INEQUALITIES FOR THE MOMENTS OF WIENER INTEGRALS WITH RESPECT TO FRACTIONAL BROWNIAN MOTIONS

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ABSTRACT. We study the possibility to control the moments of Wiener-integrals of fractional Brownian motions with respect the norm of the integrand. It turns out that when the self-similarity index $H > \frac{1}{2}$, we can have only an upper inequality, and when $H < \frac{1}{2}$ we can have only a lower inequality. As an application we obtain a maximal inequality in the case of $H > \frac{1}{2}$.

1. INTRODUCTION

1.1. **Fractional Brownian motion.** A fractional Brownian process $Z = Z^H$ with self-similarity index H, is a continuous Gaussian process with stationary increments, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, with the properties

(i) $Z_0 = 0$.
(ii) $\mathbf{E}Z_t = 0$ for every $t \geq 0$.
(iii) $\mathbf{E}Z_tZ_s = \frac{1}{2}(t^{2H} + s^{2H} - |s-t|^{2H})$ for every $s, t \geq 0$.

The standard Brownian motion is a fractional Brownian motion with index $H = 1/2$.

Fractional Brownian motion is a self similar process, and from this property we get maximal inequalities of Burkholder- Davis-Gundy type. Precisely we have the following result:

For every $T > 0$, and $p > 0$ we have

$$\mathbf{E}(Z_T^*)^p = \mathbf{E}(Z_1^*)^p T^{pH} \tag{1.1}$$

where Z^* denotes the supremum process defined by $Z_t^* = \sup_{s \leq t} |Z_s|$.

In [6], Novikov and Valkeila considered the problem of getting maximal inequalities by replacing deterministic T by a stopping time τ. In particular, for the case $H > 1/2$, they proved that for every $p > 0$ there exist constants $c(p, H)$ and $C(p, H)$ such that it holds the following :

$$c(p, H)\mathbf{E}(\tau^{pH}) \leq \mathbf{E}(Z_\tau^*)^p \leq C(p, H)\mathbf{E}(\tau)^{pH}.$$

For $H < \frac{1}{2}$ they showed that for every p there exists a constant $c(p, H)$ such that

$$c(p, H)E(\tau^{pH}) \leq E(Z_\tau^*)^p.$$

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1.2. **Classical Wiener-integrals.** Let us introduce some notations for later use. Assume that f is a measurable function with the property $||f||_{L^2(0,T)} < \infty$, where $||\cdot||_{L^p(a,b)}$ is the standard norm given by

$$||f||_{L^p(a,b)} = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}},$$

when $p \geq 1, 0 \leq a < b < \infty$. If $a = 0$ and $b = 1$ we shall write $||f||_p$ instead of $||f||_{L^p(0,1)}$. If $f \in L^2(0,\infty) \cap L^1(0,\infty)$ introduce the norm $||f||_s$ defined by

$$||f||_s = \left(\int_{\mathbb{R}} |\hat{f}(\lambda)|^2(1+\lambda^2)d\lambda\right)^{\frac{1}{2}},$$

where $\hat{f}$ is the Fourier transform of f (see [1]).

Assume that W is the standard Brownian motion. Recall that for classical Wiener integrals we have the isometry

$$E\left(\int_0^T f(s)dW_s\right)^2 = ||f||^2_{L^2(0,T)}. \tag{1.2}$$

We want to study weather it is possible to estimate $E(\int_0^T f(s)dZ_s)^2$ in terms of the norm of the function f, i.e. we are looking for upper and lower bounds of the form $||f||^r_{L^q(0,T)}$ for some $r > 0, q > 1$ for $E(\int_0^T f(s)dZ_s)^2$. By considering the function $f = 1$ and then the function $f = a$, where $a > 0$ is a constant, it is easy to see that the only possibility for such estimates is to have $r = 2$ and $q = \frac{1}{H}$. We show that it is possible to get an upper estimate in terms of this norm, when $H > \frac{1}{2}$ and not to have a lower estimate in this case, and an lower estimate, when $H < \frac{1}{2}$ and not to have an upper estimate in this case.

Recall that the quadratic variation of the standard Brownian motion is controlled by Lebesgue measure: $E(\sum_\pi (W_{s_i} - W_{s_{i-1}})^2 = T$ and $\sum_\pi (W_{s_i} - W_{s_{i-1}})^2 \xrightarrow{P} T$ as $|\pi| \to 0$. This property of Brownian motion is also connected to isometry (1.2). Above π is a subdivision of the interval $[0,T]$, $\pi = \{0 = s_0 < s_1 < \cdots < s_n = T\}$, $|\pi| = \max_{s_i \in \pi}(s_i - s_{i-1})$ and $\xrightarrow{P}$ means convergence in probability. For the $1/H$ variation of fractional Brownian motion we have $\sum_\pi |Z_{s_i} - Z_{s_{i1}}|^{1/H} \xrightarrow{P} T$ as $|\pi| \to 0$. having this property for fractional Brownian motion Z it is natural to ask, if one can use the $1/H$- norm of f to control the Wiener integrals with respect to fractional Brownian motions. As we will show, this is possible only in the case $H > \frac{1}{2}$.

1.3. **Wiener integrals with respect to fractional Brownian motions.** Wiener integration with respect to Z plays a central role below. Since Z is not a semimartingale, we refer to the integration theory of Gaussian processes (see for example [3]). We consider only deterministic integrands.

For $H > 1/2$, let Ψ denote the integral operator :

$$\Psi f(t) = H(2H-1)\int_0^\infty f(s)|s-t|^{2H-2}ds$$

and define the inner product

$$<< f, g >>_\Psi = < f, \Psi g >= H(2H-1)\int_0^\infty \int_0^\infty f(s)g(t)|s-t|^{2H-2}dsdt$$

where $< . >$ denotes the usual inner product of $L^2[0,\infty)$. Denote by L^2_Ψ (respectively : $L^2_\Psi(0,T)$) the space of equivalence classes of measurable functions f such that $<< f, f >>_\Psi < \infty$ (respectively : $<< f1_{[0,T]}, f1_{[0,T]} >>_\Psi < \infty$). The application $Z_t \to 1_{[0,t]}$ can be extended to an isometry between the Gaussian space generated by the random variables $Z_t, t \geq 0$ (respectively for $t \leq T$) and the function space L^2_Ψ, (respectively for $L^2_\Psi(0,T)$).

For $f \in L^2_\Psi$, the integral $\int_0^\infty f(t)dZ_t$ is defined as the image of f by this isometry. In particular we have, for $f, g \in L^2_\Psi(T)$

$$\mathbf{E}(\int_0^T f(u)dZ_u \int_0^T g(v)dZ_v) = \int_0^T \int_0^T f(u)g(v)|u-v|^{2H-2}dudv \tag{1.3}$$

and

$$\mathbf{E}(\int_s^t f(u)dZ_u)^2 = \int_s^t \int_s^t f(u)f(v)|u-v|^{2H-2}dudv. \tag{1.4}$$

For $H < \frac{1}{2}$ the integral in the above definition of Ψ diverges, and we have to modify the definition. If f has bounded variation, then the Wiener integral can be defined by integration by parts. To allow more general integrands, we follow the approach of Dasgupta [1]. For $f \in D(\mathbb{R}_+)$, i.e. $f \in C^\infty(\mathbb{R}_+)$ with compact support on $(0,\infty)$ put

$$\int_{\mathbb{R}_+} f(s)dZ_s = -\int_{\mathbb{R}_+} Z_s df(s).$$

If $f \in f \in L^2(0,\infty) \cap L^1(0,\infty)$ and $||f||_s < \infty$ it is shown in [1, p.15-16] that one can define $\int_{\mathbb{R}_+} f(s)dZ_s$ as $L^2(P)$ limit of the integrals of the form $\int_{\mathbb{R}_+} \phi^{(n)}(s)dZ_s$, where $\phi^{(n)} \in D(\mathbb{R}_+)$ and $||\phi^{(n)} - f||_s \to 0$ as $n \to \infty$.

1.4. **The main results.** The first result concerns the case $H > \frac{1}{2}$.

Theorem 1.1. *Let Z be a fractional Brownian motion of index $H > 1/2$. We have the inclusion : For every $T < \infty$, $L^2_\Psi(0,T) \subset L^{1/H}(0,T)$ More precisely : for every $r > 0$, for every a, b with $0 \leq a < b < \infty$, there exists a constant $c(H,r)$ such that:*

$$\mathbf{E}(|\int_a^b f(u)dZ_u|^r) \leq c(H,r)||f(u)||^r_{L^{1/H}(a,b)} \tag{1.5}$$

and

$$E|\int_a^b f(u)dZ_u \int_a^b g(u)dZ_u|^r \leq c(H,r)||f||^r_{L^{1/H}(a,b)}||g||^r_{L^{1/H}(a,b)}.$$

The next theorem shows that in the case of $H < \frac{1}{2}$ the opposite inequality takes place.

Theorem 1.2. *Assume that Z is a fractional Brownian motion with Hurst index $H < \frac{1}{2}$. Then there exists a constant $\gamma(H,r)$ such that $\forall a,b : 0 \le a < b < \infty$ and $\forall r > 0$ we have*

$$E|\int_a^b f(s)dZ_s|^r \ge \gamma(H,r)||f||^r_{L^{1/H}(a,b)} \tag{1.6}$$

in the following cases:

(i) *f has bounded variation on $[a,b]$.*
(ii) *$f \in L^1(0,\infty) \cap L^2(0,\infty)$ with $||f||_s < \infty$.*

We show that it is not possible to get reverse inequalities to (1.5) nor (1.6). This is shown in section 3.

Remark 1.1. *It is possible to define stochastic integrals with respect to fractional Brownian motion, when $H > \frac{1}{2}$. In fact, there are many different definitions (see [4] for more information on these different definitions). We do not know, whether it is possible to extend (1.5) to some of these stochastic integrals.*

2. Proofs

2.1. **The proof of Theorem 1.1.** Since, for every T $\int_0^T f(t)dZ_t$ is a centered Gaussian random variable, for every $r > 0$, there exists a constant $k(r)$ such that

$$\mathbf{E}(\int_0^T f(t)dZ_t)^r \le k(r)(\mathbf{E}(\int_0^T f(t)dZ_t)^2)^{r/2},$$

taking in account equality (1.3), the inequality (1.5) is actually implied by the following

$$\int_0^T \int_0^T f(u)f(v)|u-v|^{2H-2}dudv \le c(H,2)(\int_0^T |f(u)|^{1/H}du)^{2H}. \tag{2.1}$$

One will prove easily that (2.1) is a consequence of a classical inequality on Riesz potentials $I^\alpha f$ (see for example [8, p. 117-120]) defined formally for $0 < \alpha < 1$ by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty |x-y|^{\alpha-1} f(y)dy$$

for $f : \mathbb{R}_+ \to \mathbb{R}$ and

$$I^\alpha_+ f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)dy.$$

Precisely we have (see [8, Theorem 1, p. 119.]):

Theorem 2.1 (Hardy-Littlewood)**.** *Let $0 < \alpha < 1$ and $1 < p < q < \infty, 1/q = 1/p - \alpha$.*

a) *If $f \in L^p(0,\infty)$, then $I^\alpha f(x)$ and $I^\alpha_+ f(x)$ converge absolutely for almost every x.*
b) *We have the inequalities of norms with some constants $A_{p,q}, B_{p,q}$:*

$$||I^\alpha f||_{L^q(0,\infty)} \le A_{p,q}||f||_{L^p(0,\infty)} \text{ and } ||I^\alpha_+ f||_{L^q(0,\infty)} \le B_{p,q}||f||_{L^p(0,\infty)}. \tag{2.2}$$

We continue with the proof of inequality (2.1). By using Hölder inequality with exponent $p = 1/H$ and inequality (2.2) with $\alpha = 2H - 1$ and the same $p = 1/H$ we get:

$$\int_a^b |f(u)|(\int_a^b |f(v)||u-v|^{2H-2}dv)du$$
$$\leq (\int_a^b |f(u)|^{1/H}du)^H(\int_a^b du(\int_a^b |f(v)||u-v|^{2H-2}dv)^{\frac{1}{1-H}})^{1-H}$$
$$\leq A(\frac{1}{H}, \frac{1}{1-H})(\int_a^b |f(u)|^{1/H})^{2H}du.$$

This finishes the proof of inequality (2.1).

2.2. **Proof of Theorem 1.2.** Note that again it is sufficient to consider the case of $r = 2$ only. We start the proof with the following lemma.

Lemma 2.1. *Assume that Z is a fractional Brownian motion with Hurst index $H < \frac{1}{2}$. Then there exist a constant γ_H such that for every $f \in C^\infty(\mathbb{R}_+)$ with compact support, i.e. $f \in D(\mathbb{R}_+)$, we have*

$$E\left|\int_{\mathbb{R}_+} f(s)\,dZ_s\right|^2 \geq \gamma_H \|f\|^2_{L^{1/H}(0,\infty)}. \tag{2.3}$$

Proof of Lemma 2.1. According to [1, p.14],

$$\int_{\mathbb{R}_+} f(s)\,dZ_s = -\int_{\mathbb{R}_+} \dot{f}(s)Z_s\,ds,$$

and

$$E\left|\int_{\mathbb{R}_+} f(s)\,dZ_s\right|^2 = E\left|\int_{\mathbb{R}_+} \dot{f}(s)Z_s\,ds\right|^2 = c_H\int_R |\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda.$$

$$c_H = \frac{H\Gamma(2H)\sin\pi H}{\pi}.$$

According to [7, Chapter 2.7, p. 137],

$$|\mathcal{F}(D_+^\alpha\phi)(x)| = |x|^\alpha|\hat{\phi}(x)|,\ \ \alpha \geq 0, \tag{2.4}$$

where $\mathcal{F}(\cdot)$ is Fourier transform, $D_+^\alpha\phi$ is the fractional derivative of ϕ,

$$D_+^\alpha\phi(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{\phi(t)dt}{(x-t)^\alpha},$$

where $\phi \in D(\mathbb{R}_+)$. Therefore,

$$\int_{\mathbb{R}} |\hat{f}(\lambda)|^2|\lambda|^{1-2H}d\lambda = \int_{\mathbb{R}} |\mathcal{F}(D_+^{1/2-H}f)(\lambda)|^2d\lambda$$

and from Parceval inequality,

$$\int_{\mathbb{R}} |\mathcal{F}(D_+^{1/2-H} f)(\lambda)|^2 d\lambda = \|D_+^{1/2-H} f\|^2_{L^2(0,\infty)}.$$

Because $f \in C^\infty$, it is well-known (see, for example [7, Theorem 2.4, p. 45 or Theorem 2.1, p.31]) that

$$I_+^\alpha D_+^\alpha f(x) = f(x),$$

and according to Theorem 2.1

$$\|I_+^\alpha g\|_{L^q(0,\infty)} \le B_{p,q} \|g\|_{L^p(0,\infty)},$$

$\frac{1}{q} = \frac{1}{p} - \alpha$. Therefore

$$\|f\|_{L^{1/H}(0,\infty)} \le B_{2,1/H} \|D_+^{1/2-H} f\|_{L^2(0,\infty)}$$

(here $q = 1/H,\ \alpha = 1/2 - H,\ p = 2$). We obtain

$$\|f\|^2_{L^{1/H}(0,\infty)} \le B_{2,1/H} \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 |\lambda|^{1-2H} d\lambda = \frac{B_{2,1/H}}{c_H} E \left| \int_{\mathbb{R}_+} f(s)\, dZ_s \right|^2 .$$

This ends the proof of Lemma 2.1.

We continue with the proof of Theorem 1.2.

First we prove *(i)*. Let $f \in L_T^s$, i.e. $f = 0$ outside of $[a,b]$ and f is a simple function. Then it was proved in [4, p. 11] that there exists $\{\phi^{(n)},\ n \ge 1\} \subset D(\mathbb{R}_+)$ such that $\phi^{(n)}(t) \to f(t)$ and $E \left| \int_{\mathbb{R}_+} \phi^{(n)}(s)\, dZ_s \right|^2 \to E(\int_a^b f(s) dZ_s)^2$. Then using Lemma 2.1

$$\|f\|^2_{L^{1/H}(a,b)} = \liminf \|\phi^{(n)}\|^2_{L^{1/H}(a,b)} \le \gamma_H \liminf E \left| \int_{\mathbb{R}_+} \phi^{(n)}(s)\, dZ_s \right|^2 = \gamma_H E(\int_a^b f(s) dZ_s)^2.$$

Now, let $f \in \mathrm{BV}[a,b]$. Then

$$E \left| \int_a^b f(s)\, dZ_s \right|^2 = E \left| f(b) Z_b - f(a) Z_a - \int_a^b Z_s df(s) \right|^2 .$$

We have that $\int_a^b Z_s df(s) = \lim_{|\pi|\to 0} \sum_{i=1}^k Z_{s_i} \Delta f_{s_i}$ a.s., and $\left| \sum_{i=1}^k Z_{s_i} \Delta f_{s_i} \right| \le \sup_{a \le s \le b} |Z_s| \mathrm{var}_{[a,b]} f$, and right-hand side is uniformly integrable, using (1.1). Therefore, by Fatou lemma,

$$E \left| \int_a^b f(s)\, dZ_s \right|^2 = \lim_{|\pi|\to 0} E \left| f(b) Z_b - f(a) Z_a - \sum_{i=1}^k Z_{s_i} \Delta f_{s_i} \right|^2 =$$

$$= \lim_{|\pi|\to 0} E \left| \sum_{i=1}^k f_{s_i} \Delta Z_{s_i} \right|^2 \ge \gamma_H \lim_{|\pi|\to 0} \|f_\pi\|^2_{L^{1/H}(a,b)} = \gamma_H \lim_{|\pi|\to 0} \|\, |f_\pi|\, \|^2_{L^{1/H}(a,b)} \ge \gamma_H \|f\|^2_{L^{1/H}(a,b)},$$

where $f_\pi = \sum_{i=1}^{k} |f_{s_i}| I_{]s_{i-1},s_i]}$ with $f_\pi \to |f|$ almost surely with respect to Lebesgue measure. This finishes the proof of *(i)* in Theorem 1.2.

We continue with the proof of *(ii)*. Let $f \in L^1(0,\infty) \cap L^2(0,\infty)$ and $||f||_s < \infty$. Take $\phi^{(n)} \in D(\mathbb{R}_+)$ with $||\phi^{(n)} - f||_s \to 0$ and $\int_{\mathbb{R}_+} \phi^{(n)}(s) dZ_s \overset{L^2(P)}{\longrightarrow} \int_{\mathbb{R}_+} f(s) dZ_s$ as $n \to \infty$. Then

$$
\begin{aligned}
E\left|\int_{\mathbb{R}_+} f(t)dZ_t\right|^2 &= \lim E\left|\int_{\mathbb{R}_+} \phi^{(n)}(t)dZ_t\right|^2 \\
&= \lim \int_{\mathbb{R}_+} |\hat{\phi}^{(n)}|^2 |\lambda|^{1-2H} d\lambda \geq \gamma_H \lim \|\phi^{(n)}\|^2_{L^{1/H}(0,\infty)}.
\end{aligned} \tag{2.5}
$$

Since $||\phi^{(n)} - \phi^{(m)}||_{L^2(0,\infty)} \leq ||\phi^{(n)} - \phi^{(m)}||_s$, then $\phi^{(n)}$ is fundamental also in $L^2(0,\infty)$. So, $\phi^{(n)} \overset{L^2(0,\infty)}{\to} f$. Note also, that

$$
E\left|\int_{\mathbb{R}_+} (\phi^{(n)}(t) - \phi_m(t) dZ_t\right|^2 \geq \gamma_H \|\phi^{(n)} - \phi_m\|^2_{L^{1/H}(0,\infty)},
$$

and $\phi^{(n)}$ is fundamental in $L^{1/H}(0,\infty)$. Hence $\phi^{(n)} \overset{L^{1/H}(0,\infty)}{\longrightarrow} f$ and we have

$$
\|\phi^{(n)}\|_{L^{1/H}(0,\infty)} \to \|f\|_{L^{1/H}(0,\infty)}. \tag{2.6}
$$

From (2.5) and (2.6) we have that

$$
E\left|\int_{\mathbb{R}_+} f(t)dZ_t\right|^2 \geq \gamma_H \|f\|^2_{L^{1/H}(0,\infty)}.
$$

This finishes the proof of Theorem 1.2.

3. On reverse inequalities

3.1. **The lower inequality in the case of $H > \frac{1}{2}$.** Next we show that it is not possible to prove a reverse inequality to (1.5). Assume that Z is a fractional Brownian motion with $H > 1/2$ and consider the function $f(u) = u^{\varepsilon - H},\ 0 < u \leq 1$ with $0 < \varepsilon < H$.

Note that

$$
\|f\|^2_{1/H} = \left(\int_0^1 u^{\frac{\varepsilon}{H}-1} du\right)^{2H} = \left(\frac{1}{\frac{\varepsilon}{H}}\right)^{2H} = \frac{H^{2H}}{\varepsilon^{2H}} \tag{3.1}
$$

and by Theorem 1.1 the Wiener integral exists.

Consider now the expression

$$E\left(\int_0^1 f(u)dZ_u\right)^2 = \int_0^1\int_0^1 u^{\varepsilon-H}s^{\varepsilon-H}|u-s|^{2H-2}du\,ds =$$

$$\int_0^1 u^{\varepsilon-H}\left(\int_0^u s^{\varepsilon-H}(u-s)^{2H-2}ds + \int_u^1 s^{\varepsilon-H}(s-u)^{2H-2}ds\right)du =$$

$$\int_0^1 u^{2\varepsilon-1}\left(\int_0^1 s^{\varepsilon-H}(1-s)^{2H-2}ds\right)du + \int_0^1 u^{2\varepsilon-1}\left(\int_1^{1/u} s^{\varepsilon-H}(s-1)^{2H-2}ds\right)du =$$

(see [5, Lemma 2.2 (iii), p. 576])

$$= B(\varepsilon-H+1, 2H-1)\int_0^1 u^{2\varepsilon-1}du + \int_0^1 u^{2\varepsilon-1}\left(\int_0^{1-u} s^{2H-2}(1-s)^{-\varepsilon-H}ds\right)du =$$

$$\frac{1}{2\varepsilon}\frac{\Gamma(1-H+\varepsilon)\Gamma(2H-1)}{\Gamma(H+\varepsilon)} + \int_0^1 s^{2H-2}(1-s)^{-\varepsilon-H}\left(\int_0^{1-s} u^{2\varepsilon-1}du\right)ds =$$

$$= \frac{1}{2\varepsilon}\frac{\Gamma(1-H+\varepsilon)\Gamma(2H-1)}{\Gamma(H-\varepsilon)} + \frac{1}{2\varepsilon}\int_0^1 s^{2H-2}(1-s)^{\varepsilon-H}ds =$$

$$= \frac{1}{\varepsilon}\frac{\Gamma(1-H+\varepsilon)\Gamma(2H-1)}{\Gamma(H-\varepsilon)} \sim \frac{K_1}{\varepsilon} \text{ as } \varepsilon \to 0,$$

for constant $K_1 = B(1-H, 2H-1)$. Since $\frac{1}{\varepsilon}/\frac{1}{\varepsilon^{2H}} = \varepsilon^{2H-1}$ and we can make it as small as we want, the inequality

$$E\left|\int_0^1 f(u)dZ_u\right|^2 \geq c_H\|f\|^2_{1/H}$$

is impossible when $H > 1/2$ by (3.1).

Using this example we have the following remark:

Remark 3.1. *Assume that $H > 1/2$. Let $\phi \in D(\mathbb{R}_+)$. Then it is not possible to have an inequality of the form*

$$E\left(\int_0^T \phi_s dZ_s\right)^2 \geq b_{1/H,1/(1-H)}||\phi||^2_{L^{1/H}(0,T)} \tag{3.2}$$

with some $b_{1/H,1/(1-H)} > 0$.

Proof of Remark 3.1. Take $0 < \varepsilon < H$ and consider the function $f(u) = u^{\varepsilon - H}$. Then $f \in L^{1/H}(0,1)$ and there exists $\phi^{(n)} \in D(\mathbb{R}_+)$ such that $\phi^{(n)} \to f$ in $L^{1/H}(0,1)$. Assume that (3.2) holds. Then by (1.5)

$$\phi^{(n)} \xrightarrow{L^{1/H}} f \Leftrightarrow \int_0^1 \phi_s^{(n)} dZ_s \xrightarrow{L^2(P)} \int_0^1 f(s) dZ_s.$$

This gives

$$E\left(\int_0^1 f(s) dZ_s\right)^2 = \lim_n E\left(\int_0^1 \phi_s^{(n)} dZ_s\right)^2 \geq b_{1/H,1/(1-H)} \lim_n ||\phi^{(n)}||_{1/H} = b_{1/H,1/(1-H)} ||f||_{1/H}.$$

But this is impossible by the above counterexample.

3.2. **The upper inequality in the case of $H < \frac{1}{2}$.** We want to show that it is not possible to give a reverse inequality to (2.3). First we start with a remark about the reverse inequality to (2.2), which is probably well known.

Remark 3.2. *Put $\alpha = \frac{1}{2} - H$, where $H < \frac{1}{2}$. Then it does not exist a constant $a_{2,1/H}$ such that for every $f \in D(\mathbb{R}_+)$ we have*

$$||I_+^\alpha(f)||_{1/H} \geq a_{2,1/H} ||f||_2. \tag{3.3}$$

Proof of Remark 3.2. We consider again the function $f(u) = u^{\varepsilon - H}$ with $\varepsilon > -(\frac{1}{2} - H)$ [note that ε here can be negative]. By direct computations we get

$$||f||_2 = (1 - 2H + 2\epsilon)^{-\frac{1}{2}}$$

and

$$||I_+^\alpha(f)||_{1/H} = K_{\varepsilon,H}(1 - 2H + 2\epsilon)^{-H},$$

where

$$K_{\varepsilon,H} = \frac{\Gamma(\varepsilon - H + 1)}{\Gamma(\varepsilon - 2H + \frac{3}{2})}(2H)^H.$$

So

$$\frac{||f||_2}{||I_+^\alpha(f)||_{1/H}} \uparrow +\infty$$

when $\varepsilon \downarrow -(\frac{1}{2} - H)$. As in the Remark 3.1 we can find a sequence $(\phi^{(n)})$ of elements of $D(\mathbb{R}_+)$ such that $\phi^{(n)} \xrightarrow{L^2} f$. If there exists $a_{2,1/H}$ as in (3.3) we would have, using also Theorem 2.1, that

$$\phi^{(n)} \xrightarrow{L^2} f$$

is equivalent to

$$I_+^\alpha(\phi^{(n)}) \xrightarrow{L^{1/H}} I_+^\alpha(f);$$

passing to limit would then give $||I_+^\alpha(f)||_{1/H} \geq a_{2,1/H}||f||_2$ for every $\varepsilon > 0$, which is a contradiction. This proves remark 3.2.

By Remark 3.2 we can consider a sequence $\phi^{(n)} \in D(\mathbb{R}_+)$ such that

$$||\phi^{(n)}||_2 \geq n ||I_+^\alpha(\phi^{(n)})||_{1/H}.$$

Since $\phi^{(n)} \in D(\mathrm{IR}_+)$, there exists $g^{(n)}$ such that $\phi^{(n)} = D_+^{\frac{1}{2}-H}(g^{(n)})$. Use now the relation $I_+^\alpha D_+^\alpha(g^{(n)}) = g^{(n)}$ to get $||D_+^\alpha(g^{(n)})||_2 \geq n||g^{(n)}||_{1/H}$. But $||D_+^\alpha(g^{(n)})||_2^2 = \frac{1}{c_H}E(\int_{\mathrm{IR}_+} g^{(n)}(s)dZ_s)^2$ and hence

$$E(\int_{\mathrm{IR}_+} g^{(n)}(s)dZ_s)^2 \geq nc_H||g^{(n)}||_{1/H}^2.$$

This shows that it is not possible to obtain a reverse inequality to (2.3).

4. An application

4.1. A maximal inequality for Wiener integrals.

Theorem 4.1. *Let f be Hölder with exponent $\beta \in (-H, -H+1)$ and $f \in L^{\frac{1}{H}}(0,T)$ and let Z be fractional Brownian motion with $H > \frac{1}{2}$. Then for every $T < \infty$, the process Wiener integral $\int_0^\cdot f(s)dZ_s$ is defined on all $t \in [0,T]$. It admits a modification which have Hölderian trajectories with Hölder exponent $\lambda < H + \beta$. For every $p > 0$, for every $T > 0$, there exists a constant $C(H,p)$ such that holds the maximal inequality :*

$$\mathbf{E}(\sup_{t\leq T} |\int_0^t f(u)dZ_u|^p) \leq C(H,p)T^{pH+p\beta}.$$

Proof. From inequality (2.1) and from Hölder property of f, it follows that for every $0 \leq s < t < \infty$, and for every $p > 0$, we have:

$$\mathbf{E}(\int_s^t f(u)dZ_u)^p) \leq c(t-s)^{(\frac{\beta}{H}+1)pH}.$$

Then, using the Kolmogorov lemma, as stated in [2, Theorem 19, chapter XXIII], we get the announced results.

Remark 4.1. *Theorem 4.1 generalizes Lemma 2.1. in* [5].

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