POROUS MEASURES ON THE REAL LINE HAVE PACKING DIMENSION CLOSE TO ZERO

E. Järvenpää and M. Järvenpää

University of Jyväskylä, Department of Mathematics,
P.O. Box 35, FIN-40351 Jyväskylä, Finland
email: esaj@math.jyu.fi and amj@math.jyu.fi

Abstract. We prove that in the real line packing dimension of any Radon measure is bounded above by a function depending on porosity. This upper bound tends to zero as porosity tends to its maximum value.

1. Introduction

Many familiar fractal sets in $\mathbb{R}^n$ are constructed by removing smaller and smaller pieces out of some n-dimensional set. As a result one obtains a set which is full of holes and has usually dimension less than n. In some cases it is possible to calculate the dimension using the sizes of the removed pieces (see [BT] and [T]). Intuitively it seems quite natural to expect that if a set in $\mathbb{R}^n$ has holes of certain size at all scales then its dimension is strictly less than n, and the more porous the set is, that is, the bigger the holes are, the smaller the dimension should be. This leads to the following definition of porosity:

1.1. Definition. *The porosity of a set $A \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by*

$$\operatorname{por}(A,x) = \liminf_{r\downarrow 0} \operatorname{por}(A,x,r) \tag{1.1}$$

where

$$\operatorname{por}(A,x,r) = \sup\{p \geq 0 \mid \text{ there is } z \in \mathbb{R}^n \text{ such that } B(z,pr) \subset B(x,r) \setminus A\}.$$

Here $B(x,r)$ is the closed ball with radius r and with centre at x. The porosity of $A \subset \mathbb{R}^n$ is

$$\operatorname{por}(A) = \inf\{\operatorname{por}(A,x) \mid x \in A\}). \tag{1.2}$$

Clearly $0 \leq \operatorname{por}(A) \leq \frac{1}{2}$. It is not difficult to see that if a set in $\mathbb{R}^n$ has positive porosity then its packing dimension is strictly less than n [S]. It is also well-known that in $\mathbb{R}^n$ sets with porosity close to $\frac{1}{2}$ cannot have dimension much bigger than $n-1$. This was first proved by Mattila [M1] for Hausdorff dimension as a consequence of a conical density theorem and later showed for packing dimension, $\dim_{\mathrm{p}}$, by Salli [S] using different methods.

1991 *Mathematics Subject Classification.* 28A12, 28A80.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

1.2. Theorem. *There is a non-increasing function $\Delta_n : [0, \frac{1}{2}] \to [n-1, n]$ satisfying*

$$\lim_{p \uparrow \frac{1}{2}} \Delta_n(p) = n - 1$$

such that

$$\dim_{\mathrm{p}}(A) \leq \Delta_n(\mathrm{por}(A))$$

for all $A \subset \mathbb{R}^n$.

Replacing "lim inf" by "lim sup" in (1.1) would give a weaker form of porosity which was first studied by Denjoy [De]. Dolženko [Do] introduced the term "porosity" in connection with these quantities. For this weaker concept the analogue to Theorem 1.2 does not hold; there exists $A \subset \mathbb{R}^n$ with $\limsup_{r \downarrow 0} \mathrm{por}(A, x, r) = \frac{1}{2}$ for all $x \in A$ such that the Hausdorff dimension of A equals n [M2, 4.12]. For other related results see [Z] and [KR].

We will address the problem of proving an analogue to Theorem 1.2 for measures using the definition of porosity introduced in [EJJ]. In [EJJ] the emphasis is given to measures μ on $\mathbb{R}^n$ which satisfy the doubling condition meaning that for μ-almost all $x \in \mathbb{R}^n$

$$\limsup_{r \downarrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty.$$

In this case the porosity given in Definition 2.1 can be defined equivalently in terms of porosities of Borel sets with positive μ-measure, that is, if μ satisfies the doubling condition, then

$$\mathrm{por}(\mu) = \sup\{\mathrm{por}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\} \tag{1.3}$$

(see [EJJ, Proposition 3.1]). The methods in [EJJ] rely on (1.3) which is not necessarily valid for measures that do not satisfy the doubling condition [EJJ, Example 4]. In this paper we develop new methods for relating porosities of measures on the real line to dimensions. Unlike those in [EJJ] these methods are suitable for studying both doubling and non-doubling measures (see Corollary 3.4).

2. Preliminaries

Let μ be a Radon measure on $\mathbb{R}^n$. The packing dimension of μ is defined in terms of upper local dimensions

$$\dim_{\mathrm{p}}(\mu) = \sup\{s \geq 0 \mid \limsup_{i \to \infty} \frac{\log \mu(D_i(x))}{\log 2^{-i}} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\} \tag{2.1}$$

where $D_i(x)$ is the closed dyadic cube of side-length 2^{-i} containing x. Equivalently this definition can be given using packing dimensions of Borel sets with positive μ-measure

$$\dim_{\mathrm{p}}(\mu) = \inf\{\dim_{\mathrm{p}}(A) \mid A \text{ is a Borel set with } \mu(A) > 0\} \tag{2.2}$$

In (2.1) one can use balls with centres at x instead of dyadic cubes (see [C, Lemma 2.3]).

We denote by $\mu|_A$ the restriction of a measure μ on $\mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$, that is, $\mu|_A(B) = \mu(A \cap B)$ for all $B \subset \mathbb{R}^n$. The image of a measure μ under a map $f : X \to Y$, $X, Y \subset \mathbb{R}^n$, is denoted by $f_\sharp\mu$, so $f_\sharp\mu(A) = \mu(f^{-1}(A))$ for all $A \subset Y$.

Next we will recall the following definition from [EJJ].

2.1. Definition. *The porosity of a Radon measure μ on $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is defined by*

$$\operatorname{por}(\mu, x) = \lim_{\varepsilon \downarrow 0} \liminf_{r \downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon) \tag{2.3}$$

where for all $r, \varepsilon > 0$

$$\begin{aligned} \operatorname{por}(\mu, x, r, \varepsilon) = \sup\{p \geq 0 \mid & \text{ there is } z \in \mathbb{R}^n \text{ such that } B(z, pr) \subset B(x, r) \\ & \text{ and } \mu(B(z, pr)) \leq \varepsilon\mu(B(x, r))\}. \end{aligned} \tag{2.4}$$

The porosity of μ is

$$\begin{aligned} \operatorname{por}(\mu) &= \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \operatorname{por}(\mu, x) \\ &= \inf\{s \geq 0 \mid \operatorname{por}(\mu, x) \leq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n\}. \end{aligned} \tag{2.5}$$

Note that in (2.3) the limit as ε tends to zero exists since the function $\varepsilon \mapsto \liminf_{r \downarrow 0} \operatorname{por}(\mu, x, r, \varepsilon)$ is non-decreasing and bounded. Further $\operatorname{por}(\mu) \leq \frac{1}{2}$ [EJJ]. For other basic properties of porosities of measures see [EJJ].

Let $A_1 = \{0\}$, $A_2 = [1, 2]$, and $A = A_1 \cup A_2$. Then $\dim_{\mathrm{p}}(A) = 1$ and $\operatorname{por}(A) = 0$. Noting that $\sup_{x \in A} \operatorname{por}(A, x) = \frac{1}{2}$ this simple example shows that one cannot replace "inf" by "sup" in (1.2) when proving Theorem 1.2. Taking $\mu_1 = \delta_0$, $\mu_2 = \mathcal{L}|_{[1,2]}$, and $\mu = \mu_1 + \mu_2$, where δ_0 is Dirac measure at 0 and $\mathcal{L}$ is Lebesgue measure, we have $\operatorname{por}(\mu) = \frac{1}{2}$ and $\underline{\operatorname{por}}(\mu) := \operatorname{ess\,inf} \operatorname{por}(\mu, x) = 0$. However $\dim_{\mathrm{p}}(\mu) = 0$. Based on this example it might be possible to consider $\operatorname{por}(\mu)$ instead of $\underline{\operatorname{por}}(\mu)$ for the purpose of proving for measures on $\mathbb{R}$ an analogue to Theorem 1.2 according to which measures with porosity close to $\frac{1}{2}$ can have dimension only slightly above 0. In fact, it appears (see Corollary 3.4) that this is the case. Of course the corresponding result for $\underline{\operatorname{por}}(\mu)$ is a consequence of Corollary 3.4.

According to the next lemma we can essentially neglect points where a measure is not porous when estimating packing dimension from above.

2.2. Lemma. *Assume that μ is a Radon measure on $\mathbb{R}^n$ such that $\operatorname{por}(\mu) \geq p$. Then for all $\delta > 0$ there is a Radon measure μ_δ with compact support and with $\dim_{\mathrm{p}} \mu_\delta \geq \dim_{\mathrm{p}} \mu$ such that the following property holds: there exists $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon \leq \varepsilon_\delta$ there are a Borel set $B_{\delta,\varepsilon}$ and $r_{\delta,\varepsilon} > 0$ with $\mu_\delta(\mathbb{R}^n \setminus B_{\delta,\varepsilon}) \leq \delta\mu_\delta(\mathbb{R}^n)$ and*

$$\operatorname{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2}$$

for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$.

Proof. For fixed $\delta > 0$ let

$$A_\delta = \{x \in \mathbb{R}^n \mid \operatorname{por}(\mu, x) > p - \frac{\delta}{4}\}.$$

Since the function $x \mapsto \operatorname{por}(\mu, x)$ is a Borel function [EJJ], A_δ is a Borel set. Using the fact that $\mu(A_\delta) > 0$, we find $R > 0$ such that $\mu(A_\delta \cap B(0, R)) > 0$. Let $\mu_\delta = \mu|_{A_\delta \cap B(0,R)}$. Clearly μ_δ is a Radon measure with $\dim_{\mathrm{p}} \mu_\delta \geq \dim_{\mathrm{p}} \mu$. Further,

$\mathrm{por}(\mu_\delta, x) \geq \mathrm{por}(\mu, x) > p - \frac{\delta}{4}$ for μ_δ-almost all $x \in \mathbb{R}^n$ [EJJ]. Setting for all positive integers i

$$C_i = \{x \in A_\delta \cap B(0,R) \mid \liminf_{r \to 0} \mathrm{por}(\mu_\delta, x, r, \varepsilon) \geq p - \frac{\delta}{4} \text{ for all } 0 < \varepsilon \leq \frac{1}{i}\}$$

we find a positive integer i_δ such that $\mu_\delta(\mathbb{R}^n \setminus C_{i_\delta}) \leq \frac{\delta}{2}\mu_\delta(\mathbb{R}^n)$. Note that C_i is a Borel set since the function $x \mapsto \liminf_{r\to 0} \mathrm{por}(\mu_\delta, x, r, \varepsilon)$ is a Borel function and the definition of C_i does not change if ε is restricted to positive rationals [EJJ]. Take $\varepsilon_\delta = 1/i_\delta$. Let $0 < \varepsilon \leq \varepsilon_\delta$. For all positive integers j define a Borel set

$$B_j = \{x \in C_{i_\delta} \mid \mathrm{por}(\mu_\delta, x, r, \varepsilon) > p - \frac{\delta}{2} \text{ for all } 0 < r \leq \frac{1}{j}\}.$$

The Borel measurability of B_j follows since the function $x \mapsto \mathrm{por}(\mu_\delta, x, r, \varepsilon)$ is a Borel function and the definition of B_j remains unchanged if r is limited to rational numbers [EJJ]. Now we find a positive integer $j_{\delta,\varepsilon}$ with $\mu_\delta(\mathbb{R}^n \setminus B_{j_{\delta,\varepsilon}}) \leq \delta\mu_\delta(\mathbb{R}^n)$. Choosing $B_{\delta,\varepsilon} = B_{j_{\delta,\varepsilon}}$ and $r_{\delta,\varepsilon} = 1/j_{\delta,\varepsilon}$ gives the claim. □

For all positive integers k, denote by $\mathbf{I}_k$ the set of all k-term sequences of integers 0 and 1 and by $\mathbf{I}$ corresponding set of infinite sequences, that is,

$$\mathbf{I}_k = \{(i_1, i_2, \ldots, i_k) \mid i_l \in \{0,1\} \text{ for all } l = 1, \ldots, k\}$$

and

$$\mathbf{I} = \{(i_1, i_2, \ldots) \mid i_l \in \{0,1\} \text{ for all } l = 1, \ldots\}.$$

We use the notation $\mathbf{i}^x \in \mathbf{I}$ for the binary expansion of a point x in the closed unit interval $[0,1]$. (Note that $\mathbf{i}^x$ is unique except for dyadic rationals. The possible complications due to this countable set will be dealt with in due course.) For a positive integer k and $\mathbf{i} = (i_1, i_2, \ldots) \in \mathbf{I}$ let $\mathbf{i}|_k = (i_1, \ldots, i_k) \in \mathbf{I}_k$ be the sequence of the first k digits of $\mathbf{i}$ and let $\mathrm{n}_0(\mathbf{i}|_k)$ be the number of zeros in $\mathbf{i}|_k$.

The following lemma which we apply when proving Proposition 3.1 is verified as a part of the proof of [Fa, Proposition 10.4].

2.3. Lemma. *For $0 \leq \alpha \leq \frac{1}{2}$ we have*

$$\dim_{\mathrm{p}} E_\alpha = -\frac{1}{\log 2}(\alpha \log \alpha + (1-\alpha)\log(1-\alpha))$$

where

$$E_\alpha = \{x \in [0,1] \mid \limsup_{k\to\infty} \frac{1}{k}\,\mathrm{n}_0(\mathbf{i}^x|_k) \leq \alpha\}$$

is a Borel set.

Proof. See [Fa, Proposition 10.4]. In [Fa, Proposition 10.4] one has the limit instead of the upper limit in the definition of E_α but the argument goes through also in this more general case. □

3. Main results

Let μ be a non-atomic (that is $\mu(\{x\}) = 0$ for all $x \in [0,1]$) Radon probability measure on $[0,1]$. The measure μ gives rise to a sequence (P_k^μ) of measures on $\{0,1\}$ such that $\mathrm{P}_k^\mu(\{0\})$ gives the probability that the k-th binary digit of a random number (with respect to μ) in $[0,1]$ equals zero and $\mathrm{P}_k^\mu(\{1\})$ gives the probability that it is one, that is,

$$\mathrm{P}_k^\mu(\{0\}) = \sum_{\substack{(i_1,\dots,i_k)\in\mathbf{I}_k\\ i_k=0}} \mu(I_{i_1,\dots,i_k}) \text{ and } \mathrm{P}_k^\mu(\{1\}) = 1 - \mathrm{P}_k^\mu(\{0\})$$

where $I_{i_1,\dots,i_k}$ is the closed dyadic subinterval of $[0,1]$ of length 2^{-k} consisting of numbers whose binary expansion begins with $(i_1,\dots,i_k)$. We use the notation P^μ for the product measure $\prod_{k=1}^\infty \mathrm{P}_k^\mu$ on the code space $\mathbf{I}$.

As a consequence of Lemma 2.3 we obtain:

3.1. Proposition. *Let μ be a non-atomic Radon probability measure on $[0,1]$. Assume that $0 \le \alpha \le \frac{1}{2}$ and $\limsup_{k\to\infty} \frac{1}{k}\sum_{i=1}^k \mathrm{P}_i^\mu(\{0\}) \le \alpha$. Then*

$$\dim_{\mathrm{p}} \mu \le -\frac{1}{\log 2}(\alpha\log\alpha + (1-\alpha)\log(1-\alpha)).$$

Proof. By the strong law of large numbers [Fe, X.7.1] we have for P^μ-almost all $\mathbf{i} \in \mathbf{I}$

$$\limsup_{k\to\infty} \frac{1}{k}\,\mathrm{n}_0(\mathbf{i}|_k) \le \alpha$$

implying that $\mu(E_\alpha) = 1$ where E_α is as in Lemma 2.3. This gives the claim by Lemma 2.3 and (2.2). □

For an integer $j \ge 1$ two closed dyadic intervals of length 2^{-j} form *a couple* if they belong to the same dyadic interval of length 2^{-j+1}. Let μ be a measure on $[0,1]$. We denote by $\mathcal{D}_j^s(\mu)$ the family of closed dyadic intervals of length 2^{-j} formed by selecting from each dyadic couple the one which has smaller μ-measure.

3.2. Theorem. *Assume that μ is a non-atomic Radon probability measure on $[0,1]$ and $0 \le \beta \le 1$ such that $\mathrm{por}(\mu) \ge \frac{1}{2}(1-\beta)$. For all $0 < \delta < 1$ let μ_δ be as in Lemma 2.2. Then*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^n E_j^{\mu_\delta}(\{0\}) \le \alpha(\beta,\delta)\mu_\delta([0,1])$$

where

$$E_j^{\mu_\delta}(\{0\}) = \sum_{I\in\mathcal{D}_j^s(\mu_\delta)} \mu_\delta(I)$$

and $\alpha(\beta,\delta) \to 0$ as $\delta \to 0$ and $\beta \to 0$.

Proof. Let ε_δ be as in Lemma 2.2. Consider $0 < \varepsilon \le \varepsilon_\delta$. By Lemma 2.2 there is a Borel set $B_{\delta,\varepsilon}$ with $\mu_\delta([0,1]\setminus B_{\delta,\varepsilon}) \le \delta\mu_\delta([0,1])$ and $r_{\delta,\varepsilon} > 0$ such that

$$\mathrm{por}(\mu_\delta, x, r, \varepsilon) > \frac{1}{2}(1-\beta-\delta) \tag{3.1}$$

for all $x \in B_{\delta,\varepsilon}$ and $0 < r \leq r_{\delta,\varepsilon}$. Let i_0 be the smallest integer such that $2^{-i_0+1} < r_{\delta,\varepsilon}$. Consider a positive integer i with $i \geq i_0$.

Let $I = [a, b]$ be a dyadic subinterval of $[0, 1]$ of length 2^{-i} intersecting the set $B_{\delta,\varepsilon}$. We denote by I_L and I_R the two neighbouring dyadic intervals of I of length 2^{-i}. Let

$$M_I = \max\{\mu_\delta(I), \mu_\delta(I_L), \mu_\delta(I_R)\}.$$

For all $x \in I \cap B_{\delta,\varepsilon}$ let $r_x = \min\{x - a + 2^{-i}, b - x + 2^{-i}\}$. Setting $q = \frac{1}{2}(1 - \beta - \delta)$, the inequality (3.1) implies that for all $x \in I \cap B_{\delta,\varepsilon}$ there exists $z_x \in \mathbb{R}$ such that $B(z_x, qr_x) \subset B(x, r_x)$ and

$$\mu_\delta(B(z_x, qr_x)) \leq \varepsilon\mu_\delta(B(x, r_x)) \leq 3M_I\varepsilon. \tag{3.2}$$

We will first prove that I is a union of five (not necessarily non-empty) successive intervals

$$I = I_1 \cup P_1 \cup I_2 \cup P_2 \cup I_3 \tag{3.3}$$

where the intervals P_1 and P_2 have length at most $(\beta + \delta)2^{-i+1}$ and for $j = 1, 2, 3$ either $\mu_\delta(I_j) \leq 3M_I\varepsilon$ or $I_j \subset I \setminus B_{\delta,\varepsilon}$.

If there is $x \in I \cap B_{\delta,\varepsilon}$ such that $B(z_x, qr_x) \subset (a, b)$, where (a, b) is the open interval $\{y \in \mathbb{R} \mid a < y < b\}$, then $I \setminus B(z_x, qr_x)$ is a union of at most two intervals of length at most $2^{-i}(1 - 2q) = 2^{-i}(\beta + \delta)$. Hence (3.3) follows by (3.2).

Assume that for all $x \in I \cap B_{\delta,\varepsilon}$ we have $B(z_x, qr_x) \not\subset (a, b)$ and define

$$C = \{z_x + qr_x \mid x \in I \cap B_{\delta,\varepsilon} \text{ and } a \geq z_x - qr_x\}$$

and

$$D = \{z_x - qr_x \mid x \in I \cap B_{\delta,\varepsilon} \text{ and } b \leq z_x + qr_x\}.$$

Then either $C \neq \emptyset$ or $D \neq \emptyset$. Assume that $C \neq \emptyset$; the case where $D \neq \emptyset$ is similar to this one. Setting $s = \sup C$ and using (3.2) we obtain

$$\mu_\delta([a, s)) \leq 3M_I\varepsilon. \tag{3.4}$$

(We use the interpretation $[a, s) = \emptyset$ if $a \geq s$.)

If $D = \emptyset$, then for all $x \in I \cap B_{\delta,\varepsilon}$ with $s \leq x$ we have

$$x - s \leq x - z_x - qr_x \leq r_x(1 - 2q) \leq (\beta + \delta)2^{-i+1}.$$

Hence by (3.4) I is a union of three intervals – one of them having measure at most $3M_I\varepsilon$, one belonging to the complement of $B_{\delta,\varepsilon}$, and the remaining one having length at most $(\beta + \delta)2^{-i+1}$. This gives (3.3).

If $D \neq \emptyset$, then $t = \inf D$ exists and as in (3.4) we see that

$$\mu_\delta((t, b]) \leq 3M_I\varepsilon. \tag{3.5}$$

This together with (3.4) implies in the case $t \leq s$ that I can be covered by two intervals of μ_δ-measure at most $3M_I\varepsilon$. For $t > s$ one verifies as above that

$$I \cap B_{\delta,\varepsilon} \cap [s, t] \subset [s, s + (\beta + \delta)2^{-i+1}] \cup [t - (\beta + \delta)2^{-i+1}, t]$$

which implies by (3.4) and (3.5) that I is a union of five intervals – two of them having μ_δ-measure at most $3M_I\varepsilon$, two having length at most $(\beta+\delta)2^{-i+1}$, and the remaining one belonging to the complement of $B_{\delta,\varepsilon}$. This completes the proof of (3.3).

Consider a positive integer k with $2^{-k-1}\le 2(\beta+\delta)<2^{-k}$. For $l=1,\dots,k$ we say that a dyadic couple $\{D^0_{i+l},D^1_{i+l}\}$ of length 2^{-i-l} *satisfies property (A) with respect to* P_j, where $j=1,2$, if $P_j\cap D^0_{i+l}\ne\emptyset$ and $P_j\cap D^1_{i+l}\ne\emptyset$. Further, we say that $\{D^0_{i+l},D^1_{i+l}\}$ *satisfies property (B)* if P_1 intersects one of the members of $\{D^0_{i+l},D^1_{i+l}\}$ and P_2 meets the other one. For $j=1,2$ there is at most one integer $1\le l_j\le k$ for which there exists a dyadic couple of length 2^{-i-l_j} satisfying property (A) with respect to P_j. It is also easy to see that there are at most three integers $1\le l_j\le k$, $j=3,4,5$, such that for each l_j, $j=3,4,5$, there exists a dyadic couple of length 2^{-i-l_j} satisfying property (B). Since

$$E^{\mu_\delta|_I}_{i+l_j}(\{0\})\le\frac{1}{2}\mu_\delta(I)$$

for $j=1,\dots,5$, and

$$E^{\mu_\delta|_I}_{i+l}(\{0\})\le 6M_I\varepsilon+\mu_\delta(I\setminus B_{\delta,\varepsilon})$$

for $l\ne l_j$, $j=1,\dots,5$, we have

$$\sum_{l=1}^{k}E^{\mu_\delta|_I}_{i+l}(\{0\})\le\frac{5}{2}\mu_\delta(I)+k(6M_I\varepsilon+\mu_\delta(I\setminus B_{\delta,\varepsilon})).$$

Note that if $I\cap B_{\delta,\varepsilon}=\emptyset$, then $\sum_{l=1}^{k}E^{\mu_\delta|_I}_{i+l}(\{0\})\le k\mu_\delta(I\setminus B_{\delta,\varepsilon})$. When summing over intervals I each M_I may appear at most three times giving the upper bound

$$\sum_{l=1}^{k}E^{\mu_\delta}_{i+l}(\{0\})\le\mu_\delta([0,1])\Big(\frac{5}{2}+18k\varepsilon+k\delta\Big).$$

This implies for $n=i_0+mk$ where m is a positive integer that

$$\frac{1}{n}\sum_{j=1}^{n}E^{\mu_\delta}_{j}(\{0\})\le\frac{\mu_\delta([0,1])}{n}\Big(\frac{i_0}{2}+\frac{5m}{2}+18mk\varepsilon+mk\delta\Big).$$

Letting $n\to\infty$ and $\varepsilon\to 0$ we obtain $\alpha(\beta,\delta)=\frac{5}{2k}+\delta$. $\square$

We denote by D the set of end-points of dyadic subintervals of $[0,1]$ and by $\mathbf{D}$ the set of binary expansions of points belonging to D, so

$$D=\{l2^{-k}\mid k=0,1,\dots\text{ and }l=0,\dots,2^k\}$$

and

$$\mathbf{D}=\{(i_1,i_2,\dots)\in\mathbf{I}\mid\text{ there is }k\text{ such that either }i_j=0\text{ for all }j\ge k\text{ or }i_j=1\text{ for all }j\ge k\}.$$

Let μ be a non-atomic Radon probability measure on $[0,1]$. Next we will define a function $G^\mu : [0,1] \setminus D \to [0,1]$ which will be used to establish a connection between Theorem 3.2 and Proposition 3.1 by comparing the relative weights of dyadic couples. Let $\mathbf{i} = (i_1, i_2, \dots) \in \mathbf{I} \setminus \mathbf{D}$. Recalling the notation $I_{i_1,\dots,i_j}$ for the closed dyadic subinterval of $[0,1]$ of length 2^{-j} consisting of numbers whose binary expansions begin with $(i_1, \dots, i_j)$, set

$$(F^\mu(\mathbf{i}))_j = \begin{cases} i_j & \text{if } \mu(I_{i_1,\dots,i_{j-1},0}) \le \mu(I_{i_1,\dots,i_{j-1},1}) \\ i_j^c & \text{if } \mu(I_{i_1,\dots,i_{j-1},0}) > \mu(I_{i_1,\dots,i_{j-1},1}) \end{cases}$$

where

$$i_j^c = \begin{cases} 1 & \text{if } i_j = 0 \\ 0 & \text{if } i_j = 1. \end{cases}$$

Let $H : \mathbf{I} \to [0,1]$ be the natural projection. Note that $H^{-1} : [0,1] \setminus D \to \mathbf{I} \setminus \mathbf{D}$ is well-defined. Define a continuous function $G^\mu : [0,1] \setminus D \to [0,1]$ by

$$G^\mu = H \circ F^\mu \circ H^{-1}.$$

The functions F^μ and G^μ have the following properties:

3.3. Lemma. *Let μ be a non-atomic Radon probability measure on $[0,1]$.*

(1) If $\mathbf{i},\mathbf{j} \in \mathbf{I} \setminus \mathbf{D}$ such that $\mathbf{i} \neq \mathbf{j}$ and $F^\mu(\mathbf{i}) = F^\mu(\mathbf{j})$, then $F^\mu(\mathbf{i}) \in \mathbf{D}$.

(2) $(F^\mu)^{-1}(\mathbf{i})$ contains at most two points for all $\mathbf{i} \in \mathbf{D}$.

(3) F^μ is an injection in the set $\mathbf{I} \setminus (\mathbf{D} \cup (F^\mu)^{-1}(\mathbf{D}))$ whose complement is countable.

(4) $\dim_{\mathrm{p}}((G^\mu)_\sharp(\mu|_{[0,1]\setminus D})) = \dim_{\mathrm{p}}(\mu)$.

Proof. (1) follows easily from the definition of F^μ.

For (2), let $\mathbf{i} \in \mathbf{D}$. Then there are a positive integer k and closed dyadic intervals D_k^R and D_k^L of length 2^{-k} such that $H(\mathbf{i})$ is the right-hand side end-point of D_k^R and the left-hand side end-point of D_k^L. The definition of F^μ implies that for all $j \ge k+1$ there are closed dyadic intervals $D_j^R \subset D_{j-1}^R$ and $D_j^L \subset D_{j-1}^L$ of length 2^{-j} with $(F^\mu)^{-1}(\mathbf{i}) \subset H^{-1}(D_j^R) \cup H^{-1}(D_j^L)$. This gives the claim.

(3) is a consequence of (1) and (2).

(4) follows from (2.1) and from the fact that $D_i(G^\mu(x)) = G^\mu(D_i(x))$ for all positive integers i where $D_i(x)$ is the closed dyadic interval of length 2^{-i} containing $x \in [0,1]$. Note that by (3) G^μ is an injection in a set whose complement is countable and therefore has μ-measure zero. □

3.4. Corollary. *Assume that ν is a Radon measure on the real line. If $0 \le \beta \le 1$ such that $\mathrm{por}(\nu) \ge \frac{1}{2}(1-\beta)$, then $\dim_{\mathrm{p}}(\nu) \le d(\beta)$ where $d(\beta) \to 0$ as $\beta \to 0$.*

Proof. We may assume that ν is non-atomic. Let $\eta > 0$. By (2.5) there is $A_\eta \subset \mathbb{R}$ with $\nu(A_\eta) > 0$ such that $\mathrm{por}(\nu, x) \ge \frac{1}{2}(1-\beta-\eta)$ for all $x \in A_\eta$. Taking $R_\eta > 0$ with $\nu(A_\eta \cap [-R_\eta, R_\eta]) > 0$ we have

$$\mathrm{por}(\nu|_{[-R_\eta,R_\eta]}) \ge \frac{1}{2}(1-\beta-\eta).$$

Mapping $\mu = \nu|_{[-R_\eta,R_\eta]}$ into $[0,1]$ by an affine bijection which preserves both porosity and packing dimension, and normalizing the image measure, we may assume that μ is a non-atomic Radon probability measure on $[0,1]$ [M, Theorem 1.18].

Let $0<\delta<1$ and let μ_δ be as in Lemma 2.2. Set $N=\mu_\delta([0,1])$. Then Theorem 3.2 implies that

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\mathrm{P}_i^{(G^{\frac{\mu_\delta}{N}})_\sharp(\frac{\mu_\delta}{N}|_{[0,1]\setminus D})}(\{0\})=\limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}E_i^{\frac{\mu_\delta}{N}|_{[0,1]\setminus D}}(\{0\})$$
$$=\frac{1}{N}\limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}E_i^{\mu_\delta}(\{0\})\le\alpha(\beta-\eta,\delta)$$

where α is as in Theorem 3.2. Since $(G^{\frac{\mu_\delta}{N}})_\sharp(\frac{\mu_\delta}{N}|_{[0,1]\setminus D})$ is a non-atomic Radon probability measure [M, Theorem 1.18], we have by Lemmas 2.2 and 3.3 (4), and Proposition 3.1

$$\dim_{\mathrm{p}}(\nu)\le\dim_{\mathrm{p}}(\mu)\le\dim_{\mathrm{p}}(\frac{\mu_\delta}{N})=\dim_{\mathrm{p}}((G^{\frac{\mu_\delta}{N}})_\sharp(\frac{\mu_\delta}{N}|_{[0,1]\setminus D}))$$
$$\le-\frac{1}{\log 2}\big(\alpha(\beta-\eta,\delta)\log\alpha(\beta-\eta,\delta)+(1-\alpha(\beta-\eta,\delta))\log(1-\alpha(\beta-\eta,\delta))\big).$$

Letting η and δ tend to zero gives the claim. □

Remarks. 1. We expect that Corollary 3.4 holds in higher dimensions as well despite the fact that the extension of the above methods is not straightforward. Our proof relies on the geometry of the real line. Intuitively, for a measure with porosity close to $\frac{1}{2}$ small balls around typical points are such that either their left or right half is practically empty. Stating this property in higher dimensions seems to be unclear.

2. It is possible to use the above methods to prove that if μ is a measure on $\mathbb{R}^n$ with positive porosity then its packing dimension is strictly less than n. For this choose a positive integer k depending on the porosity of the measure such that the ball with small measure given by (2.4) always contains at least one cube of side-length k^{-l} where l depends on the scale. In this way one obtains the upper bound $\log(k^n-1)/\log k$ for the packing dimension (compare the dimension formula for self-similar sets [Fa, Theorem 2.7]).

3. Let μ be a non-atomic Radon probability measure on $[0,1]$. Our proof originated from the idea to view μ as sequences of relative weights. For all $\mathbf{i}=(i_1,i_2,\dots)\in\mathbf{I}$ there is a sequence $(p_k^{\mathbf{i}}(\mu))$ that gives the relative weights of the dyadic couples of length 2^{-k} containing the point whose binary expansion is $\mathbf{i}$. More precisely, let $p_1^{\mathbf{i}}(\mu)=\mu(I_0)$. Then $1-p_1^{\mathbf{i}}(\mu)=\mu(I_1)$. Define inductively

$$\mu(I_{i_1,\dots,i_{j-1},0})=p_j^{\mathbf{i}}(\mu)\mu(I_{i_1,\dots,i_{j-1}})$$

giving

$$\mu(I_{i_1,\dots,i_{j-1},1})=(1-p_j^{\mathbf{i}}(\mu))\mu(I_{i_1,\dots,i_{j-1}}).$$

Now the space of non-atomic Radon probability measures on $[0,1]$ can be identified with the space

$$X=\{(p_k^{\mathbf{i}})\in[0,1]^{\mathbb{N}\times\mathbf{I}}\mid \text{If } \mathbf{i}=(i_1,i_2,\dots),\mathbf{j}=(j_1,j_2,\dots)\in\mathbf{I}\text{ such that }\exists\, n\text{ with }$$
$$i_l=j_l\,\forall\, l=1,\dots,n\text{ and }i_{n+1}\neq j_{n+1}\text{ then}$$
$$p_l^{\mathbf{i}}=p_l^{\mathbf{j}}\,\forall\, l=1,\dots,n\text{ and }p_{n+1}^{\mathbf{i}}=1-p_{n+1}^{\mathbf{j}}\}.$$

The condition guarantees that points in the same dyadic interval have the same history of relative weights.

Acknowledgements

We thank Claude Tricot for his interest in our work. We acknowledge the financial support of the Academy of Finland (projects 46208 and 38955). EJ is also grateful to the European Science Foundation programme Probabilistic Methods in Non-hyperbolic Dynamics.

References

[BT] A. S. Besicovitch and S. J. Taylor, *On the complementary intervals of a linear closed set of zero Lebesgue measure*, J. London Math. Soc. **29** (1954), 449–459.

[C] C. D. Cutler, *Strong and weak duality principles for fractal dimension in Euclidean space*, Math. Proc. Cambridge Phil. Soc. **118** (1995), 393–410.

[De] A. Denjoy, *Sur une propriété des séries trigonométriques*, Verlag v.d.G.V. der Wis-en Natuur. Afd., 30 Oct. (1920).

[Do] E. P. Dolženko, *Boundary properties of arbitrary functions (in Russian)*, Izv. Akad. Nauk SSSR Ser. Mat. **31** (1967), 3–14.

[EJJ] J.-P. Eckmann, E. Järvenpää, and M. Järvenpää, *Porosities and dimensions of measures*, submitted to Nonlinearity.

[Fa] K. J. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, Chichester, 1997.

[Fe] W. Feller, *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons, New York, 1950.

[KR] P. Koskela and S. Rohde, *Hausdorff dimension and mean porosity*, Math. Ann. **309** (1997), 593–609.

[M1] P. Mattila, *Distribution of sets and measures along planes*, J. London Math. Soc. (2) **38** (1988), 125–132.

[M2] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

[S] A. Salli, *On the Minkowski dimension of strongly porous fractal sets in* $\mathbb{R}^n$, Proc. London Math. Soc. (3) **62** (1991), 353–372.

[T] C. Tricot, *Porous Surfaces*, Constr. Approx. **5** (1989), 117–136.

[Z] L. Zajiček, *Porosity and* σ*-porosity*, Real Anal. Exchange **13** (1987-88), 314–350.