theorem the mapping f is a.e. differentiable in G. The theorem follows.

Theorem 2.2 *Suppose that a homeomorphism $f\colon D \to \mathbf{R}^n$ satisfies $H(x,f) \in L^s_{loc}(D)$, $s \in [1,\infty]$. Then $f' \in L^p_{loc}(D)$ with $p = sn/(n-1+s)$ and $p = n$ if $s = \infty$.*

Proof. We may assume that f is sense-preserving. Since $H(x,f) \le \infty$ a.e. in D, Theorem 2.1 implies that $f'(x)$ exist a.e. If f is differentiable at x and $H(x,f) \le \infty$, then an elementary argument shows that

$$\|f'(x)\|^n \le H(x,f)^{n-1}J(x,f) \tag{2.3}$$

where $J(x,f)$ is the jacobian determinant of $f'(x)$.
Fix an open set $G \subset\subset D$. For $s < \infty$ (2.3) and the Hölder inequality imply

$$\begin{aligned}\int_G |f'(x)|^p\,dx &\le \int_G H(x,f)^{\frac{p(n-1)}{n}} J(x,f)^{\frac{p}{n}}\,dx \\ &\le \left[\int_G H(x,f)^{\frac{p(n-1)}{(n-p)}}\,dx\right]^{\frac{(n-p)}{n}} \left[\int_G J(x,f)\,dx\right]^{\frac{p}{n}} \\ &\le \left[\int_G H(x,f)^s\,dx\right]^{\frac{(n-p)}{n}} |f(G)|^{\frac{p}{n}} < \infty\end{aligned}$$

as required. For $s = \infty$ the proof is similar. Note that the inequality

$$\int_G J(x,f)\,dx \le |f(G)|$$

always holds for an a.e. differentiable homeomorphism, see [RR].

Corollary 2.4 *Under the condition of Theorem 1.2 f is a.e. differentiable and $f' \in L^p_{loc}(D)$, $p = sn/(n-1+s)$. In particular f is ACL^p.*

3 Proof for Theorem 1.2

We prove Theorem 1.2 in the case $S = \emptyset$. By the theorem of Gross, see e.g. [V, p. 103], the condition (1.6) implies that S meets almost every line parallel to some coordinate axis in a countable set only. For a continuous function a countable set E does not destroy absolute continuity if an estimate like (3.8) below holds for compact sets F in the complement of E. Thus the case $S \neq \emptyset$ does not lead to essential difficulties, see [G1].

Pick a closed cube $Q \subset\subset D$ whose sides are parallel to the coordinate axes and write $Q' = \frac{1}{2}Q$ for the cube with the same center as Q and side length half of that of Q. In order to show that f is ACL it suffices to show that f is absolutely continuous on almost every line segment of Q' parallel to the coordinate axes. Renormalizing we may assume that $Q = [-2,2]^n$ and by symmetry it is sufficient to consider segments parallel to the x_n-axis. Let $P\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ denote the projection $P(x) = x - x \cdot e_n e_n$ and for $y \in P(Q) \subset \mathbf{R}^{n-1}$ write $I = I(y) = Q' \bigcap P^{-1}(y)$ for the line segment parallel to the x_n-axis in Q'.

Next for a Borel set $E \subset P(Q)$ write

$$\Phi(E) = |f(Q \cap P^{-1}(E))| \leq |f(Q)| < \infty.$$

Then Φ is a finite Borel measure on $P(Q)$ and hence it has a finite derivative $\Phi'(y)$ for almost all $y \in P(Q')$. We choose $y \in P(Q')$ such that (i) $\Phi'(y)$ exists and (ii) $H(x,f) \in L^s(I(y))$. The last assertion follows from the Fubini theorem. It suffices to show that f is absolutely continuous on $I(y)$.

To this end let $F \subset I(y)$ be a compact set. For each $k = 0,1,2,\ldots$ write

$$F_k = \{x \in F : 2^k \leq H(x,f) < 2^{k+1}\}.$$

Then F_k is a Borel set and $F = \bigcup F_k$ because of (1.4) and our assumption $S \neq \emptyset$. Note also that $H(x,f) \geq 1$ for every x. We first derive the following estimate

$$H^1(fF_k) \leq C 2^k H^1(F_k)^{\frac{n-1}{n}} \tag{3.1}$$

where $C = (2^{2n+1}\Phi'(y))^{1/n}$.

For (3.1) fix k and for each $j = 1,2,\ldots$ consider the set

$$F_{k,j} = \{x \in F_k : L(x,f,r)^n \leq 2^{n(k+1)}|fB(x,r)|/\Omega_n \text{ for } 0 < r < 1/j\}$$

where $\Omega_n = |B(0,1)|$. The sets $F_{k,j}$ are Borel sets and $F_{k,j} \subset F_{k,j+1}$ with

$$F_k = \bigcup_{j=1}^{\infty} F_{k,j}. \tag{3.2}$$

To see (3.2) let $x \in F_k$. Then $H(x,f) < 2^{k+1}$ and hence there is j such that

$$L(x,f,r)/l(x,f,r) < 2^{k+1}$$

for all $0 < r < 1/j$ and we obtain

$$L(x,f,r)^n < 2^{n(k+1)} l(x,f,r)^n \le 2^{n(k+1)} |fB(x,r)|/\Omega_n.$$

This shows that $x \in F_{k,j}$ and (3.2) follows.
By the monotonicity and (3.2) it suffices to prove (3.1) for $F_{k,j}$ instead of F_k. Fix j and let F' be an arbitrary compact subset of $F_{k,j}$. Let $\epsilon > 0$ and $t > 0$. The continuity of the mapping $(x,r) \mapsto L(x,f,r)$ gives δ, $0 < \delta < 1/j$, such that $L(x,f,r) < t/2$ for $0 < r < \delta$ and for all $x \in F'$. By a well known covering lemma for sets on a real line, see [G1, Lemma 1, p.6], for each sufficiently small $r > 0$, $0 < r < \delta$, there exists a covering of F' by a finite number of open balls $B_i = B(x_i, r)$, $i = 1, \ldots, l$, where (a) $x_i \in F'$, $i = 1, \ldots, l$, (b) each point of $\mathbf{R}^n$ lies in at most two B_i and (c) $lr \le H^1(F') + \epsilon$. Note that the normalizing condition gives

$$B_i \subset Q \cap P^{-1}(B) \tag{3.3}$$

where $B = B^{n-1}(y,r)$.
The union of the sets $f(B_i)$ covers $f(F')$ and

$$\mathrm{dia}(fB_i) \le 2L(x_i,f,r) < t.$$

Hence

$$H^1_t(fF') \le \sum_{i=1}^{l} \mathrm{dia}(fB_i)$$

where

$$H^1_t(A) = \inf\{\Sigma\, \mathrm{dia}(A_i) : \cup A_i \supset A,\ \mathrm{dia}(A_i) < t\}$$

and the Hölder inequality together with the definition of $F_{k,j}$ yields

$$\begin{aligned} H^1_t(fF')^n &\le \left(\sum_{i=1}^{l} \mathrm{dia}(fB_i)\right)^n \le l^{n-1} \sum_{i=1}^{l} \mathrm{dia}(fB_i)^n \\ &\le l^{n-1} 2^n \sum_{i=1}^{l} L(x_i,f,r)^n \le \frac{l^{n-1} 2^n 2^{n(k+1)}}{\Omega_n} \sum_{i=1}^{l} |fB_i|. \end{aligned} \tag{3.4}$$

Since f is a homeomorphism, we obtain from (b) and (3.3) that

$$\sum_{i=1}^{l} |fB_i| \le 2 \left|\bigcup_{i=1}^{l} fB_i\right| \le 2\Phi(B)$$

and thus (3.4) and (c) yield

$$\begin{aligned} H_t^1(fF')^n &\leq 2^{n(k+2)+1}(H^1(F')+\epsilon)^{n-1}\Phi(B)/H^{n-1}(B) \\ &\leq 2^{n(k+2)+1}(H^1(F_{k,j})+\epsilon)^{n-1}\Phi(B)/H^{n-1}(B). \end{aligned}$$

Since $H_t^1(fF') \to H^1(fF')$ as $t \to 0$, letting first $r \to 0$, then $\epsilon \to 0$, and finally $t \to 0$ we obtain

$$H^1(fF')^n \leq 2^{n(k+2)+1}H^1(F_{k,j})^{n-1}\Phi'(y). \tag{3.5}$$

Now F' is an arbitrary compact subset of $F_{k,j}$ and hence (3.5) holds for $F_{k,j}$ on the left hand side of (3.5). This leads to the estimate (3.1).
Since $fF = \cup fF_k$, (3.1) implies

$$H^1(fF) \leq \sum H^1(fF_k) \leq C\sum 2^k H^1(F_k)^{\frac{n-1}{n}}. \tag{3.6}$$

The sets F_k, $k = 1, \ldots$, are disjoint and hence the integral estimate

$$\sum_{k=0}^{\infty} 2^{ks}H^1(F_k) \leq \int_F H(x,f)^s\,dx_n \tag{3.7}$$

is elementary. From (3.6), (3.7) and from the Hölder inequality we obtain

$$\begin{aligned} H^1(fF) &\leq C_1\left(\sum_{k=0}^{\infty} 2^{ks}H^1(F_k)\right)^{\frac{n-1}{n}}\left(\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}\right)^{\frac{1}{n}} \\ &\leq C_2\left(\int_F H(x,f)^s\,dx_n\right)^{\frac{n-1}{n}} \end{aligned} \tag{3.8}$$

where C_2 depends only on n, s and $\Phi'(y)$. Note that the series

$$\sum_{k=0}^{\infty} 2^{k(n-s(n-1))}$$

converges because $s > n/(n-1)$ and hence $n - s(n-1) < 0$. The inequality (3.8) shows that f is absolutely continuous on $I(y)$ as required.

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