MAPPINGS of FINITE DISTORTION: Monotonicity and Continuity

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1 Introduction

We study mappings $f = (f_1, ..., f_n) : \Omega \to \mathbb{R}^n$ in the Sobolev space $W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$, where Ω is a connected, open subset of $\mathbb{R}^n$ with $n \geq 2$. Thus, for almost every $x \in \Omega$, we can speak of the linear transformation $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$, called differential of f at x. Its norm is defined by $|Df(x)| = \sup\{|Df(x)h| : h \in S^{n-1}\}$. We shall often identify $Df(x)$ with its matrix, and denote by $J(x, f) = \det Df(x)$ the Jacobian determinant. Thus, using the language of differential forms, we can write

$$J(x, f)dx = df_1 \wedge ... \wedge df_n$$

Most of the time the Jacobian determinant will be nonnegative and we shall refer to such mappings as orientation preserving (this need not have a topological interpretation).

Definition 1.1 *A Sobolev mapping $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ is said to have finite distortion if there is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that*

$$|Df(x)|^n \leq K(x)J(x, f) \qquad a.e. \tag{1}$$

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We call (1) the distortion inequality for f. It is worth recalling that the smallest such function K, refered to as *outer dilatation*, is defined by

$$K_O(x,f) = \begin{cases} \frac{|Df(x)|^n}{J(x,f)} & \text{if } J(x,f) \neq 0 \\ 1 & \text{if } J(x,f) = 0 \end{cases} \tag{2}$$

Geometrically this means that at the points where $J(x,f) > 0$ the differential takes the unit ball to an ellipsoid E and we have $K_O(x,f) = \text{vol } B_O \ / \text{ vol } E$, where B_O is the ball circumscribed about E. Similarly, the ratio $K_I(x,f) = \text{vol } E \ / \text{ vol } B_I$, where B_I is the ball inscribed in E, is known as the *inner dilatation* of f. Precisely, we have

$$K_I(x,f) = \begin{cases} |Df(x)^{-1}| \ J(x,f) & \text{if } J(x,f) \neq 0 \\ 1 & \text{if } J(x,f) = 0 \end{cases} \tag{3}$$

There are, of course, other distortion functions of interest but we shall confine ourselves to only these basic ones. There have been remarkable advances made towards understanding and developing a theory of mappings with finite distortion. The noticeable growth of applications, especially in nonlinear elasticity, necessitates thorough revision of the foundation of this theory. It is the objective of the present paper to give a complete account of the continuity properties of mappings with finite distortion. We investigate them under minimal possible assumptions on the degree of integrability of the differential. The main trust of our results is that no additional assumptions on the distortion function are made here.

We shall take a little time now to point out some of the advances in the current literature. The origin of mappings with finite distortion can be traced back to the work of Yu. G. Reshetnyak [26] but our account begins with the paper of V. Goldshtein and S.K. Vodopyanov [8], in which they showed that mappings of finite distortion in the Sobolev class $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ are actually continuous. Most recently, F.W. Gehring and T. Iwaniec [7] verified that the limit mapping f of a weakly convergent sequence of mappings $f_j \in W^{1,n}(\Omega, \mathbb{R}^n)$ with finite distortion $K = K(x)$ also has finite distortion. While substantial progress has been made on the limit theorems, many questions still remain unanswered. Let us next discuss some results for mappings in $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ under suitable integrability conditions on the distortion function. First, T. Iwaniec and V. Šverák [17] proved that non-constant mappings in $W^{1,2}_{loc}(\Omega, \mathbb{R}^2)$ with locally integrable distortion are open and discrete. In higher dimensions, the analog of this holds when $K_O \in L^p_{loc}(\Omega)$ for some $p > n-1$ and $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. It fails if $p < n-1$ (see [3]), though it remains unknown in the critical case of $p = n-1$. Let us observe that $K_I(x,f) \leq K_O^{n-1}(x,f)$, a.e. We believe that in all dimensions the nonconstant mappings $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$ with $K_I \in L^1_{loc}(\Omega)$ are open and discrete. The initial steps towards solution of this problem were made by J. Heinonen and P. Koskela [12] while more definite answers were given later by J. Manfredi and E. Villamor [20], [21].

The natural Sobolev setting for mappings of finite distortion is in the space $W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$, largely due to the wish to integrate the Jacobian determinant by parts. However, matters are quite complicated if one does not know a priori that the Jacobian is locally integrable or, even if so, whether it coincides with the so-called distributional Jacobian. The first regularity results below the natural setting were recently established by T. Iwaniec, P. Koskela and G. Martin [14]. Assuming that $J(x,f) \in L^1_{loc}(\Omega)$ and $e^{\lambda K} \in L^1_{loc}(\Omega)$ for some sufficiently large $\lambda = \lambda(n)$ they proved, among other things, that in fact $f \in W^{1,n}_{loc}(\Omega, \mathbb{R}^n)$. Also see [1] for further developments. The standing conjecture is that one can take $\lambda = \lambda(n) = 1$ as the critical exponent for the regularity conclusions; it is known that the L^n-integrability of the differential fails for any $\lambda < 1$. The relevant examples are homeomorphic maps in $W^{1,1}_{loc}(\Omega, \mathbb{R}^n)$ and, therefore, have locally integrable Jacobian determinants.

One message from the present paper is that mappings of exponentially integrable distortion, regardless of the size of λ, always have a continuous representative. More precisely, such mappings coincide almost everywhere with a continuous map. Here and in the sequel, for brevity, we simply say that the mappings in question are continuous.

Theorem 1.2 *Let $f \in W^{1,1}(B, \mathbb{R}^n)$ satisfy the distortion inequality*

$$|Df(x)|^n \leq K(x)J(x,f) \qquad a.e.$$

in a ball $B = B(0,R)$, where $e^{\lambda K}$ is integrable for some $\lambda > 0$. If the Jacobian determinant is integrable, then f is continuous and we have the modulus of *continuity estimate*

$$|f(x) - f(y)|^n \leq \frac{C_K(n,\lambda) \ \int_B J(x,f) \ dx}{\log\log\left(e + \frac{R}{|x-y|}\right)} \tag{4}$$

for all $x, y \in B(0, \frac{R}{2})$.

An explicit bound for the constant here is:

$$C_K(n,\lambda) \leq C(n) \fint_B e^{\lambda K(x)} \ dx \tag{5}$$

where $\fint_B$ stands for the integral average over the ball B. Theorem 1.2 is a consequence of our more general results concerning continuity of mappings (of arbitrary finite distortion) in the Orlicz-Sobolev classes $W^{1,\Phi}(\Omega, \mathbb{R}^n)$. The point is that the assumption $e^{\lambda K} \in L^1(\Omega)$, together with the integrability of the Jacobian determinant, implies that $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$ with the Orlicz function $\Phi(t) = \frac{t^n}{\log(e+t)}$. This is exactly what we need for Theorem 1.2. In this connection, we should point out that the Jacobian determinant of an orientation preserving mapping in $W^{1,\Phi}(\Omega, \mathbb{R}^n)$ is always locally integrable [16], and coincides with the distributional Jacobian [9]. The summability of nonnegative Jacobians is the heart of our ideas. Two categories of Sobolev mappings are well suited for the question of summability of the Jacobian; the Orlicz-Sobolev spaces $W^{1,\Phi}(\Omega, \mathbb{R}^n)$ with

$$\int_1^\infty \frac{\Phi(t) \ dt}{t^{n+1}} = \infty \tag{6}$$

and the so-called grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$. The latter consists of mappings whose differential belongs to the space $VL^n(\Omega)$ of vanishing mobulus of integrability. Without getting into technicalities, this means that

$$\lim_{\epsilon \to 0+} \epsilon \int_\Omega |Df(x)|^{n-\epsilon} \ dx = 0 \tag{7}$$

see Section 2 for details. Let us first consider continuity in the case of the grand Sobolev space.

Theorem 1.3 *Mappings $f \in W^{1,n)}(B, \mathbb{R}^n)$ of finite distortion in a ball $B = B(0, R)$ satisfy the continuity estimate*

$$|f(x) - f(y)| \leq C(n) R \; \mathcal{L}^n \left(Df; \log^{-1} \frac{R}{|x-y|} \right) \tag{8}$$

for all $x, y \in B(0, \frac{R}{10})$. Here, $\mathcal{L}^n(Df; t)$ stands for the modulus of integrability of Df:

$$\mathcal{L}^n(Df;t) = \left[t \fint_B |Df|^{n-t} \right]^{\frac{1}{n-t}} \qquad 0 < t \leq n-1 \tag{9}$$

In many ways this result is sharp. To see this, consider the mapping $f(x) = x + \frac{x}{|x|}$. An elementary computation reveals that $|Df(x)| = 1 + \frac{1}{|x|}$, $J(x,f) = \left(1 + \frac{1}{|x|}\right)^{n-1}$ and $K_I(x,f) = 1 + \frac{1}{|x|}$. We then see that the differential belongs to the Marcinkiewicz space $weak - L^n(B)$ and, consequently, has bounded (though not vanishing) modulus of integrability. The reader is referred to Section 3 for the discussion of this example.

Theorem 1.3 covers Orlicz Sobolev mappings $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$ as well, but only if

$$\Phi(t) \geq \frac{c \; t^n}{\log(e+t)}, \tag{10}$$

confront this with Proposition 2.1.

This and other continuity results for Orlicz-Sobolev mappings with finite distortion require estimates below the dimension. For an orientation preserving mapping $f = (f_1, ..., f_n)$ in $W^{1,n-\epsilon}(B)$ we have

$$\int_B |Df|^{-\epsilon} df_1 \wedge ... \wedge df_n \leq C(n)\epsilon \int_B |Df(x)|^{n-\epsilon} \, dx \tag{11}$$

provided one of the coordinate functions vanishes on ∂B in the sense of distributions. This inequality is part of a more general spectrum of integral estimates concerning wedge products of differential forms [16], [10] and [13]. The key is to average (11) with respect to ϵ to gain estimates that yield continuity of mappings $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$ with finite distortion. This can be done for a class of the Orlicz functions Φ satisfying (6). Notice that (6) is necessary because of the example $f(x) = x + \frac{x}{|x|}$. The reason why we have not been able to deal with all Orlicz functions Φ satisfying (6) is basically that it is not known if the pointwise Jacobian of an orientation preserving Orlicz-Sobolev mapping always coincides with the distributional Jacobian under condition (6). To illustrate our results let us state here some of them.

Theorem 1.4 *Mappings $f \in W^{1,\Phi}(B, \mathbb{R}^n)$ with finite distortion are continuous if*

$$\Phi(t) \geq \frac{C\ t^n}{\log(e+t)\log\log(e+t)} \tag{12}$$

Sharp estimates on the modulus of continuity are also available. For instance, if (12) holds as equality, then

$$|f(x) - f(y)| \leq \frac{C(n)R\ ||Df||_\Phi}{\log\log\log^{\frac{1}{n}}\left(e + \frac{R}{|x-y|}\right)} \tag{13}$$

If $\Phi(t) = t^n \log^{\alpha-1}(e+t)$ and $\alpha > 0$, then

$$|f(x) - f(y)| \leq \frac{C(n)R\ ||Df||_\Phi}{\log^{\frac{\alpha}{n}}\left(e + \frac{R}{|x-y|}\right)} \tag{14}$$

If $\Phi(t) = t^n \log^{-1}(e+t)$, then

$$|f(x) - f(y)| \leq \frac{C(n)R\ ||Df||_\Phi}{\log\log^{\frac{1}{n}}\left(e + \frac{R}{|x-y|}\right)} \tag{15}$$

Here $x, y \in B(0, \frac{R}{2})$, where $||Df||_\Phi$ stands for the Luxemburg Φ-norm of the differential.

A very powerful method when dealing with continuity properties of functions is furnished by the notion of monotonicity, which goes back to H. Lebesgue [18] in 1907. Monotonicity for a continuous function u in a domain $\Omega \subset \mathbb{R}^n$ simply means that

$$\operatorname{osc}(u, B) \leq \operatorname{osc}(u, \partial B) \tag{16}$$

for every ball $B \subset \Omega$. Roughly speaking, u satisfies both the maximum and minimum principles in Ω. Hence the relevance for elliptic PDEs is clear. However, to effectively handle very weak solutions to the differential inequalities such as (1) one needs to adopt a definition of the so-called weakly monotone functions, due to J. Manfredi [19].

Definition 1.5 *A real valued function $u \in W^{1,\Phi}(\Omega)$ is said to be weakly monotone if for every ball $B \subset \Omega$ and all constants $m \leq M$ such that*

$$\varphi := (u - M)^+ - (m - u)^+ \in W_0^{1,\Phi}(B), \tag{17}$$

we have

$$m \leq u(x) \leq M$$

for almost every $x \in B$.

Without saying it so every time, we are working under the integral condition (6) on the Orlicz function Φ. It will also be required that the function $\tau \to \Phi(\sqrt[p]{\tau})$ is convex for some $p > n - 1$. Note that (6) is equivalent to

$$\int_0^1 \Phi\left(\frac{1}{s}\right) \, ds^n = \infty \tag{18}$$

To accomodate explicit bounds we define the Φ-modulus of continuity

$$\omega = \omega_\Phi(t), \qquad 0 < t \leq 1$$

where ω is determined uniquely from the equation

$$1 = \int_t^1 \Phi\left(\frac{\omega}{s}\right) \, ds^n \tag{19}$$

For example, if $\Phi(t) = t^n \log^{\alpha-1}(e + t)$, $\alpha > 0$, then $\omega_\Phi(t) \approx \log^{-\frac{\alpha}{n}}\left(e + \frac{1}{t}\right)$.

Theorem 1.6 *Let $u \in W^{1,\Phi}(B)$ be weakly monotone in the ball $B = B(0, R)$. Then for all Lebesgue points $x, y \in B(0, \frac{R}{2})$ it holds that*

$$|u(x) - u(y)| \leq C(p, n) \; R \; ||\nabla u||_\Phi \; \omega_\Phi\left(\frac{|x - y|}{R}\right). \tag{20}$$

In particular, u has a continuous representative.

The use of weakly monotone functions is essential for our approach: Theorem 1.4 and a more general version of it will be deduced from Theorem 1.6 and the weak monotonicity of mappings of finite distortion in the Orlicz-Sobolev class $W^{1,\Phi}(\Omega, \mathbb{R}^n)$. Also the proof of Theorem 1.3 is based on similar reasoning. For a discussion of weak monotonicity of mappings of finite distortion in Orlicz-Sobolev classes see Section 4 that also deals with the question of sharpness.

There are many related works that have not been mentioned above. First of all the interesting paper [5] of G. David gives existence theorems for mappings of exponentially integrable distortion in the plane. These results are obtained using the Beltrami equation. Theorem 1.2 gives new information even in this setting. Papers related to David's result include [4], [24], [27] and [29]. Also see the references in these papers. In higher dimensions references not discussed yet include [11] and [22].

2 Grand and Orlicz spaces

In this section we briefly review some known function spaces that will be used in the sequel. We will give sharp results on the inclusions between Orlicz spaces and grand Lebesgue spaces.

Let Ω be an open bounded region in $\mathbb{R}^n$. The average of a function $u \in L^1(\Omega)$ is denoted by

$$u_\Omega = \fint_\Omega u(x)dx.$$

For $p > 1$ we shall consider functions

$$u \in \bigcap_{1 \leq s < p} L^s(\Omega),$$

and define their *modulus of integrability* by

$$\mathcal{L}^p(u;\epsilon) = \left[\epsilon \fint_\Omega |u|^{p-\epsilon}\right]^{\frac{1}{p-\epsilon}}, \qquad 0 < \epsilon \leq p-1.$$

The space $BL^p(\Omega)$ of functions with bounded modulus of integrability, also known as the *grand Lebesgue space* $L^{p)}(\Omega)$ [16], is furnished with the norm

$$||u||_{p)} = \sup_{0<\epsilon\leq p-1} \mathcal{L}^p(u;\epsilon).$$

$BL^p(\Omega)$ is a Banach space. It then follows that the closure of $L^p(\Omega)$ with respect to this norm, denoted by $VL^p(\Omega)$, consists of functions with vanishing modulus of integrability. Namely

$$\lim_{\epsilon\to 0+} \mathcal{L}^p(u;\epsilon) = 0$$

whenever $u \in VL^p(\Omega)$.

Next recall the Marcinkiewicz space $weak - L^p(\Omega)$ which consists of all measurable functions $u : \Omega \to \mathbb{R}^n$ such that

$$\sup_{t>0} |\{x \in \Omega : u(x) > t\}| t^p < \infty.$$

We note that $weak - L^p(\Omega)$ is contained in $BL^p(\Omega)$ but not in $VL^p(\Omega)$, see [16]. The spaces $weak - L^p(\Omega)$ and $VL^p(\Omega)$ are not comparable [9].

A continuous and strictly increasing function $\Phi : [0,\infty] \to [0,\infty]$ with $\Phi(0) = 0$ and $\Phi(\infty) = \infty$, is called an *Orlicz function*. The *Orlicz space* $L^\Phi(\Omega)$ is made up of all measurable functions u on Ω such that

$$\int_\Omega \Phi\left(\frac{|u|}{k}\right) < \infty$$

for some positive $k = k(u)$. $L^\Phi(\Omega)$ is equipped with the nonlinear *Luxemburg functional*

$$||u||_\Phi = \inf\{k > 0 : \fint_\Omega \Phi\left(\frac{|u|}{k}\right) \leq 1\}.$$

If Φ is convex, then $||\cdot||_\Phi$ defines a norm in $L^\Phi(\Omega)$. The Orlicz space $L^\Theta(\Omega)$ with

$$\Theta(t) = t^p log^\alpha(e + t),$$

also denoted by $L^p \log^\alpha L(\Omega)$, will play special role in our investigation. In [9] L. Greco examined relations between $L^p \log^\alpha L(\Omega)$ and the grand Lebesgue spaces. He proved that

$$L^p(\Omega) \subsetneq L^p \log^{-1} L(\Omega) \subsetneq VL^p(\Omega) \subsetneq BL^p(\Omega) \subsetneq \bigcap_{\alpha<-1} L^p \log^\alpha L(\Omega).$$

The aim of this section is to improve on this result. The interplay between the Orlicz spaces and $BL^p(\Omega)$ is both satisfying and indispensible for the forthcoming results. In this regard, we give the following sharp inclusions.

Proposition 2.1 *If $1 < p < \infty$, then*

$$L^p \log^{-1} L(\Omega) \subset VL^p(\Omega) \subset BL^p(\Omega) \subset L^{\Psi}(\Omega),$$

for all Orlicz functions Ψ such that the function $t \to \Psi(t)t^{-p}\log t$, with large t, decreases to zero fast enough to satisfy

$$\int_1^{\infty} \frac{\Psi(t)}{t^{p+1}} dt < \infty. \tag{21}$$

Convergence of this integral is necessary for the latter inclusion. There are no Orlicz spaces between $L^p \log^{-1}$ and BL^p.

Proof. We will first show that $L^p \log^{-1} L(\Omega)$ is property contained in $VL^p(\Omega)$. Let $h \in L^p \log^{-1} L(\Omega)$. Fix $\delta > 0$ and $a \in (1, \infty)$ so big that

$$log(e+1) \int_{\{x\in\Omega:|h(x)|>a\}} \frac{|h|^p\ dx}{log(e+|h|)} < \delta.$$

Then, by the elementary inequality $(e+t)^{\epsilon} < e + t^{\epsilon}$, we obtain

$$\begin{aligned}
\epsilon \int_{\Omega} |h|^{p-\epsilon} dx &= \epsilon \int\limits_{\{x\in\Omega:h(x)\le a\}} |h|^{p-\epsilon} + \epsilon \int\limits_{\{x\in\Omega:h(x)>a\}} |h|^{p-\epsilon} \\
&\le \epsilon a^{p-\epsilon}|\Omega| + \epsilon \int\limits_{\{x\in\Omega:h(x)>a\}} \frac{|h|^p}{\log(e+|h|)} \frac{\log(e+|h|)}{|h|^{\epsilon}} \\
&\le \epsilon a^{p-\epsilon}|\Omega| + \int\limits_{\{x\in\Omega:h(x)>a\}} \frac{|h|^p}{\log(e+|h|)} \frac{\log(e+|h|^{\epsilon})}{|h|^{\epsilon}} \\
&\le \epsilon a^{p-\epsilon}|\Omega| + \sup_{t>1} \frac{\log(e+t)}{t} \frac{\delta}{\log(e+1)} \\
&\le \epsilon a^{p-\epsilon}|\Omega| + \delta.
\end{aligned}$$

Thus

$$0 \le \limsup_{\epsilon\to 0} \epsilon \int_{\Omega} |h|^{p-\epsilon} dx \le \delta.$$

Since $\delta > 0$ was arbitrary, we conclude with $\lim\limits_{\epsilon\to 0} \epsilon \int_{\Omega} |h|^{p-\epsilon} dx = 0$, as desired.

We shall now show that

$$L^{\Phi}(\Omega) \not\subset VL^p(\Omega).$$

for any Orlicz function $\Phi(t) = \frac{H(t)\ t^p}{\log t}$ such that $\liminf_{t\to\infty} H(t) = 0$. For this, we must construct a function $f \in L^\Phi(\Omega)$ which is not in $VL^p(\Omega)$.

Since $\liminf_{t\to\infty} H(t) = 0$, one can find $k_0 \in \mathbb{N}$ and numbers $t_k > e$ such that $H(t_k) = 2^{-k}$, for $k \geq k_0$. Clearly, there is a subsequence $\{k_j\}_{j=1}^\infty$ for which

$$\sum_{j=1}^\infty \frac{\log t_{k_j}}{t_{k_j}^{p-1}} \leq |\Omega|.$$

For instance, choose t_{k_j} large enough to satisfy $t_{k_j}^{1-p} \log t_{k_j} \leq 2^{-j}|\Omega|$.

To each t_{k_j} we associate the number

$$a_j = \min\{k_j, t_{k_j}\}.$$

It is clear that there exist mutually disjoint measurable subsets $E_j \subset \Omega$ such that

$$|E_j| = a_j t_{k_j}^{-p} \log t_{k_j}.$$

Having disposed of these terms we are now in a position to define the function f by the formula

$$f(x) := \sum_{j=1}^\infty t_{k_j} \chi_{E_j}(x), \qquad \text{for all } x \in \Omega.$$

Then we have that

$$\begin{aligned}
\int_\Omega \Phi(|f|)dx &= \int_\Omega \frac{|f|^p H(|f|)\ dx}{\log|f|} \\
&= \sum_{j=1}^\infty |E_j| \frac{t_{k_j}^p\ H(t_{k_j})}{\log t_{k_j}} \\
&= \sum_{j=1}^\infty a_j\ H(t_{k_j}) \leq \sum_{j=1}^\infty \frac{k_j}{2^{k_j}} \leq 2
\end{aligned}$$

On the other hand

$$\begin{aligned}
\limsup_{\epsilon\to 0}\ \epsilon \int_\Omega |f|^{p-\epsilon} dx &\geq \lim_{j\to\infty} \frac{p}{\log t_{k_j}} \int_{E_j} |f|^{p-\frac{p}{\log t_{k_j}}} \\
&= \lim_{j\to\infty} \frac{p|E_j|}{\log t_{k_j}} \left[t_{k_j}\right]^{p-\frac{p}{\log t_{k_j}}} \\
&= \lim_{j\to\infty} p a_j e^{-p} = \infty,
\end{aligned}$$

Hence $f \notin VL^p(\Omega)$, as required.

Next suppose that an Orlicz function Ψ satisfies the conditions stated in Proposition 2.1. For notational convenience we write $F(t) = \Psi(t)t^{-p}\log t$. Thus

$$\int_e^\infty \frac{F(t)\ dt}{t\log t} \leq \int_1^\infty \frac{\Psi(t)}{t^{p+1}} < \infty$$

We notice, by using the Integral Test, that for sufficiently large N

$$\sum_{k=N}^{\infty} F\left(e^{e^k}\right) \leq \int_0^\infty F\left(e^{e^x}\right)\ dx = \int_e^\infty \frac{F(t)\ dt}{t\log t} < \infty$$

Now, consider an arbitrary $u \in BL^p(\Omega)$ and its level sets

$$\Omega_k = \{x \in \Omega : e^{e^k} \leq |u(x)| < e^{e^{k+1}}\} \quad (k = 0, 1, 2, ...).$$

It involves no loss of generality in assuming that $||u||_{p)} = 1$.
For every $x \in \Omega_k$, we have $e \leq |u(x)|^{e^{-k}} < e^e$. Therefore, for $k \geq N$, we can write

$$\int_{\Omega_k} \Psi(u) = \int_{\Omega_k} \frac{u^p\ F(u)}{\log u} \leq \int_{\Omega_k} \frac{u^p F(e^{e^k})}{e^k}$$

since $F(u)$ is decreasing for $u \geq e^{e^N}$.
By the definition of the grand L^p-norm we obtain

$$\begin{aligned}\int_{\Omega_k} \Psi(u)dx \ &\leq \ u^{e^{-k}}\ F\left(e^{e^k}\right)\ e^{-k}\int_{\Omega_k} u^{p-e^{-k}} \\ &\leq \ e^e\ ||u||_{p)}^{p-e^{-k}}\ F\left(e^{e^k}\right) = e^e F\left(e^{e^k}\right).\end{aligned}$$

Summing up we conclude with the desired estimate

$$\int_{|u|\geq e^{e^N}} \Psi(u) \leq e^e \int_e^\infty \frac{F(t)\ dt}{t\log t} < \infty$$

which shows that $u \in L^\Psi(\Omega)$.

In order to see that condition (21) is necessary, we consider the function $u(x) = |x|^{-\frac{n}{p}}$ in the unit ball $\Omega = \{x : |x| \leq 1\}$. Note that

$$\epsilon \fint_\Omega |u(x)|^{p-\epsilon}\ dx = p.$$

Hence u belongs to $BL^p(\Omega)$. Suppose now that $u \in L^\Psi(\Omega)$ for some Orlicz function Ψ. An elementary computation reveals that

$$\int_1^\infty \frac{\Psi(t)}{t^{p+1}} = \frac{1}{p} \fint_\Omega \Psi(u) < \infty.$$

We end this section with two more definitions. The completion of the Sobolev space $W^{1,p}(\Omega)$ under the norm

$$||\cdot||_p + ||\nabla \cdot||_{p)}$$

will be called the grand Sobolev space and denoted by $W^{1,p)}(\Omega)$. Thus

$$\lim_{\epsilon \to 0} \epsilon \int_\Omega |\nabla u|^{p-\epsilon} = 0$$

whenever $u \in W^{1,p)}(\Omega)$. We do not introduce the space of functions with $|\nabla u|$ in $BL^p(\Omega)$ as the need will not arise. The Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ consists of function $u \in W^{1,1}(\Omega)$ such that $\nabla u \in L^\Phi(\Omega)$.

3 Example

Mappings with finite distortion in the Sobolev space $W^{1,n}(\Omega, \mathbb{R}^n)$ are known to be continuous. At this point it is worthwhile to recall the familiar example of a map that forms a cavity at the origin [2]

$$f : \mathbb{B} \setminus \{0\} \to \mathbb{R}^n : x \mapsto x + \frac{x}{|x|}$$

where $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$. This example indicates that, in order to preserve continuity, one should not go too far from $W^{1,n}(\Omega)$.
Clearly, the function f cannot be redefined in a set of measure zero to become continuous at the origin. By an easy calculation, we get

$$Df(x) = (1 + \frac{1}{|x|})I - \frac{x \otimes x}{|x|^3},$$

where $x \otimes x$ is the $n \times n$ matrix whose i, j-entry equals $x_i x_j$. Hence

$$|Df(x)| = 1 + \frac{1}{|x|}$$

and

$$\det\, Df(x) = (1 + \frac{1}{|x|})^{n-1}.$$

Thus the outer dilatation function $K_O(x) = 1 + \frac{1}{|x|}$ belongs to $weak - L^n(\mathbb{B})$. We see at once that $|Df(x)| \in BL^n(\mathbb{B})$ and

$$\lim_{\epsilon \to 0} \epsilon \fint_{\mathbb{B}} |x|^{n-\epsilon} = 1$$

Thus $f \notin W^{1,n)}(\mathbb{B}, \mathbb{R}^n)$. Moreover,

$$\fint_{\mathbb{B}} \Phi(\frac{1}{|x|})dx = \int_1^\infty \frac{\Phi(t)}{t^{n+1}} dt.$$

Hence $|Df(x)| \in L^\Phi(\mathbb{B}, \mathbb{R})$ if and only if the defining Orlicz fuction Φ satisfies

$$\int_1^\infty \frac{\Phi(t)}{t^{n+1}} dt < \infty. \tag{22}$$

This example suggests that, in order to conclude with any sort of continuity estimate for a mapping f of finite distortion, we must assume that $f \in W^{1,n)}(\Omega, \mathbb{R}^n)$ or in $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$, with $\int_1^\infty \frac{\Phi(t)}{t^{n+1}} dt = \infty$.

4 Monotonicity in $W^{1,n}(\Omega, \mathbb{R}^n)$

Unlike in the classical approach, we can easily prove without getting into PDEs that the coordinate functions of the map $f = (f_1, ..., f_n) \in W^{1,n}(\Omega, \mathbb{R}^n)$ of finite distortion are weakly monotone. It is rewarding and illuminating to outline this simple proof.

Let B be a ball in Ω and suppose that for some coordinate function, say the first one, we have

$$v := (f_1 - M)^+ - (m - f_1)^+ \in W_0^{1,n}(B).$$

Then

$$\nabla v(x) = \begin{cases} 0 & \text{if } m \leq f^1(x) \leq M \\ \nabla f_1(x), & \text{otherwise (say, on the set } E \subset B). \end{cases}$$

Hence, in view of the distortion inequality (1), we can write

$$\begin{aligned}
\int_\Omega \frac{|\nabla v(x)|^n}{K(x)} dx &\leq \int_E \frac{|Df(x)|^n}{K(x)} dx \\
&\leq \int_E df_1 \wedge df_2 \wedge ... \wedge df_n \\
&\leq \int_B dv \wedge df_2 \wedge ... \wedge df_n = 0,
\end{aligned}$$

by Stokes´ Theorem. Thus $v \equiv 0$ on B, which simply means that $m \leq f_1(x) \leq M$ for almost every $x \in B$, as desired.
Similar considerations apply to the John Ball class of mappings:

$$|Df| \in L^{n-1}(\Omega) \text{ and } |Adj Df| \in L^{\frac{n}{n-1}}(\Omega) \tag{23}$$

which has been studied by several people in nonlinear elasticity and calculus of variations [28], [23], [15], [12] and [11].
A slight change in the above proof leads to a result that is known to specialists in nonlinear elasticity.

Proposition 4.1 *Under the integrability hypotheses stated at (23), mappings of finite distortion are weakly monotone.*

The key here is the isoperimetric type inequality for mappings in the John Ball class:

$$\int_B \det Df(x) dx \leq C(n) \prod_{k=1}^{n} \left(\int_{\partial B} |df_1 \wedge ... \wedge \widehat{df_k} \wedge ... \wedge df_n|^{\frac{n}{n-1}} \right)^{\frac{1}{n}}$$

for almost every ball $B \in \Omega$ centered at the given point of Ω. This follows from an analogous estimate in [23] by analysis of homogeneity. As before, we find that $\int_B dv \wedge df_2 \wedge ... \wedge df_n = 0$, by applying this isoperimetric inequality instead of Stokes´ Theorem, details being left to the reader.

Matters are quite different if $f \notin W^{1,n}(\Omega, \mathbb{R}^n)$ because one does not know, a priori, that the Jacobian determinant is locally integrable. First we take on stage the mappings of finite distortion in the grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$.

5 Monotonicity in $W^{1,n)}(\Omega, \mathbb{R}^n)$

Essential to our development is to establish inequality (11).

Lemma 5.1 *Let $f : B \to \mathbb{R}^n$, be a mapping of Sobolev class $W^{1,n-\epsilon}(B, \mathbb{R}^n)$, $0 \leq \epsilon \leq 1$, in a ball B. Then*

$$\int_B |Df(x)|^{-\epsilon} \ J(x,f) \ dx \leq C(n)\epsilon \int_B |Df(x)|^{n-\epsilon} dx$$

provided one of the coordinate functions for $f = (f_1, ..., f_n)$ belongs to $W_0^{1,n-\epsilon}(B)$.

This inequality is part of a more general spectrum of integral estimates concerning wedge products of differential forms [13] and [16].

Proof. We may certainly assume that $0 \leq \epsilon \leq \frac{1}{2}$, since the inequality always holds with constant 1 in place of $C(n)\epsilon$. Suppose $f_1 \in W_0^{1,n-\epsilon}(B)$. As in [16] we begin with the Hodge decomposition

$$|df_1|^{-\epsilon} \ df_1 \ = \ d\varphi + \gamma$$

where $\varphi \in W_0^{1,\frac{n-\epsilon}{1-\epsilon}}(B)$ and $\gamma \in L^{\frac{n-\epsilon}{1-\epsilon}}(B)$. It is important to realize that γ becomes small as ϵ tends to zero. Precisely, we have

$$||\gamma||_{L^{\frac{n-\epsilon}{1-\epsilon}}(B)} \leq C(n)\epsilon \ ||df_1||^{1-\epsilon}_{L^{n-\epsilon}(B)} \leq C(n)\epsilon \ ||Df||^{1-\epsilon}_{L^{n-\epsilon}(B)}$$

see [16].

By Stokes' Theorem we find that

$$\begin{aligned} \int_B |df_1|^{-\epsilon} \ df_1 \wedge ... \wedge df_n \quad &= \quad \int_B \gamma \wedge df_2 ... \wedge df_n \leq \int_B |\gamma||Df|^{n-1} \\ &\leq \quad ||\gamma||_{\frac{n-\epsilon}{1-\epsilon}} ||Df||^{n-1}_{n-\epsilon} \leq \ C(n)\epsilon \int_B |Df(x)|^{n-\epsilon} \ dx \end{aligned}$$

It remains to observe that

$$\begin{aligned} |Df|^{-\epsilon} J(x,f) \ dx \quad &= \quad |df_1|^{-\epsilon} df_1 \wedge ... \wedge df_n - \left(|df_1|^{-\epsilon} - |Df|^{-\epsilon}\right) df_1 \wedge ... \wedge df_n \\ &\leq \quad |df_1|^{-\epsilon} df_1 \wedge ... \wedge df_n + \left(\frac{|Df|^{\epsilon}}{|df_1|^{\epsilon}} - 1\right) |df_1| \ |Df|^{n-1-\epsilon} \\ &\leq \quad |df_1|^{-\epsilon} df_1 \wedge ... \wedge df_n + \epsilon \ |Df|^{n-\epsilon} \end{aligned}$$

which follows from the elementary inequality $\lambda^\epsilon - 1 \leq \epsilon\lambda$, for $\lambda \geq 1$. The lemma is proved.

Now, we are ready to prove the following extension of V. Goldstein, Yu. G. Reshetnyak and S.K. Vodop'yanov result [25] and [8]; L^n-integrability of the differential being relaxed.

Proposition 5.2 *Coordinate functions of mappings with finite distortion in the grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$ are weakly monotone.*

Let us stress explicitly that this result covers mappings for which

$$\int_\Omega \frac{|Df(x)|^n}{log(e + |Df(x)|)} dx < \infty \tag{24}$$

as it is easy to see from Proposition 2.1.

Proof. By symmetry, it suffices to prove that f_1 is weakly monotone.
Let B be a ball in Ω and $m \leq M$ be real numbers such that $v = (f_1 - M)^+ - (m - f_1)^+ \in W_0^{1,n-\epsilon}(B)$ for every $\epsilon \in (0,1)$. We have to show that

$$m \leq f_1(x) \leq M$$

for almost every $x \in B$.
As in Section 4, we calculate

$$dv = \begin{cases} 0 & \text{if } m \leq f^1 \leq M \\ df_1 & \text{otherwise (say, on a set } E \subset B) \end{cases}$$

Applying Lemma 5.1 to the mapping $(v, f_2, ..., f_n) : B \to \mathbb{R}^n$, we obtain

$$\begin{aligned} \int_E \frac{|Df(x)|^{n-\epsilon}}{K(x)} dx &\leq \int_E |Df(x)|^{-\epsilon} J(f,x)\, dx \\ &\leq C(n)\epsilon \int_B |Df(x)|^{n-\epsilon}\, dx \end{aligned}$$

Therefore, since $\epsilon \int_B |Df|^{n-\epsilon} dx \to 0$, as $\epsilon \to 0$ and $1 \leq K(x) < \infty$ for almost every x, by the Monotone Convergence Theorem we infer that Df vanishes almost everywhere in E. This implies that $v = 0$, completing the proof of Proposition 5.2.

6 Monotonicity in $W^{1,\Phi}(\Omega, \mathbb{R}^n)$

The Jacobian determinants of orientation preserving mappings in $W^{1,\Phi}(\Omega, \mathbb{R}^n)$ are known to be locally integrable for a considerable class of Orlicz functions $\Phi(t) = o(t^n)$. What we actually need, to prove monotonicity, is the identity

$$\int_B df_1 \wedge ... \wedge df_n \;=\; 0 \tag{25}$$

for $f = (f_1, ..., f_n) \in W^{1,\Phi}(B, \mathbb{R}^n)$, with one coordinate function in $W_0^{1,\Phi}(B)$. Since we assume that $J(x, f) \geq 0$, identity (25) simply means that $df_1 \wedge ... \wedge df_n \;=\; J(x, f)\; dx \equiv 0$ in B. The affirmative answer to this question follows from Lemma 5.1 and it is our goal here to show how.

Consider a nonnegative decreasing function $\varphi \in C^1(0, 1]$ such that $\lim\limits_{s\to 0+} \varphi(s) = \infty$. Having disposed of Lemma 5.1 we can write, as a starting point, the inequality

$$\int_B \frac{-J(x,f)}{\varphi(\epsilon)} \left(\int_\epsilon^1 |Df(x)|^{-s} \; d\varphi(s) \right) \; dx$$

$$\leq C(n) \int_B \frac{-|Df(x)|^n}{\varphi(\epsilon)} \left(\int_\epsilon^1 \; s\; |Df(x)|^{-s} \; d\varphi(s) \right) \; dx \tag{26}$$

for all $0 < \epsilon \leq 1$. We want to pass to the limit as $\epsilon \to 0$. By the Fatou Lemma and L'Hôpital's Rule this procedure is perfectly valid for the left hand side, yielding the estimate

$$\int_B J(x, f)\; dx \leq C(n) \lim_{\epsilon\to 0} \int_B \frac{-|Df(x)|^n}{\varphi(\epsilon)} \left(\int_\epsilon^1 \; s\; |Df(x)|^{-s} \; d\varphi(s) \right) \; dx \tag{27}$$

The latter limit is equal to zero once we can use the Lebesgue Dominated Convergence Theorem. Indeed, by L'Hôpital's Rule we would obtain

$$\int_B J(x, f)\; dx \leq C(n) \int_B \lim_{\epsilon\to 0} \frac{\epsilon \varphi'(\epsilon)}{\varphi'(\epsilon)} |Df(x)|^{n-\epsilon} \; dx = 0 \tag{28}$$

We are, therefore, left with the task of justifying the use of the LDCT. To this effect, we only need to show that there is a function $M \in L^1(B)$ such that

$$-\frac{1}{\varphi(\epsilon)} \int_\epsilon^1 s|Df(x)|^{n-s} \; ds \leq M(x) \tag{29}$$

for sufficiently small $\epsilon > 0$. This is certainly true if $f \in W^{1,\Phi}(B)$, where the defining Orlicz function Φ is given by

$$\Phi(t) = \sup_{0<\epsilon<1} -\frac{1}{\varphi(\epsilon)} \int_\epsilon^1 st^{n-s}\, d\varphi(s) \tag{30}$$

Clearly Φ is increasing and convex. The question arises as to which Orlicz functions can be obtained by formula (30). One necessary condition on Φ is that

$$\int_1^\infty \frac{\Phi(t)\, dt}{t^{n+1}} = \infty \tag{31}$$

Indeed, for all $0 < \epsilon \leq 1$ we have

$$-\frac{1}{\varphi(\epsilon)} \int_\epsilon^1 \frac{s\, d\varphi(s)}{t^{1+s}} \leq \frac{\Phi(t)}{t^{n+1}}$$

Fixing an arbitrary $N \geq 1$ and integrating this inequality with respect to t from N to ∞ we arrive at

$$-\frac{1}{\varphi(\epsilon)} \int_\epsilon^1 \frac{d\varphi(s)}{N^s} \leq \int_N^\infty \frac{\Phi(t)\, dt}{t^{n+1}}$$

For every $0 < \delta < 1$ we can write

$$\frac{\varphi(\epsilon) - \varphi(\delta)}{N^\delta \varphi(\epsilon)} \leq -\frac{1}{\varphi(\epsilon)} \int_\epsilon^\delta \frac{d\varphi(s)}{N^s} \leq \int_N^\infty \frac{\Phi(t)\, dt}{t^{n+1}}$$

provided $0 < \epsilon \leq \delta$. Letting ϵ go to zero we obtain

$$\frac{1}{N^\delta} \leq \int_N^\infty \frac{\Phi(t)\, dt}{t^{n+1}}$$

Since δ can be arbitrarily small this estimate yields

$$1 \leq \int_N^\infty \frac{\Phi(t)\, dt}{t^{n+1}}$$

for all $N \geq 1$, which is possible only when integral at (31) diverges.

With the introduction of the Orlicz function Φ by the rule (30), we can now formulate the following result.

Theorem 6.1 *The coordinate functions of a mapping $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$ of finite distortion, with Φ given by (30), are weakly monotone.*

It is rewarding to make an explicit calculation for $\varphi(s) = \log \frac{e}{s}$, $0 < s \le 1$:

$$\begin{aligned} \Phi(t) &= \sup_{0<\epsilon<1} \log^{-1} \frac{e}{\epsilon} \int_\epsilon^1 t^{n-s}\, ds \\ &\le \sup_{0<\epsilon<1} \frac{t^n}{\left(t^\epsilon \log \frac{e}{\epsilon}\right) \log t} \\ &\le \frac{t^n}{\log t \ \log\log t} \end{aligned}$$

Here we have assumed that $t > e$, to claim the inequality

$$t^\epsilon \log \frac{e}{\epsilon} \ge \log\log t \tag{32}$$

for all $0 < \epsilon \le 1$. Indeed, the function $\epsilon \mapsto t^\epsilon \log \frac{e}{\epsilon}$ has exactly one critical point ϵ_0, which satisfies the equation $\epsilon_0 \log \frac{e}{\epsilon_0} = \log^{-1} t$. In particular, $\epsilon_0 \le \log^{-1} t$ and we obtain $t^\epsilon \log \frac{e}{\epsilon} \ge t_0^\epsilon \log \frac{e}{\epsilon_0} \ge t^{\epsilon_0} \log(e \log t) \ge \log\log t$. This results in the following preliminary step for the proof of Theorem 1.4.

Corollary 6.2 *The coordinate functions of a mapping $f \in W^{1,\Phi}(\Omega, \mathbb{R}^n)$ of finite distortion, with $\Phi(t) \ge \frac{C\ t^n}{\log(e+t)\ \log\log(e+t)}$, are weakly monotone.*

Formula (30) remains far from being conclusive for a general function $\varphi = \varphi(s)$, but we shall not enter into these quite involved computations. The Banach spaces that one naturaly encounters in this connection are equipped with the norm

$$[F]_\varphi = \inf\{K > 0 : -\int_\epsilon^1 sK^s \left(\int_B |F|^{n-s}\right)\ d\varphi(s) \le K^n \varphi(\epsilon) \text{ for all } 0 < \epsilon \le 1\} \tag{33}$$

We denote them by $L_\varphi(B)$ and leave it for the interested reader to verify that

$$[F]_\varphi \le ||F||_\Phi \tag{34}$$

where $||\cdot||_\Phi$ stands for the Luxemburg norm determined by the function Φ at (30). We also have

$$[F]_\varphi \le ||F||_{n)} \le C(n) ||F||_{L^n \log^{-1} L} \tag{35}$$

for every φ as described above.

7 The Oscillation Lemma

There is a particularly elegant geometric approach to the continuity estimate of monotone functions. The idea goes back to the oscillation lemma by F.W. Gehring [6]. While many interesting implications of Gehring's lemma have been discussed in the literature, the fact that one can use it for weakly monotone functions seems to be less familiar. It is surprising that the usual convolution procedure with mollifiers of Dirac distribution has little effect on the monotonicity of functions. Consequently, we take the time here to state and give a rigorous proof of this fact, as it might be of independent interest. Let $u \in W^{1,p}(B)$ be a Sobolev function in a ball $B(a,R)$. Fix a nonnegative $\chi \in C_0^\infty(B)$ supported in the unit ball such that $\int_B \chi(y)\, dy = 1$. The mollifiers $\chi_j(y) = j^n \chi(jy)$, $j = 1, ...,$ give rise to the sequence $u_j \in C^\infty(\mathbb{R}^n)$ defined by

$$u_j(x) = \chi_j * u(x) := \int_B u(y)\chi_j(x-y)\, dy \tag{36}$$

It is well known that $\{u_j\}$ converges to u in $W^{1,p}_{loc}(B)$ and $u_j(x_0) \to u(x_0)$, $u_j(y_0) \to u(y_0)$ at the Lebesgue points $x_0, y_0 \in B$. Here is the presice statement concerning monotonicity.

Lemma 7.1 *Let $u \in W^{1,p}(B)$ be weakly monotone in a ball $B = B(a,R)$ and x_0, y_0 be Lebesgue points in $B(a,r)$, $r < R$. For each $\delta > 0$ there exists N such that*

$$|u_j(x_0) - u_j(y_0)|^p \le \operatorname{osc}(u_j, \partial B(a,t)) + 2\delta \tag{37}$$

for all $j \ge N$ and every $r \le t \le R$.

Proof. We claim that the estimates

$$u_j(x_0) \le \max\{u_j(x) : x \in \partial B(a,t)\} + \delta \tag{38}$$

and

$$u_j(y_0) \ge \min\{u_j(y) : y \in \partial B(a,t)\} - \delta \tag{39}$$

hold for all $r \leq t \leq R$, whenever j is sufficiently large. We only need to show the first inequality. The second inequality follows by applying the first one to the function $-u$ and to the point y_0 in place of x_0. Suppose, to the contrary, that there exists a sequence $\{j_k\}$ and radii $r \leq t_k \leq R$, $k = 1, 2, 3, ...$, such that

$$u_{j_k}(x_0) > \max\{u_{j_k}(x) : x \in \partial B(a, t_k)\} + \delta.$$

We may assume that $\{t_k\}$ converges to some number $t \in [r, R]$. Since

$$u_{j_k}(x) - u_{j_k}(x_0) + \delta < 0$$

on $\partial B(a, t_k)$, we have

$$(u_{j_k} - u_{j_k}(x_0) + \delta)^+ \in W_0^{1,p}(B(a, t_k)).$$

Passing to the limit as $k \to \infty$ yields

$$(u - u(x_0) + \delta)^+ \in W_0^{1,p}(B(a, t)).$$

That this function vanishes on $\partial B(a, t)$ in the sense of distributions can be seen in various ways; details are left to the reader. As u is weakly monotone it follows that $u(x) \leq u(x_0) - \delta$ for almost every $x \in B(a, t)$. But this is impossible since x_0 is a Lebesgue point of u in $B(a, r) \subset B(a, t)$.

Now inequalities (38) and (39) imply

$$u_j(x_0) - u_j(y_0) \leq \operatorname{osc}(u_j, \partial B(a, t)) + 2\delta$$

One may interchange x_0 with y_0 to conclude with inequality (37), completing the proof of Lemma (7.1).

By Fubini's theorem we observe that the function $t \to \int_{\partial B(a,t)} |\nabla u|^p$ belongs to $L^1_{loc}(0, R)$. Consequently, its Lebesgue points form a set of full linear measure on the interval $(0, R)$. With these preliminaries, we can now prove the following variant of the oscillation lemma.

Lemma 7.2 *Let $u \in W^{1,p}(B)$, $n - 1 < p < n$, be weakly monotone in a ball $B = B(a, R)$ and x_0, y_0 be the Lebesgue points of u in $B(a, r)$, $r < R$. Then*

$$|u(x_0) - u(y_0)| \leq C(p, n) t \left(\fint_{\partial B(a,t)} |\nabla u|^p \right)^{\frac{1}{p}} \tag{40}$$

for almost every $t \in (r, R)$.

Proof. We apply Sobolev's inequality on spheres for the function $u_j \in C^\infty(B)$ at (37):

$$|u_j(x_0) - u_j(y_0)| \le C(p,n)t \left(\fint_{\partial B(a,t)} |\nabla u_j|^p \right)^{\frac{1}{p}} + 2\delta$$

for all $r \le t \le R$. Fix a Lebesgue point $r < t_0 < R$ of the function $t \to \int_{\partial B(a,t)} |\nabla u|^p$. For sufficiently small $\epsilon > 0$, upon integration over the interval $t_0 - \epsilon < t < t_0 + \epsilon$, we obtain

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \left(\frac{|u_j(x_0) - u_j(y_0)| - 2\delta}{C(p,n)t} \right)^p \omega_{n-1} t^{n-1} \, dt \le \int_{t_0-\epsilon \le |x| \le t_0+\epsilon} |\nabla u_j|^p$$

Now we can pass to the limit as $j \to \infty$. It is also legitimate to take $\delta = 0$ in this limit inequality

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{|u(x_0) - u(y_0)|^p}{C^p(p,n)t^p} \omega_{n-1} t^{n-1} \, dt \le \int_{t_0-\epsilon}^{t_0+\epsilon} \left(\int_{\partial B(a,t)} |\nabla u|^p \right)$$

Divide by 2ϵ and let ϵ go to zero to obtain

$$\frac{|u(x_0) - u(y_0)|^p \omega_{n-1} t^{n-1}}{C^p(p,n) t_0^p} \, dt \le \int_{\partial B(a,t_0)} |\nabla u|^p$$

This means that

$$|u(x_0) - u(y_0)| \le C(p,n) t_0 \left(\fint_{\partial B(a,t_0)} |\nabla u|^p \right)^{\frac{1}{p}}$$

as desired.

8 Modulus of Continuity

Here we prove Theorem 1.6. Recall that we are working under the condition that the function $\tau \to \Phi(\sqrt[p]{\tau})$ is convex for some $p > n-1$. Thus, Φ is also convex and $\Phi(s) \ge c \cdot s^p$ for large s. In particular, $u \in W^{1,p}(B)$ on the ball $B = B(0,R)$. Because of homogeneity at (20) we may assume that $||\nabla u||_\Phi = 2^{-n}$, that is

$$1 = \fint_{B(0,R)} \Phi\left(2^n |\nabla u|\right) \ge 2^n \fint_{B(0,R)} \Phi(|\nabla u|) \tag{41}$$

since $\Phi(K\tau) \geq K\Phi(\tau)$ for $K \geq 1$, as Φ is convex and vanishes at zero.

Fix Lebesgue points $x_0, y_0 \in B(0, \frac{1}{2}R)$ and consider the concentric balls $B(a,r) \subset B(a, \frac{1}{2}R) \subset B(0,R)$, where $a = \frac{x_0+y_0}{2}$ and $r = \frac{|x_0-y_0|}{2}$. By the Lemma 7.2 we have

$$\frac{|u(x_0)-u(y_0)|}{C(p,n)t} \leq \left(⨍_{\partial B(a,t)} |\nabla u|^p \right)^{\frac{1}{p}}$$

for almost all $r < t < \frac{1}{2}R$. Jensen's inequality applied to the convex function $\tau \to \Phi(\sqrt[p]{\tau})$ yields

$$\Phi\left(\frac{|u(x_0)-u(y_0)|}{C(p,n)t} \right) \leq ⨍_{\partial B(a,t)} \Phi(|\nabla u|)$$

We multiply by $\omega_{n-1}t^{n-1}$ and integrate over the interval $(r, \frac{1}{2}R)$ to obtain

$$\omega_{n-1} \int_r^{\frac{1}{2}R} \Phi\left(\frac{|u(x_0)-u(y_0)|}{C(p,n)t} \right) t^{n-1}\, dt \leq \int_{B(a,\frac{1}{2}R)} \Phi(|\nabla u|) \leq \int_{B(0,R)} \Phi(|\nabla u|) \tag{42}$$

Next we make the substitution $t = \frac{sR}{2}$ with $\frac{2r}{R} < s < 1$ and arrived at

$$\int_{\frac{2r}{R}}^1 \Phi\left(\frac{2|u(x_0)-u(y_0)|}{C(p,n)sR} \right)\, ds^n \leq 2^n ⨍_{B(0,R)} \Phi(|\nabla u|) \leq 1 \tag{43}$$

By the definition of the Φ-modulus of continuity this means that

$$\begin{aligned} |u(x_0)-u(y_0)| &\leq \frac{C(p,n)R}{2}\omega_\Phi\left(\frac{2r}{R}\right) \\ &= 2^{n-1}C(p,n)||\nabla u||_\Phi\ \omega_\Phi\left(\frac{|x_0-y_0|}{R}\right) \end{aligned}$$

completing the proof of Theorem 1.6.

Drawing on the notation developed above, we give the following formulas.

Examples 8.1

$$\Phi_1(t) = t^n \log^{\alpha-1}(1+t), \quad \alpha > 0 \qquad \omega_{\Phi_1}(t) = [\log^\alpha(e+\frac{1}{t})]^{-\frac{1}{n}}$$

$$\Phi_2(t) = \frac{t^n}{\log(1+t)} \qquad \omega_{\Phi_2}(t) = [\log\log(e+\frac{1}{t})]^{-\frac{1}{n}}$$

$$\Phi_3(t) = \frac{t^n}{\log(1+t)\log\log(e+t)} \qquad \omega_{\Phi_3}(t) = [\log\log\log(e+\frac{1}{t})]^{-\frac{1}{n}}$$

Elementary computation of these formulas are left for the reader.

Similar considerations for weakly monotone functions apply when the gradient belongs to $BL^n(B)$. Recall the modulus of integrability

$$\mathcal{L}^n(\nabla u;\epsilon) = \left(\epsilon \fint_\Omega |\nabla u|^{n-\epsilon}\right)^{\frac{1}{n-\epsilon}}, \qquad 0 < \epsilon \leq n-1.$$

On the other hand using (42) for the function $\Phi(\tau) = \tau^{n-\epsilon}$ with $0 < \epsilon \leq \frac{4}{5}$, we obtain

$$|u(x_0) - u(y_0)| \leq \frac{C(n)\ R}{\left[1 - \left(\frac{2r}{R}\right)^\epsilon\right]^{\frac{1}{n-\epsilon}}} \left(\epsilon \fint_B |\nabla u|^{n-\epsilon}\right)^{\frac{1}{n-\epsilon}}$$

We choose $\epsilon = \log^{-1}\frac{R}{2r}$, which is legitimate if $R \geq 10r$. Recall that $2r = |x_0 - y_0|$, while the ball $B(a,r)$ centered at $a = \frac{1}{2}(x_0 + y_0)$ must lay in $B(0, \frac{1}{2}R)$. This certainly holds if $x_0, y_0 \in B(0, \frac{1}{10}R)$. We then conclude with the following continuity estimate.

Proposition 8.2 *Suppose u is a weakly monotone function in the grand Sobolev space $W^{1,n)}(B)$ on a ball $B = B(0,R) \subset \mathbb{R}^n$. Then for all Lebesgue points $x_0, y_0 \in B(a, \frac{1}{10}R)$ we have*

$$|u(x_0) - u(y_0)| \leq C(n)\ R\ \mathcal{L}^n\left(\nabla u; log^{-1}\frac{R}{|x_0 - y_0|}\right) \tag{44}$$

Thus u has a continuous representative.

Remark. Observe that estimate (44) remains valid if $|\nabla u| \in BL^n(B)$, in which case we conclude that $u \in L^\infty_{loc}(B)$ and obtain the uniform bound

$$\text{ess osc}\left(u; \frac{1}{10}B\right) \leq C(n)\ R\ ||\nabla u||_{BL^n(B)} \tag{45}$$

9 Final Arguments

Coming to an end, we are in a position to complete the proofs of the results stated in Introduction.

Theorem 1.6 was completely established in Section 8. Theorem 1.3 follows by combining Proposition 5.2 and 8.2. Concerning Theorem 1.4, we see from

Corollary 6.2 that the coordinate functions of f are weakly monotone. Then Theorem 1.6 provides us with the modulus of continuity estimates. According to Examples 8.1 these estimates take the form (13), (14) and (15), as desired.

It remains to complete the proof of Theorem 1.2. First we observe that f belongs to $W^{1,\Theta}(B)$ with $\Theta(t) = t^n \log^{-1}(e+t)$, and we have

$$||Df||_\Theta^n \leq C(n) \fint_B J(x,f)\ dx \fint_B e^{\lambda K(x)}\ dx \tag{46}$$

Indeed,

$$\begin{aligned} ||Df||_\Theta^n &\leq n^n||\ |Df|^n||_{L\log^{-1}L} \leq n^n||K\ J(x,f)||_{L\log^{-1}L} \\ &\leq C_\lambda n^n \left(\fint_B J(x,f)\ dx\right)\left(\fint_B e^{\lambda K(x)}\ dx\right) \end{aligned}$$

Finally, we use Theorem 1.4 and estimate (15) to conclude with the inequality

$$\begin{aligned} |f(x)-f(y)|^n &\leq \frac{C(n)R^n||Df||_\Theta^n}{\log\log\left(e+\frac{R}{|x-y|}\right)} \\ &\leq \frac{C_K(n,\lambda)\int_B J(x,f)\ dx}{\log\log\left(e+\frac{R}{|x-y|}\right)} \end{aligned}$$

References

[1] Astala, K. Iwaniec, T. Koskela, P. and Martin, G. Mappings of BMO-bounded distortion. Math. Ann. to appear.

[2] Ball, J. (1978). Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. 63, 337-403.

[3] Ball, J. (1981). Global invertibility of Sobolev functions and the interpenetration of matter. Proc. Roy. Soc. Edinburgh Sect. A 88, no. 3-4, 315–328.

[4] Brakalova, M.A. and Jenkins, J.A. (1998). On solutions of the Beltrami equation. J. Anal. Math. 76, 67-92.

[5] David, D. (1988). Solutions de l'equation de Beltrami avec $||\mu||_\infty = 1$. Ann Acad. Sci. Fenn. Ser. A I, Math. 13, No.1, 25-70.

[6] Gehring, F.W. (1962). Rings and quasiconformal mappings in space. Trans. Amer. Math. Soc. 103, 353-393.

[7] Gehring, F.W. and Iwaniec, T. (1999). The limit of mappings with finite distortion. Ann. Acad. Sci. Fenn. Math. 24, no. 1, 253–264.

[8] Gol'dstein V.M. and Vodop'yanov S.K. (1976). Quasiconformal mappings and spaces of functions with generalized first derivatives. Sibirsk. Mat. Z. 17, 515-531.

[9] Greco, L. (1993). A remark on the equality $\det Df = \operatorname{Det} Df$. Differential Integral Equations. No 5, 1089-1100.

[10] Greco, L. Iwaniec, T. and Moscariello, G. (1995). Limits of the improved integrability of the volume forms. Indiana Univ. Math. J. 44, no. 2, 305-339.

[11] Hajłasz, P. (1995). A note on weak approximation of minors. Ann. Inst. H. Poincaré Anal. Non Linéaire 12, no. 4, 415–424.

[12] Heinonen, J. and Koskela, P. (1993). Sobolev mappings with integrable dilatations. Arch. Rational Mech. Anal. 125, no. 1, 81–97.

[13] Iwaniec, T. (1999). Nonlinear Differential Forms, Lectures at the International Summer School in Jyväskylä, 1998, Editor: P. Mattila University of Jyväskylä, Department of Mathematics Report 80, pp 1-207.

[14] Iwaniec, T. Koskela, P. and Martin, G. Mappings of BMO-distortion and Beltrami type operators. Preprint.

[15] Iwaniec, T. and Lutoborski, A. (1993). Integral estimates for null Lagrangians. Arch. Rational Mech. Anal. 125, no. 1, 25–79.

[16] Iwaniec, T. and Sbordone, C. (1992). On the integrability of the Jacobian under minimal hypotheses. Arch. Rational Mech. Anal. 119, no. 2, 129-143.

[17] Iwaniec, T and Šverák, V. (1993). On mappings with integrable dilatation. Proc. Amer. Math. Soc. 118, 181-188.

[18] Lebesgue, H. (1907). Sur le problème de Dirichlet. Rend. Circ. Palermo 27, 371-402.

[19] Manfredi, J. (1994). Weakly Monotone Functions. The Journal of Geometric Analysis. 3, 393-402.

[20] Manfredi, J. and Villamor, E. (1995). Mappings with integrable dilatation in higher dimensions. Bull. Amer. Math. Soc. (N.S.) 32, no. 2, 235–240.

[21] Manfredi, J. and Villamor, E. (1998). An extension of Reshetnyak's theorem. Indiana Univ. Math. J. 47, no. 3, 1131–1145.

[22] Migliaccio, L. and Moscariello, G. (1997). Mappings with unbounded dilatation. Preprint n.3. Universita'degli studi di Salerno.

[23] Müller, S. Qi, T. and Yan, B.S. (1994). On a new class of elastic deformations not allowing for cavitation. Ann. Ins. H. Poincaré.

[24] Potemkin, V. and Ryazanov, V. (1998). On the noncompactness of David classes. Ann. Acad. Sci. Fenn. Math. 23, 191-204.

[25] Reshetnyak, Yu. G. (1966). Bounds on moduli of continuity for certain mappings. (Russian). Sibirsk. Mat. Sh. 7, 1106-1114.

[26] Reshetnyak, Yu. G. (1989). Space Mappings with Bounded Distortion. Trans. of Mathematical Monographs, Amer. Math. Soc, vol. 73.

[27] Ryazanov, V. Srebro, U. and Yakubov, E. BMO-Quasiconformal mappings. Preprint.

[28] Šverák, V. (1988). Regularity properties of deformations with finite energy. Arch. Rational Mech. Anal. 100, no. 2, 105–127.

[29] Tukia, P. (1991). Compactness properties of μ -homeomorphisms. Ann. Acad. Sci. Fenn. Ser. A I. Math. 16, 47-69.

[30] Vodop'yanov, S. K. (1999). Mappings with bounded distortion and with finite distortion on Carnot groups. Siberian Mathematical Journal, 33, no. 4, 764-804.

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