Sobolev inequalities on sets with irregular boundaries

Tero Kilpeläinen* and Jan Malý†

March 11, 1999

1 Introduction

It is well known that the Sobolev space $W^{1,p}(\Omega)$ is continuously embedded into $L^q(\Omega)$ if Ω is a nice bounded domain in $\mathbf{R}^n$ and

$$1 \leq p < \infty, \qquad q(n-p) \leq np. \tag{1.1}$$

This fact, originally due to Sobolev, to Gagliardo and to Nirenberg, can nowadays be found in textbooks (cf. [M3], [Z]) and it is stated as the Sobolev-Poincaré inequality

$$\Big(\int_\Omega |u-u_\Omega|^q\,dx\Big)^{1/q} \leq C\Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}. \tag{1.2}$$

The weighted case of Sobolev's imbedding has been developed by by Nečas [N], Besov, Ilin, and Nikolskii [BIN1, BIN2], Kufner [K], Maz'ya [M3], and others.

It is not very difficult to give examples of domains having cusps for which the Sobolev-Poincaré inequality (1.2) fails to hold or the range for its validity differs from (1.1). The question of this embedding in nonsmooth domains Ω is addressed by many authors. To mention but a few, we would like to refer to the books [M3] and [MP], and point out that Besov [B1, B2]

*The research is supported by the Academy of Finland (Project #8597).
†Supported by the grants GAČR 201/97/1161 and GAUK 186/96.

obtained embeddings in domains satisfying "flexible cone conditions", Smith and Stegenga [SS] proved Poincaré inequality with $q = p$ for s-John domains (that allow for twisted cusps of the type t^s with certain $s \geq 1$). Maz'ya [M1] (see also Labutin [L]) established the optimal embedding for s-cusps.

Hajłasz and Koskela [HK] proved the optimal Sobolev-Poincaré inequality in s-John domains with $p = 1$ and the next to the optimal one for $p > 1$. Their result also involves weights. We refer to [HK] also for further historical notes and references.

In this note we complete the picture for s-John domains and give a proof for the optimal Sobolev-Poincaré inequality in s-John domains, thus improving the results in [HK] (see Theorem 2.3). We study also the weigted case where the weight is a power of the distance to the boundary. The result is obtained as a consequence of a slightly more general criterion, which may be used to illustrate why the optimal exponent for s-John domains is worse than the optimal exponent for domains with a single s-cusp.

We use Hedberg's trick on the maximal operator [He], a truncation argument due to Maz'ya [M2] and some ideas from Hajłasz and Koskela [HK]. The main new ingredient of our proof is a careful grouping of chains around a curve which we call a worm.

Lebesgue measure on $\mathbf{R}^n$ is denoted by $\boldsymbol{\lambda}$, and we write

$$|E| = \boldsymbol{\lambda}(E)$$

for a measurable set $E \subset \mathbf{R}^n$. If u is an integrable function defined at least on E, we let u_E stand for the average

$$u_E = \fint_E u\,dx = \frac{1}{|E|}\int_E u\,dx\,.$$

The open n-dimensional ball with center at x and radius r is written as $B(x,r) = B_n(x,r)$. We use $\sharp F$ for the cardinality of a set F.

2 Main results

This section contains the results with proofs. We start with a general theorem and deduce the s-John domain result from it.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. Given an exponent $a \geq 0$, let $\boldsymbol{\mu}$ the measure on $\mathbf{R}^n$ with

$$\frac{d\boldsymbol{\mu}}{d\boldsymbol{\lambda}} = \begin{cases} \rho^a & \text{in } \Omega, \\ 0 & \text{outside } \Omega\,; \end{cases}$$

here and in what follows $\rho(x) = \operatorname{dist}(x, \mathbf{R}^n \setminus \Omega)$.

We shall define a *worm*. It is a pair (γ, Δ), where $\gamma : [0, \ell] \to \Omega$ is a curve joining $y = \gamma(0)$ to $x_0 = \gamma(\ell)$, parametrized by its arc-length, and $\Delta = \{\xi_k\}$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_m = \ell$, is a finite partition of $[0, \ell]$. With each worm we associate its *parameters:* the number m of the partition intervals $[\xi_{k-1}, \xi_k]$, and for $k = 1, \ldots, m$ the quantities

$$\begin{aligned} \ell_k &= \xi_k - \xi_{k-1}, \\ R_k &= \sup\{|\gamma(t) - y| :\ t \in [\xi_{k-1}, \xi_k]\}, \\ r_k &= \inf\{\rho(\gamma(t)) :\ t \in [\xi_{k-1}, \xi_k]\}. \end{aligned}$$

Theorem 2.1 *Let $1 \leq p \leq q < \infty$ such that $q(n-p) \leq np$ and let*

$$1 - n \leq b \leq p\Big(\frac{a+n}{q} + 1 - \frac{n}{p}\Big). \tag{2.1}$$

Suppose that there is a constant $A > 0$ and a point $x_0 \in \Omega$ such that for each $y \in \Omega \setminus B(x_0, \rho(x_0)/2)$ there is a worm (γ, Δ) joining y to x_0, with parameters m, $\{\ell_k\}$, $\{R_k\}$, $\{r_k\}$ and constants $\tau_1, \ldots, \tau_m \in (0,1]$ (both parameters and τ_k's may depend on y), such that

$$\rho(y) \leq 3R_k, \quad k = 1, \ldots, m, \tag{2.2}$$

$$(1 + A^{-1})\tau_{k-1} \leq \tau_k \leq A\tau_{k-1}, \quad k = 2, \ldots, m \tag{2.3}$$

and

$$A^{-1}\big(\boldsymbol{\mu}(B(y, 3R_k))\big)^{1/q} \leq \tau_k \leq A r_k^{(n+b-1)/p} \ell_k^{(1-p)/p}. \tag{2.4}$$

Then there is a constant $C = C(n, p, a, b, A, \Omega) > 0$ such that

$$\Big(\int_\Omega |u - \bar{u}_a|^q \rho^a\, dx\Big)^{1/q} \leq C\Big(\int_\Omega |\nabla u|^p \rho^b dx\Big)^{1/p}$$

for each $u \in C^1(\Omega)$; here

$$\bar{u}_a = \fint_\Omega u\, d\boldsymbol{\mu} = \frac{1}{\boldsymbol{\mu}(\Omega)} \int_\Omega u\, d\boldsymbol{\mu}\,.$$

We start the proof with the following lemma.

Lemma 2.2 *Suppose that Ω is a bounded open set. Let z, $z' \in \Omega$ and let $\gamma : [\xi, \xi'] \to \mathbf{R}^n$ be a path of the length ℓ that joins z and z'. Suppose that $b \geq 1-n$ and that $\rho \geq r$ on γ. Let $u \in C^1(\Omega)$. Then*

$$|u_{B(z,\rho(z)/2)} - u_{B(z',\rho(z')/2)}| \leq C r^{(1-b-n)/p}\, \ell^{(p-1)/p} \int_{D_\gamma} |\nabla u|^p\, \rho^b\, dx\,, \tag{2.5}$$

where

$$D_\gamma = \bigcup_{t\in[\xi,\xi']} B(\gamma(t), \rho(\gamma(t))/2).$$

Proof. Write $B = B(z, \rho(z)/2)$ and $B' = B(z', \rho(z')/2)$. We construct a *chain* $\{B_i\}$, $B_i \equiv B(z_i, \rho(z_i)/2)$ of balls and denote $\hat{B}_i = B(z_i, \rho(z_i)/4)$. For the construction, it is enough to determine the points t_i such that $z_i = \gamma(t_i)$. If $t_1, \ldots, t_{j-1}$ are selected, we find next as

$$t_j = \sup\bigl\{t \in [t_{i-1}, \xi'] : B(\gamma(t), \rho(\gamma(t)/4)) \cap \hat{B}_{j-1} \neq \emptyset\bigr\}.$$

If $t_j = \xi'$, we set $j_{\max} = j$, $t_j = \xi'$ and terminate the construction.

We observe that the balls $B(z_i, \rho(z_i)/4)$, $i < j_{\max}$, are disjoint, and since their radii are bounded away from zero and Ω is bounded, the sequence really terminates after a finite number of steps. Fix $x \in \Omega$ and denote $I(x) = \{i < j_{\max} : x \in B_i\}$. Let $i \in I(x)$. Then

$$\rho(z_i) \leq \rho(x) + |x - z_i| \leq \rho(x) + \frac{1}{2}\rho(z_i),$$
$$\rho(x) \leq \rho(z_i) + |x - z_i| \leq \rho(z_i) + \frac{1}{2}\rho(z_i),$$

and thus

$$\rho(z_i) \leq 2\rho(x), \qquad \rho(x) \leq 2\rho(z_i). \tag{2.6}$$

For any $y \in \hat{B}_i$ we have

$$|y - x| \leq \rho(z_i) \leq 2\rho(x),$$

which means that

$$\bigcup_{i\in I(x)} \hat{B}_i \subset B(x, 2\rho(x))$$

Since $\hat{B}_i$, , $i \in I(x)$, are disjoint, we have

$$|B(x, \rho(x)/8)| \, \sharp I(x) \leq \sum_{i \in I(x)} |\hat{B}_i| \leq |B(x, 2\rho(x))|,$$

which implies

$$\sharp I(x) \leq 16^n.$$

Thus we have proven that

$$\sum \chi_{B_i} \leq 16^n + 1. \tag{2.7}$$

Next, consider $i \in \{1, \dots, j_{\max}\}$ and notice that there is a point $x \in \overline{\hat{B}_{i-1}} \cap \overline{\hat{B}_i}$. Then, as above, we infer that (2.6) holds and

$$\begin{gathered} B(x, \rho(x)/8)) \subset B(x, \rho(z_{i-1})/4)) \cap B(x, \rho(z_i)/4)) \subset B_{i-1} \cap B_i, \\ B_{i-1} \cup B_i \subset B(x, \rho(z_{i-1})) \cup B(x, \rho(z_i)) \subset B(x, 2\rho(x)), \end{gathered}$$

so that

$$|B_{i-1} \cup B_i| \leq 16^n |B_{i-1} \cap B_i|. \tag{2.8}$$

Also it is clear that

$$\sum_{i=1}^{j_{\max}} \rho(z_i) \leq C\ell. \tag{2.9}$$

Using (2.8) and the Poincaré inequality we have

$$\begin{aligned} &|u_{B_i} - u_{B_{i-1}}| \leq |u_{B_i} - u_{B_i \cap B_{i-1}}| + |u_{B_i \cap B_{i-1}} - u_{B_{i-1}}| \\ &\leq \fint_{B_i \cap B_{i-1}} |u - u_{B_i}| \, dx + \fint_{B_i \cap B_{i-1}} |u - u_{B_{i-1}}| \, dx \\ &\leq \frac{|B_i|}{|B_i \cap B_{i-1}|} \fint_{B_i} |u - u_{B_i}| \, dx + \frac{|B_{i-1}|}{|B_i \cap B_{i-1}|} \fint_{B_{i-1}} |u - u_{B_{i-1}}| \, dx \\ &\leq C \, \rho(z_i) \Big(\fint_{B_i} |\nabla u|^p \, dx \Big)^{1/p} + C \, \rho(z_{i-1}) \Big(\fint_{B_{i-1}} |\nabla u|^p \, dx \Big)^{1/p}. \end{aligned}$$

Hence we can estimate by using (2.7) and (2.9) that

$$
\begin{aligned}
|u_{B'} - u_B| &\le \sum_{i=2}^{j_{\max}} |u_{B_i} - u_{B_{i-1}}| \\
&\le C \sum_{i=1}^{j_{\max}} \rho(z_i)^{1-n/p} \Big(\int_{B_i} |\nabla u|^p \, dx \Big)^{1/p} \\
&\le C \sum_{i=1}^{j_{\max}} \rho(z_i)^{1-\frac{1}{p}+\frac{1-n-b}{p}} \Big(\int_{B_i} \rho(z_i)^b |\nabla u|^p \, dx \Big)^{1/p} \\
&\le C \sum_{i=1}^{j_{\max}} r^{\frac{1-n-b}{p}} \rho(z_i)^{1-\frac{1}{p}} \Big(\int_{B_i} \rho^b |\nabla u|^p \, dx \Big)^{1/p} \\
&\le C r^{\frac{1-n-b}{p}} \Big(\sum_{i=1}^{j_{\max}} \rho(z_i) \Big)^{1-\frac{1}{p}} \Big(\sum_{i=1}^{j_{\max}} \int_{B_i} \rho^b |\nabla u|^p \, dx \Big)^{1/p} \\
&\le C r^{(1-b-n)/p} \ell^{(p-1)/p} \Big(\int_{D_\gamma} \rho^b |\nabla u|^p \, dx \Big)^{1/p},
\end{aligned}
\tag{2.10}
$$

since $b + n \ge 1$. The lemma is proven.

Proof of Theorem 2.1. Denote $B_0 = B(x_0, \rho(x_0)/2)$. Let $u \in C^1(\Omega)$. We may assume that

$$
|\{u \ge 0\} \cap B_0| \ge \frac{1}{2}|B_0| \quad \text{and} \quad |\{u \le 0\} \cap B_0| \ge \frac{1}{2}|B_0|. \tag{2.11}
$$

We will also assume as we may that

$$
\int_\Omega |\nabla u|^p \rho^b \, dx = 1\,. \tag{2.12}
$$

We shall first establish a weak type estimate:

$$
\mu(A_\lambda) \le C\lambda^{-q}, \tag{2.13}
$$

where

$$
A_\lambda = \{x \in \Omega : |u(x)| > \lambda\}
$$

and $\lambda > 0$. First observe that since the median of u is zero in B_0 by (2.11), we have that

$$
\int_{B_0} |u|^p \, dx \le c \int_{B_0} |\nabla u|^p \, dx\,, \tag{2.14}
$$

see [Z, Theorem 4.4.4]. Hence

$$|u_{B_0}| \le \left(\fint_{B_0} |u|^p \, dx \right)^{1/p} \le c \left(\fint_{B_0} |\nabla u|^p \, dx \right)^{1/p} \le c_0 \,, \tag{2.15}$$

where c_0 is independent of u. Since $\mu(\Omega) < \infty$ it suffices to establish (2.13) for $\lambda > 3c_0$. To do so, we fix $\lambda > 3c_0$ and divide A_λ into three parts: write $B_y = B(y, \rho(y)/2)$ and let

$$E_\lambda = \{ y \in A_\lambda \setminus B_0 : |u_{B_y}| > \frac{\lambda}{2} \}$$

and

$$F_\lambda = A_\lambda \setminus (B_0 \cup E_\lambda) \,.$$

The third part is $A_\lambda \cap B_0$. We treat E_λ first. Fix $y \in E_\lambda$ and let $(\gamma, \{\xi_k\})$ be a worm in Ω that connects y to x_0, with parameters $m, \{\ell_k\}, \{R_k\}, \{r_k\}$, and obeys the bounds of the theorem. We apply Lemma 2.2 to paths

$$\gamma_k = \gamma|_{[\xi_{k-1}, \xi_k]}$$

and points $z = z_k = \gamma(\xi_{k-1})$, $z' = z'_k = \gamma(\xi_k)$. Let $x = \gamma(t)$ with $t \in [\xi_{k-1}, \xi_k]$. Then by (2.2)

$$\rho(x) \le \rho(y) + |y - x| \le 4R_k$$

and thus

$$B(x, \rho(x)/2) \subset B(y, R_k + 2R_k),$$
$$D_{\gamma_k} \subset B(y, 3R_k).$$

Since $\lambda > 3c_0$, we have that

$$\begin{aligned} \lambda \le 6\, |u_{B_y} - u_{B_0}| &\le 6 \sum_{k=1}^{m} |u_{B_{z'_k}} - u_{B_{z_k}}| \\ &\le C \sum_k r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \left(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p \, d\boldsymbol{\mu} \right)^{1/p}. \end{aligned}$$

We split the last sum into two parts by $K = K(y)$ that is to be determined. First we notice that by (2.3)

$$\sum_{k>K} \tau_k^{-1} \le C \tau_{K+1}^{-1}, \qquad \sum_{k \le K} \tau_k^{q/p-1} \le C \tau_K^{q/p-1}. \tag{2.16}$$

If $K < m$, due to our normalization of u, (2.4) and (2.16) we have that

$$\begin{aligned} &\sum_{k>K} r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \Big(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p \, d\boldsymbol{\mu} \Big)^{1/p} \\ &\le \Big(\int_\Omega \rho^b |\nabla u|^p \, dx \Big)^{1/p} \sum_{k>K} r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \\ &= \sum_{k>K} r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \le C \sum_{k>K} \tau_k^{-1} \le C \tau_{K+1}^{-1}. \end{aligned} \tag{2.17}$$

Before treating the second part of the sum, we set

$$f = |\nabla u|^p \rho^{b-a}$$

and

$$g(x) = \Big(\sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\boldsymbol{\mu} \Big)^{1/p}.$$

Since the maximal operator with respect to $\boldsymbol{\mu}$ is of weak type $(1,1)$ (see e.g. [M, Theorem 2.19] or [S, I.8.17, p. 44]) and $\|f\|_{L^1(\boldsymbol{\mu})} = 1$, we have

$$\boldsymbol{\mu}(\{g^p > t\}) \le C/t, \qquad 0 < t < \infty. \tag{2.18}$$

We estimate

$$\begin{aligned} &\sum_{k\le K} r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \Big(\int_{B(y,3R_k)} \rho^{b-a} |\nabla u|^p \, d\boldsymbol{\mu} \Big)^{1/p} \\ &\le \sum_{k\le K} r_k^{(1-b-n)/p} \ell_k^{(p-1)/p} \big(\boldsymbol{\mu}(B(y,3R_k))\big)^{1/p} g(y) \\ &\le C \sum_{k\le K} \tau_k^{-1} \tau_k^{q/p} g(y) \le C \tau_K^{-1+q/p} g(y). \end{aligned} \tag{2.19}$$

Now we specify the choice of K, distinguishing three cases. If

$$\tau_1^{-q/p} \le g(y),$$

we choose $K = 0$. Then the sum over all $k = 1, \ldots, m$ reduces to (2.17) and we have

$$\lambda \le C \tau_1^{-1} \le C \, g(y)^{p/q}.$$

If

$$\tau_m^{-q/p} \ge g(y),$$

we choose $K = m$. Then the sum over $k = 1, \ldots, m$ is treated in (2.19), and we have

$$\lambda \le C\tau_m^{-1+q/p}\, g(y) \le C g(y)^{(p/q)-1}\, g(y) = C\, g(y)^{p/q}.$$

The remaining case is that

$$\tau_m^{-q/p} < g(y) < \tau_1^{-q/p}$$

Then we choose the integer $K < m$ so that

$$\tau_{K+1}^{-q/p} \le g(y) < \tau_K^{-q/p}.$$

Using (2.17) and (2.19) we obtain

$$\lambda \le C\tau_{K+1}^{-1} + C\tau_K^{-1+q/p} g(y) \le C g(y)^{p/q}.$$

Hence we always have that

$$\lambda \le C g(y)^{p/q}$$

for every $y \in E_\lambda$. Therefore by (2.18)

$$\boldsymbol{\mu}(E_\lambda) \le \boldsymbol{\mu}(\{g^p > (\lambda/C)^q\}) \le C\lambda^{-q}. \tag{2.20}$$

Next, we estimate the measure of F_λ. Using the Besicovitch covering theorem (cf. [M, 2.7]) we can cover F_λ with balls $B_{x_i} = B(x_i, \rho(x_i)/2)$ so that $x_i \in F_\lambda$ and

$$\sum_i \chi_{B_{x_i}} \le N\,.$$

Then

$$|u - u_{B_{x_i}}| \ge \frac{\lambda}{2} \quad \text{on } F_\lambda$$

whence we have by using the Sobolev-Poincaré inequality that

$$
\begin{aligned}
\boldsymbol{\mu}(F_\lambda) &\le \sum_i \boldsymbol{\mu}(B_{x_i} \cap F_\lambda) \\
&\le \sum_i \int_{B_{x_i}\cap F_\lambda} \rho^a dx \\
&\le C \sum_i \rho(x_i)^a \int_{B_{x_i}\cap F_\lambda} dx \\
&\le C\lambda^{-q} \sum_i \rho(x_i)^a \int_{B_{x_i}\cap F_\lambda} |u - u_{B_{x_i}}|^q dx \\
&\le C\lambda^{-q} \sum_i \rho(x_i)^{a+q+n(1-q/p)} \big(\int_{B_{x_i}} |\nabla u|^p dx \big)^{q/p} \\
&\le C\lambda^{-q} \sum_i \Big(\int_{B_{x_i}} |\nabla u|^p \rho^{p\big((a+n)/q+1-n/p\big)} dx \Big)^{q/p} \\
&\le C\lambda^{-q} \Big(\int_\Omega |\nabla u|^p \rho^b dx \Big)^{q/p} \\
&\le C\lambda^{-q},
\end{aligned}
\tag{2.21}
$$

since $p\big((a+n)/q + 1 - n/p\big) \ge b$ by (2.1).

Finally, combining (2.14) and the usual Sobolev inequality in the ball B_0, we obtain the weak type estimate

$$\boldsymbol{\mu}(A_\lambda \cap B_0) \le C\lambda^{-q}.$$

Hence by the estimates (2.20) and (2.21)

$$\boldsymbol{\mu}(A_\lambda) \le \boldsymbol{\mu}(E_\lambda) + \boldsymbol{\mu}(F_\lambda) + \boldsymbol{\mu}(A_\lambda \cap B_0) \le C\lambda^{-q}.$$

In conclusion, (2.13) holds for all $\lambda > 0$, or without normalization (2.12),

$$\sup_{\lambda>0} \lambda\, \boldsymbol{\mu}(\{|u| > \lambda\})^{1/q} \le C\Big(\int_\Omega |\nabla u|^p \rho^b dx \Big)^{1/p}. \tag{2.22}$$

A truncation argument shows that the weak type estimate (2.22) implies the desired embedding. Indeed, for each $t > 0$ the truncated functions

$$u_t(x) = \begin{cases} t/2 & \text{if } u(x) > t, \\ u(x) - t/2 & \text{if } t/2 < u(x) < t, \\ 0 & \text{if } u(x) < t/2, \end{cases}$$

satisfy (2.11). Thus we may use (2.22) to conclude

$$\begin{aligned}\Big(\int_{\{t<u\le 2t\}}|u|^q\,d\boldsymbol{\mu}\Big)^{1/q} &\le Ct\,\boldsymbol{\mu}(\{|u|>t\})^{1/q}\\ &\le Ct\,\boldsymbol{\mu}(\{u_t\ge t/2\})^{1/q}\\ &\le C\Big(\int_\Omega|\nabla u_t|^p\rho^b dx\Big)^{1/p}\\ &= C\Big(\int_{\{t/2<|u|\le t\}}|\nabla u|^p\rho^b dx\Big)^{1/p}.\end{aligned}$$

Hence

$$\begin{aligned}\int_\Omega|u|^q\rho^a\,dx &\le \sum_{j=-\infty}^{\infty}\int_{\{2^j<|u|\le 2^{j+1}\}}|u|^q\rho^a\,dx\\ &\le C\sum_{j=-\infty}^{\infty}\Big(\int_{\{2^{j-1}<|u|\le 2^j\}}|\nabla u|^p\rho^b dx\Big)^{q/p}\\ &\le C\Big(\int_\Omega|\nabla u|^p\rho^b dx\Big)^{q/p},\end{aligned}$$

and the theorem is proved, since

$$\int_\Omega|u-\bar u_a|^q\rho^a\,dx\le C\int_\Omega|u|^q\rho^a\,dx.$$

Following Smith and Stegenga [SS] we call a bounded domain Ω an *s-John domain*, $s\ge 1$, if there is a point $x_0\in\Omega$ and a constant $c_0\ge 1$ such that each point $x\in\Omega$ can be joined to x_0 in Ω by a rectifiable curve (called an *s-John core*) $\gamma\,[0,\ell]\to\Omega$, such that γ is parametrized by the arc length, $\gamma(0)=x$, $\gamma(\ell)=x_0$, and

$$\operatorname{dist}\big(\gamma(t),\partial\Omega\big)\ge c_0^{-1}t^s$$

for all $t\in[0,\ell]$. The next theorem improves the main result of [HK].

Theorem 2.3 *Suppose that Ω is an s-John domain and $b\ge 1-n$. Then there is a constant $C=C(n,p,q,\Omega)>0$ such that*

$$\big(\int_\Omega|u-\bar u_a|^q\rho^a\,dx\big)^{1/q}\le C\big(\int_\Omega|\nabla u|^p\rho^b dx\big)^{1/p}$$

for each $u \in C^1(\Omega)$; here the Sobolev exponent is

$$q = \frac{p(n+a)}{s(n+b-1)-p+1}.$$

Proof. We will verify the assumptions of Theorem 2.1. First we notice that the inequalities $s \geq 1$ and $b \geq 1-n$ imply

$$p\Big(\frac{a+n}{q}+1-\frac{n}{p}\Big) = s(n+b-1)+1-n \geq b,$$

so that (2.1) is true. For fixed $y \in \Omega \setminus B(x_0, \rho(x_0)/2)$, the s-John core γ on $[0,\ell]$ gives us the desired worm: Let

$$d = \sup\big\{|\gamma(t)-y| : \ t \in [0,\ell]\big\}.$$

Find the integer m with

$$3d < 2^m \rho(y) \leq 6d.$$

Since

$$\rho(y) \leq \rho(x_0) + |y-x_0| \leq 3|y-x_0| \leq 3d,$$

we have $m \geq 1$. Set

$$\xi_k = \sup\{t \in [0,\ell] : |\gamma(s)-y| \leq 2^{k-m}d \text{ for all } s \in [0,t]\}.$$

Then $(\gamma, \{\xi_k\})$ is a worm with parameters m, $\{\ell_k\}$, $\{R_k\}$, $\{r_k\}$, and

$$\begin{aligned} &\ell_k \leq \xi_k, \\ &\xi_k \geq R_k \ = 2^{k-m}d, \\ &r_k \geq c_0 \xi_k^s. \end{aligned}$$

The inequality

$$\rho(y) \leq 6 \cdot 2^{-m} d \leq 3R_k$$

verifies (2.2). Since

$$(n+a)/q = (s(n+b-1)+1-p)/p$$

we have by choosing $\tau_k = 2^{(k-m)(s(n+b-1)+1-p)/p}$ that

$$\boldsymbol{\mu}(B(y,R_k))^{1/q} \leq R_k^{(n+a)/q} \leq C\tau_k$$

and

$$\begin{aligned} r_k^{-(n+b-1)/p}\ell_k^{(p-1)/p} &\le \bigl(c_0\xi_k\bigr)^{-s(n+b-1)/p}\xi_k^{(p-1)/p} \\ &= C\xi_k^{-(n+a)/q} \\ &\le C\tau_k^{-1} \end{aligned}$$

Hence the claim follows from Theorem 2.1.

Remark. The exponent q of Theorem 2.3 is the best possible in the class of s-John domains, see [HK].

Example 2.4 An example of an s-John domain is an s-*cusp* domain. Surprisingly, the optimal embedding exponent for the s-cusp obtained in [M1], [L], [MP] is better than that for general s-John domains. The reason is that complicated s-John domains may contain "rooms and corridors" so that the upper estimate for $\boldsymbol{\mu}(B(y,R)\cap\Omega)$ must be more carefully examined. We show that the optimal embedding for s-cusp domains can be deduced from Theorem 2.1. Let us write $x\in\mathbf{R}^n$ as $x=(\hat{x},x^*)$, where $\hat{x}\in\mathbf{R}^{n-1}$ and x^* is the last coordinate of x. We will consider the s-cusp domain

$$\Omega=\{x\in\mathbf{R}^n : |\hat{x}|\le(x^*)^s,\ 0<x^*<2\}$$

and show that Theorem 2.1 yields embedding of $W^{1,p}(\Omega,\rho^b)$ to $L^q(\Omega,\rho^a)$, where the Sobolev exponent is

$$q=\frac{p\bigl(s(n+a-1)+1\bigr)}{s(n+b-1)-p+1}.$$

We choose $x_0=\boldsymbol{e}_n=(0,1)$. If $y\in\Omega\setminus B(x_0,\rho(x_0)/2)$, we set

$$\ell=\ell(y)=|\hat{y}|+|y^*-1|$$

and define the worm curve $\gamma:[0,\ell]\to\Omega$ as

$$\gamma(t)=\begin{cases}\Bigl((1-\frac{t}{|\hat{y}|})\hat{y},\ y^*\Bigr) & 0\le t\le|\hat{y}|,\\ \Bigl(1+\frac{\ell-t}{\ell-|\hat{y}|}\,(y^*-1)\Bigr)\,\boldsymbol{e}_n, & |\hat{y}|\le t\le\ell.\end{cases}$$

In other words, worm curve starts at y, goes first on line segment connecting y with $y^*\boldsymbol{e}_n$ and then turns to the line segment connecting $y^*\boldsymbol{e}_n$ with $\boldsymbol{e}_n$. We find a partition $\{\xi_0,\dots,\xi_m\}$ of $[0,\ell]$ in such a way that $\xi_0=0$,

$$\xi_k=2^{k-m}\ell,\quad k=1,\dots,m, \tag{2.23}$$

$$\rho(y)<\xi_1<2\rho(y), \tag{2.24}$$

where (2.24) is what determines m and guarantees (2.2). From now we treat only the interesting case that $y^* < 1$. Then

$$\ell_k = \xi_k/2, \quad k = 2, \dots, m, \quad \ell_1 = \xi_1, \tag{2.25}$$

$$\ell_k^s \le r_k, \tag{2.26}$$

$$\xi_k \le R_k \le 2\xi_k, \tag{2.27}$$

$$B(y, R_k) \cap \Omega \subset B_{n-1}(\hat{y}, Cr_k) \times (y^* - R_k,\, y^* + R_k), \tag{2.28}$$

$$\rho \le Cr_k \quad \text{ on } B(y, R_k). \tag{2.29}$$

Set $\tau_k = (\xi_k^{n+a-1}\ell_k)^{1/q}$. It is easy to observe that τ_k satisfy (2.3). From (2.26) we obtain

$$\begin{aligned} r_k^{(n+b-1)/p} \ell_k^{(1-p)/p} &\ge r_k^{(n+a-1)/q} \ell_k^{1/q} \\ &\ge C^{-1}\tau_k. \end{aligned}$$

The additional information provided by (2.28) and (2.29) has no counterpart in the case of a general s-John domain. We use it to estimate $\boldsymbol{\mu}(B(y, 3R_k))$:

$$\begin{aligned} C\boldsymbol{\mu}(B(y, R_k))^{1/q} &\le C\big(R_k r_k^{n-1+a}\big)^{1/q} \\ &\le C\big(\xi_k\, r_k^{n-1+a}\big)^{1/q} \le \\ &\le C\tau_k. \end{aligned}$$

Hence (2.4) is verified and Theorem 2.1 yields the result.

References

[B1] Besov, O. V., *Integral representations of functions and embedding theorems for a domain with a flexible horn condition*, Trudy Mat. Inst. Steklov **170** (1984), 12–30.

[B2] Besov, O. V., *Embeddings of Sobolev spaces in domains with a splitting flexible cone condition*, Trudy Mat. Inst. Steklov **173** (1986), 14–31.

[BIN1] Besov, O. V, Ilin, V. P. and Nikolskii, S. M., "*Integral representations of functions and imbedding theorems. Vol I.*", Halsted Press, New York, 1978.

[BIN2] Besov, O. V, Ilin, V. P. and Nikolskii, S. M., "*Integral representations of functions and imbedding theorems. Vol II.*", Halsted Press, New York, 1979.

[HK] Hajłasz, P. and Koskela, P., *Isoperimetric inequalities and imbedding theorems in irregular domains*, J. London Math. Soc. (1998?), to appear.

[He] Hedberg, L. I., *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.

[K] Kufner, A., "*Weighted Sobolev Spaces*", Teubner, Leipzig, 1980 (first eddition), J. Wiley and Sons, Chichester, 1985 (second eddition).

[L] Labutin, D.A., *Integral representations of functions and embeddings of Sobolev spaces on cuspidal domains*, Mathematical Notes **61** (1997), 164–179.

[M] Mattila, P., "*Geometry of Sets and Measures in Euclidean Spaces, Fractals and rectifiability*", Cambridge University Press, Cambridge, 1995.

[M1] Maz'ya, V. G., *Classes of domains and imbedding theorems for function spaces*, Soviet Math. Dokl. **1** (1960), 882–885.

[M2] Maz'ya, V. G., *A theorem on the multidimensional Schrödinger operator* (Russian), Izv. Akad. Nauk. **28** (1964), 1145-1172.

[M3] Maz'ya, V. G., "*Sobolev spaces*", Springer-Verlag, Berlin, 1985.

[MP] Maz'ya, V. G. and Poborchi, S. V., "*Differentiable functions on bad domains*", World Scientific, Singapore, 1997.

[N] Nečas, J., *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle*, Ann. Scuola Norm. Sup. Pisa **16** (1962), 305–326.

[SS] Smith, W. and Stegenga, D. A., *Hölder domains and Poincaré domains*, Trans. Amer. Math. Soc **319** (1990), 67–100.

[S] Stein, E. M., "*Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*", Princeton University Press, Princeton, NJ, 1993.

[Z] Ziemer, W. P., "*Weakly Differentiable Functions*", Graduate Text in Mathematics 120, Springer-Verlag, 1989.

Department of Mathematics
University of Jyväskylä
P.O. Box 35 (Mattilanniemi D)
40351 Jyväskylä, Finland
e-mail: terok@math.jyu.fi

Department of Mathematical Analysis
Charles University
Sokolovská 83
18675 Praha, Czech Republic
maly@karlin.mff.cuni.cz