MENGER CURVATURE AND C^1 REGULARITY OF FRACTALS

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Abstract. We show that if E is an s-regular set in $\mathbf{R}^2$ for which the triple integral $\int_E \int_E \int_E c(x,y,z)\,d\mathcal{H}^s x\,d\mathcal{H}^s y\,d\mathcal{H}^s z$ of the Menger curvature c is finite and if $0 < s \leq 1/2$, then $\mathcal{H}^s$ almost all of E can be covered with countably many C^1 curves. We give an example to show that this is false for $1/2 < s < 1$.

1. Introduction. The Menger curvature $c(x,y,z)$ of three points x, y and z in the plane $\mathbf{R}^2$ is defined as the reciprocal of the radius of the circle passing through these points. For a historical background, see [K]. In [Me] Melnikov found a remarkable connection between the Menger curvature and the Cauchy kernel $1/z$, $z \in \mathbf{C}$. This lead to a rapid development on singular integrals over 1-dimensional subsets of $\mathbf{R}^2$ and on removable sets of bounded analytic functions, see [MV], [MMV], [D], and for a survey [M2].

Another aspect of the Menger curvature is that its integrals can be used to measure smoothness properties of subsets of $\mathbf{R}^n$. Note that $c(x,y,z) = 0$ if and only if the points x, y and z lie on the same line. Let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure. For $\mathcal{H}^s$ measurable sets $E \subset \mathbf{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$ the proper quantity to use is

$$c^{2s}(E) = \int_E \int_E \int_E c(x,y,z)^{2s}\,d\mathcal{H}^s x\,d\mathcal{H}^s y\,d\mathcal{H}^s z.$$

Léger proved in [L] that if $\mathcal{H}^1(E) < \infty$ and $c^2(E) < \infty$, then there are rectifiable curves $\Gamma_1, \Gamma_2, \dots$ such that

$$\mathcal{H}^1\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

Sets with this property are called 1-rectifiable in [M1] and countably $(\mathcal{H}^1, 1)$ rectifiable in [F].

In this paper we study analogous questions for other values of s. It was shown in [Li, Theorem 1.4] that if $E \subset \mathbf{R}^n$ is $\mathcal{H}^s$ measurable and $0 < \mathcal{H}^s(E) < \infty$ for

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some $1 < s \leq n$, then $c^{2s}(E) = \infty$. Hence we restrict to $0 < s < 1$. We also study only subsets of $\mathbf{R}^2$, although with slight modifications the results would extend to $\mathbf{R}^n$. For reasons indicated in Example 2.5 we restrict to the so-called s-regular sets. This means that there is a constant C such that

$$r^s/C \leq \mathcal{H}^s\big(E \cap B(x,r)\big) \leq Cr^s \quad \text{for } x \in E,\ 0 < r < d(E). \tag{1.1}$$

Here $B(x,r)$ is the closed ball with centre x and radius r, and $d(E)$ stands for the diameter of E.

When $0 < s < 1$ and E is compact, (1.1) alone implies that E is contained in one rectifiable curve Γ, see the proof of Theorem 4.1 in [MM]. A rectifiable curve is the same as a Lipschitz image of the interval $[0,1]$. We study here whether Lipschitz images can be replaced by C^1 curves. By a C^1 curve we mean a curve with continuously varying tangent. It is the same as the image of an interval under a regular C^1 map, that is, a C^1 map with non-vanishing derivative. We shall prove in Corollary 2.2 that if E satisfies (1.1), $c^{2s}(E) < \infty$ and $0 < s \leq 1/2$, then there are C^1 curves $\Gamma_1, \Gamma_2, \dots$ such that

$$\mathcal{H}^s\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

We give in 2.4 an example showing that this is false if $1/2 < s < 1$. For $s = 1$ it is again true due to [L], even with the weaker condition $\mathcal{H}^1(E) < \infty$ instead of (1.1), because then covering $\mathcal{H}^1$ almost everything with Lipschitz images or C^1 curves are equivalent as a consequence of Rademacher's theorem, see [F, 3.2.29].

There are also many other characterizations for 1-rectifiable sets. In [MM] analogous conditions for s-dimensional sets were investigated and this paper can be considered as a further contribution to that study.

2. Covering with C^1 curves. We begin with the following result on the existence of tangents. We say that a set $E \subset \mathbf{R}^2$ has a tangent L at x if L is a line through x such that for any $\alpha > 0$, $E \cap B(x,r) \subset C(x,\alpha)$ for all sufficiently small $r > 0$, where $C(x,\alpha)$ is the double-cone with centre x, axis L and angle α.

2.1. Theorem. *Let $0 < s \leq 1/2$ and let $E \subset \mathbf{R}^2$ be $\mathcal{H}^s$ measurable and s-regular. If $x \in E$ and*

$$c^{2s}(E,x) := \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z < \infty, \tag{2.1}$$

then E has a tangent at x.

Proof. Let $x \in E$. By the s-regularity of E there are positive numbers b and $d < d(E)$ such that for $i = 1, 2, \dots$,

$$\mathcal{H}^s(A_i) \geq b d^{is} \tag{2.2}$$

where

$$A_i = \big\{y \in E : d^{i+1} < |x-y| \leq d^i\big\}.$$

Set

$$\gamma_i = \int_{A_i} \int_{A_i} c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z.$$

Then

$$\sum_i \gamma_i \le \int_E \int_E c(x,y,z)^{2s}\, d\mathcal{H}^s y\, d\mathcal{H}^s z < \infty. \tag{2.3}$$

We shall show that for each i there is a line L_i through x such that

$$\mathcal{H}^s\big(A_i \cap L_i(\eta_i\, d^i)\big) \ge bd^{is}/16, \tag{2.4}$$

where

$$\eta_i = (16b^{-1})^{1/s}\, \gamma_i^{1/(2s)}, \tag{2.5}$$

and

$$B(\delta) = \big\{x \in \mathbf{R}^2 : \operatorname{dist}(x,\beta) \le \delta\big\} \quad \text{for } B \subset \mathbf{R}^2,\ \delta > 0.$$

Suppose (2.4) fails for some i. By (2.2) there is closed quarter-plane Q (a sector with angle $\pi/2$) with vertex at x such that $\mathcal{H}^s(A_i \cap Q) \ge bd^{is}/4$. Further, there is a line L through x such that

$$\mathcal{H}^s(A_i \cap Q \cap H_j) \ge bd^{is}/8 \quad \text{for } j = 1, 2,$$

where H_1 and H_2 are the two closed half-planes whose boundary is L. Since $\mathcal{H}^s\big(A_i \cap L(\eta_i\, d^i)\big) < bd^{is}/16$, we have

$$\mathcal{H}^s\big(A_i \cap Q \cap H_j \setminus L(\eta_i\, d^i)\big) > bd^{is}/16 \quad \text{for } j = 1, 2. \tag{2.6}$$

Let $x_j \in A_i \cap Q \cap H_j \setminus L(\eta_i\, d^i)$ for $j = 1, 2$. We use the formula, which is an exercise in elementary geometry,

$$c(x, x_1, x_2) = \frac{2\operatorname{dist}(x_2, L_{x,x_1})}{|x - x_2|\,|x_1 - x_2|}, \tag{2.7}$$

where $L_{y,z}$ denotes the line through two points y and z. This gives

$$c(x, x_1, x_2) \ge \frac{2\eta_i\, d^i}{d^i \cdot d^i} = \frac{2\eta_i}{d^i}.$$

Thus by (2.6) and (2.5)

$$\gamma_i > \left(\frac{\eta_i}{d^i}\right)^{2s} (bd^{is}/16)^2 = (b/16)^2\, \eta_i^{2s} = \gamma_i,$$

which is a contradiction proving (2.4).

Next we show that if

$$\zeta_i = \max\big\{12\eta_i/d,\ \big(16 \cdot 50^{2s} C b^{-1} d^{-2s} (\gamma_{i-1} + \gamma_i + \gamma_{i+1})\big)^{1/(3s)}\big\}, \tag{2.8}$$

and if $\zeta_i < d$, then

$$A_i \subset L_i(\zeta_i\, d^i). \tag{2.9}$$

Suppose this fails and let $y_1 \in A_i \setminus L_i(\zeta_i\, d^i)$. Then $\zeta_i < 1$ and $B(y_1, \frac{1}{2}\zeta_i\, d^i) \subset B(x, 2d^i) \setminus L_i(\frac{1}{2}\zeta_i\, d^i)$. Thus for all $y \in B(y_1, \frac{1}{2}\zeta_i\, d^i)$ and $z \in A_i \cap L_i(\frac{1}{12}\zeta_i\, d^{i+1})$ we have by some elementary geometry $d(x, L_{y,z}) \geq \frac{1}{24}\zeta_i\, d^{i+1}$. Hence by (2.7),

$$c(x,y,z) \geq \frac{\zeta_i\, d^{i+1}/12}{d^i \cdot 2d^i} = \frac{\zeta_i\, d}{24 d^i}.$$

Since by (1.1) $\mathcal{H}^s\big(E \cap B(y_1, \frac{1}{2}\zeta_i\, d^i)\big) \geq (\frac{1}{2}\zeta_i\, d^i)^s/C$, $B(y_1, \frac{1}{2}\zeta_i\, d^i) \subset A_{i-1} \cup A_i \cup A_{i+1}$ (as $\zeta_i < d$) and by (2.4) $\mathcal{H}^s\big(A_i \cap L_i(\frac{1}{12}\zeta_i\, d^{i+1})\big) \geq \frac{1}{16} b d^{is}$ (as $L_i(\frac{1}{12}\zeta_i\, d^{i+1}) \supset L_i(\eta_i\, d^i)$ by (2.8)), we get

$$\begin{aligned}\gamma_{i-1} + \gamma_i + \gamma_{i+1} &\geq \Big(\frac{\zeta_i\, d}{24 d^i}\Big)^{2s} C^{-1} \Big(\frac{1}{2}\zeta_i\, d^i\Big)^s \frac{1}{16} b d^{is} \\ &> \frac{b d^{2s}\, \zeta_i^{3s}}{16 \cdot 50^{2s} C} \geq \gamma_{i-1} + \gamma_i + \gamma_{i+1},\end{aligned}$$

which proves (2.9).

Let $\alpha_i \in [0, \pi)$ be the angle between the lines L_i and L_{i+1}. We claim that

$$\alpha_i \leq \max\big\{8d^{-1}\eta_i, 8d^{-1}\eta_{i+1}, 8(16/b)^{1/s} d^{-3} (\gamma_i + \gamma_{i+1})^{1/(2s)}\big\}. \tag{2.10}$$

Suppose this is false. Then if $y \in A_i \cap L_i(\eta_i\, d^i)$ and $z \in A_{i+1} \cap L_{i+1}(\eta_{i+1}\, d^{i+1})$, we have by simple elementary geometry $\operatorname{dist}(z, L_{x,y}) \geq \frac{1}{4}\alpha_i\, d^{i+2}$. Hence (2.7) gives

$$c(x,y,z) \geq \frac{\frac{1}{2}\alpha_i\, d^{i+2}}{(2d^i)^2} = \frac{\alpha_i\, d^2}{8 d^i}.$$

Integrating over $A_i \cap L_i(\eta_i\, d^i)$ and $A_{i+1} \cap L_{i+1}(\eta_{i+1}\, d^{i+1})$ and using (2.4), we obtain

$$\gamma_i + \gamma_{i+1} \geq \Big(\frac{\alpha_i\, d^2}{8 d^i}\Big)^{2s} \Big(\frac{1}{16} b d^{(i+1)s}\Big)^2 = \frac{d^{6s}\, b^2\, \alpha_i^{2s}}{16^2 \cdot 8^{2s}} > \gamma_i + \gamma_{i+1};$$

a contradiction proving (2.10).

By the definition (2.5) of η_i and by (2.10) we have for some $C_1 < \infty$ for all i,

$$\alpha_i \leq C_1(\gamma_i + \gamma_{i+1})$$

since $0 < s \leq 1/2$. Using (2.3) we find that $\sum_i \alpha_i < \infty$. This means that the lines L_i converge to a line L through x. Applying (2.9) and the fact that $\zeta_i \to 0$, we see that L is a tangent to E at x. This completes the proof.

2.2. Corollary. *If $0 < s \leq 1/2$, $E \subset \mathbf{R}^2$ is s-regular, $\mathcal{H}^s$ measurable and $c^{2s}(E) < \infty$, then there are C^1 curves $\Gamma_1, \Gamma_2, \dots$ such that*

$$\mathcal{H}^s\Big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\Big) = 0.$$

Proof. This follows from Theorem 2.1 and [MM, Theorem 3.9(1)]. The proof is a relatively easy application of Whitney's extension theorem.

2.3. Remark. Even if we would assume that the integral in (2.1) is uniformly bounded for $x \in \mathbf{R}^2$, E is not necessarily contained in a single C^1 curve, that is, the tangent need not vary continuously. For example, let C be a compact s-regular set lying on the unit circle S^1 and let D be an s-regular Cantor set on $\{(x, y) : 0 \leq x \leq 1, y = 0\}$ with $0 \in D$. Choose a sequence I_j of complementary intervals of D with mid-points x_j and lengths l_j in such a way that $x_j \to 0$, and $l_j/x_j \to 0$ very quickly. Let

$$E = D \cup \bigcup_{j=1}^{\infty} \Big(\frac{1}{4} l_j\, C + x_j\Big).$$

Then E is s-regular. If $l_j/x_j \to 0$ sufficiently quickly, then $c^{2s}(E, \ )$ is uniformly bounded, but E is not contained in any C^1 curve.

We now give an example to show that Theorem 2.1 and Corollary 2.2 fail for $1/2 < s < 1$.

2.4. Example. *Let $1/2 < s < 1$. Then there is a compact s-regular set $E \subset \mathbf{R}^2$ such that $c^{2s}(E, x)$ is uniformly bounded for $x \in E$, but $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ.*

Proof. We shall construct E with a von Koch-type construction similar to that used in [DS, §20]. Define $r \in (0, 1)$ by

$$2r^s = 1.$$

Let $J_{0,1}$ be a closed oriented line-segment of length 1 in $\mathbf{R}^2$. Let $J_{1,1}$ and $J_{1,2}$ be the closed oriented line-segments of length r in $\mathbf{R}^2$ such that the initial point of $J_{1,1}$ is the initial point of $J_{0,1}$, the initial point of $J_{1,2}$ is the mid-point of $J_{0,1}$, and the oriented angle from $J_{0,1}$ to both $J_{1,1}$ and $J_{1,2}$ is 1. Suppose we have constructed the closed oriented line-segments $J_{k,1}, \dots, J_{k,2^k}$ of length r^k. We apply the above operation to each $J_{k,i}$ with the angle 1 replaced by $1/(k+1)$ to obtain the line-segments $J_{k+1,1} \dots,\ J_{k+1,2^{k+1}}$ of length r^{k+1}. It is clear that the unions $\bigcup_{i=1}^{2^k} J_{k,i}$ converge as $k \to \infty$ to a compact s-regular set E. For each k and j we denote by $E_{k,j}$ the subset of E generated by $J_{k,j}$ (in the obvious way). Then for all k,

$$E = \bigcup_{j=1}^{2^k} E_{k,j}.$$

Since $\sum_k k^{-1} = \infty$, one sees easily that E has tangent at none of its points. In fact, E approaches all of its points along all directions in the sense that for any $x \in E$ and any line L through x there is a sequence $x_i \in E \setminus \{x\}$ such that $x_i \to x$ and $\operatorname{dist}(x_i, L)/|x_i - x| \to 0$. This together with the s-regularity of E implies that $\mathcal{H}^s\big(E \cap f([0,1])\big) = 0$ for any regular C^1 mapping $f : [0,1] \to \mathbf{R}^2$. This can be checked by using the regularity of f to write $[0,1] = \bigcup_{k=1}^{\infty} A_k$ where each A_k is a Borel set such that for some $e_k \in S^1$

$$\big|(f(x) - f(y))/|f(x) - f(y)| - e_k\big| < 1/2 \quad \text{for } x, y \in A_k.$$

Then $\mathcal{H}^s\big(E \cap f(A_k)\big) = 0$ for all k by the above scatteredness property of E. It remains to show that $c^{2s}(E, \ )$ is uniformly bounded.

Fix $x \in E$. For $y \in E$, $y \neq x$, let $k(y)$ be the largest k such that $x, y \in E_{k-1,j}$ for some j. Here $E_{0,j} = E$. Let $y, z \in E \setminus \{x\}$ with $y \neq z$. Denote $k = k(y)$, $l = k(z)$ and assume that $k \leq l$.

Suppose first that $k = l$. Then for some j, $y, z \in E_{k,j}$ whereas $x \notin E_{k,j}$. Let $m = m(y, z)$ be the largest m such that $y, z \in E_{m-1,j}$ for some j. Then $m > k$. It follows from the construction that there is a positive number b, depending only on r, such that

$$\begin{aligned} &|x - z| \geq b^{-1} r^k, \qquad |y - z| \geq b^{-1} r^m, \\ &\operatorname{dist}(z, L_{x,y}) \leq b r^m. \end{aligned}$$

If m is not much bigger than k, we can improve the last estimate. Since for $m \leq 2k$ the angle between the lines $L_{x,y}$ and $L_{y,z}$ is at most constant times

$$\sum_{j=k+1}^{m} \frac{1}{j} \approx \log \frac{m}{k} \approx \frac{m-k}{k}$$

we can choose b so that also

$$\operatorname{dist}(z, L_{x,y}) \leq b \frac{m-k}{k} r^m.$$

Consequently by (2.7) we have both

$$\begin{aligned} &c(x, y, z) \leq 2b^3 r^{-k} \qquad \text{and} \\ &c(x, y, z) \leq 2b^3 \frac{m-k}{k} r^{-k}. \end{aligned}$$

If $k < l$ we get in the same way interchanging x and y in the above argument

$$\begin{aligned} &c(x, y, z) \leq 2b^3 r^{-k} \qquad \text{and} \\ &c(x, y, z) \leq 2b^3 \frac{l-k}{k} r^{-k}. \end{aligned}$$

Set

$$\begin{aligned} &F_k = \{y \in E : k(y) = k\} \qquad \text{and} \\ &F_m(y) = \{z \in E : k(z) = k(y), k(y, z) = m\}. \end{aligned}$$

Then

$$\begin{aligned} &\mathcal{H}^s(F_k) \leq C_1 r^{sk} \qquad \text{and} \\ &\mathcal{H}^s\big(F_m(y)\big) \leq C_1 r^{sm} \end{aligned}$$

where C_1 depends only on r. Therefore, changing $m-k$ to n,

$$\begin{aligned}
c^{2s}(E,x) &= \int_E \int_E c(x,y,z)^{2s} d\mathcal{H}^s z \, d\mathcal{H}^s y \\
&\le 2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{\infty} \int_{F_k} \int_{F_m(y)} c(x,y,z)^{2s} d\mathcal{H}^s z \, d\mathcal{H}^s y \\
&\quad + 2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \int_{F_k} \int_{F_l} c(x,y,z)^{2s} d\mathcal{H}^s z \, d\mathcal{H}^s y \\
&\le 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=k+1}^{2k} \left(\frac{m-k}{k}\right)^{2s} r^{-2sk} r^{sk} r^{sm} \\
&\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{m=2k+1}^{\infty} r^{-2sk} r^{sk} r^{sm} \\
&= 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{n=1}^{\infty} r^{ns} n^{2s} \sum_{k=n}^{\infty} k^{-2s} \\
&\quad + 4 \cdot 2^{2s} b^{6s} C_1^2 \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} r^{sn} < \infty
\end{aligned}$$

since $2s > 1$. Thus $c^{2s}(E, \ )$ is bounded.

We now show that Theorem 2.1 and Corollary 2.2 fail if we replace the regularity assumption (1.1) by $\mathcal{H}^s(E) < \infty$.

2.5. Example. *Given $0 < s < 1$ there exists a compact set $E \subset \mathbf{R}^2$ such that $0 < \mathcal{H}^s(E) < \infty$, $c^{2s}(E,x)$ is uniformly bounded for $x \in \mathbf{R}^2$ and $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ.*

Proof. Choose an integer $n_1 > 1$, let $J = \{(x,y) \in \mathbf{R}^2 : 0 \le x \le 1, y = 0\}$ and

$$J_i = \{(x,y) \in \mathbf{R}^2 : x = i/n_1, 0 \le y \le r_1\}, \quad i = 1, \dots, n_1,$$

where r_1 is determined by

$$n_1 r_1^s = 1.$$

Let μ_1 be the normalized, i.e., $\mu_1\left(\bigcup_i J_i\right) = 1$, length measure on $\bigcup_{i=1}^{n_1} J_i$. If we choose n_1 very large, r_1 will be very small and the distance $1/n_1 = r_1^s$ between any J_i and J_{i+1} will be much bigger than r_1. From this we see easily that choosing n_1 large enough, we have

$$c^{2s}(\mu_1, x) = \iint c(x,y,z)^{2s} \, d\mu_1 y \, d\mu_1 z \le C_0$$

for all $x \in \mathbf{R}^2$, where C_0 is an absolute constant.

Next we replace each vertical line segment J_i by n_2 horizontal line segments $J_{i,j}$ of length r_2 such that $n_2 r_2^s = r_1^s$ in the same way. Let μ_2 be the normalized length measure on $\bigcup_{i,j} J_{i,j}$. Choosing n_2 sufficiently large we can keep $c^{2s}(\mu_2, x)$ as close to $c^{2s}(\mu_1, x)$, uniformly, as we want. We choose it so that $c^{2s}(\mu_2, x) \le C_0 + \frac{1}{2}$ for

$x \in \mathbf{R}^2$. The point here is that for some small $\delta > 0$ looking from any $x \in \mathbf{R}^2$ the part of μ_2 outside $B(x, \delta)$ looks very much like μ_1 and the contribution of μ_2 in $B(x, \delta)$ to $c^{2s}(\mu_2, x)$ is very small.

Continuing this we get unions E_k of line segments $J_{i_1,\dots,i_k}$ of length r_k. Every second time these line segments are horizontal and every second time vertical. We also have uniformly distributed probability measures μ_k on E_k satisfying $c^{2s}(\mu_k, x) \leq C_0 + \sum_{i=2}^{k} 2^{-i}$ for all k and $x \in \mathbf{R}^2$. The sets E_k converge to a compact set E with $0 < \mathcal{H}^s(E) < \infty$ and the measures μ_k converge weakly to a probability measure μ supported on E such that

$$c^{2s}(\mu, x) \leq 2 \quad \text{for } x \in \mathbf{R}^2.$$

It is easy to check that μ is comparable with $\mathcal{H}^s$ restricted to E. Thus $c^{2s}(E, x)$ is uniformly bounded.

Finally that $\mathcal{H}^s(E \cap \Gamma) = 0$ for every C^1 curve Γ can be checked with the help of decompositions such as in the proof of Example 2.4.

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