On functions with derivatives in a Lorentz space

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January 6, 1999

Abstract

We establish a sharp integrability condition on the partial derivatives of a Sobolev mapping to guarantee that sets of measure zero get mapped to sets of measure zero. This condition is sharp also for continuity and differentiability almost everywhere.

1 Introduction

Mappings $f : \Omega \to \mathbf{R}^m$, where Ω is a domain in $\mathbf{R}^n$, arise naturally in many different situations. It is often desirable for f to have properties similar to those of an absolutely continuous function of a single variable or of a Lipschitz mapping. The properties we have in mind are: continuity, differentiability a.e., and the Lusin N-condition that requires the n-dimensional Hausdorff measure of $f(E)$ to be zero whenever E is of n-measure zero. It is well known that the N-property with differentiability a.e. is sufficient for validity of various change-of-variable formulas, including the area formula. In mathematical models for nonlinear elasticity such properties are of interest, for example, regarding cavitation and creation of matter, see [9].

In this note we address the following question: What are the minimal analytic assumptions on f to guarantee the above mentioned properties? As previously known, it suffices to assume that f belongs to the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^m)$ for some $p > n$, cf. [7]. Here $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbf{R}^m)$ consists of mappings of Ω into $\mathbf{R}^m$ whose coordinate functions belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$; that is, they together with their first order weak partial derivatives are locally p-integrable. We will show that this condition can be sharpened to a very precise integrability condition.

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Mathematics Subject Classification (1991): 26B10, 26B35, 46E35, 73C50.

As a tool we employ the concept of n-absolute continuity recently introduced by Malý [5]. We say that $f : \Omega \to \mathbf{R}^m$ is n-absolutely continuous if for each $\varepsilon > 0$ there is $\delta > 0$ such that for any family $\{B_i\}$ of pairwise disjoint balls in Ω we have

$$\sum_i \mathcal{L}^n(B_i) < \delta \implies \sum_i \bigl(\operatorname{osc}_{B_i} f\bigr)^n < \varepsilon.$$

It is an easy observation that a mapping $f : \Omega \to \mathbf{R}^m$ is n-absolutely continuous if and only if its coordinate functions are n-absolutely continuous. We define the class $AC^n(\Omega)$ as the family of all n-absolutely continuous functions $u : \Omega \to \mathbf{R}$ for which the seminorm

$$\|u\|_{AC^n} := \Bigl(\sup\Bigl\{ \sum_i \bigl(\operatorname{osc}_{B_i} u\bigr)^n : \{B_i\} \text{ is a disjoint family of balls in } \Omega \Bigr\}\Bigr)^{1/n}$$

is finite. It was shown in [5] that n-absolutely continuous mappings enjoy the above listed properties and that $\nabla f \in L^p(\Omega)$ for some $p > n$ guarantees that f has an n-absolutely continuous representative.

Our first result gives a very sharp sufficient condition for n-absolute continuity for a function $u \in W^{1,1}_{\text{loc}}(\Omega)$.

Theorem A. *Suppose that $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a function whose weak partial derivatives belong to $\mathrm{L}^{n,1}(\Omega)$. Then there is a representative of u that belongs to $AC^n(\Omega)$. The embedding of $\{u : \nabla u \in \mathrm{L}^{n,1}(\Omega)\}$ into $AC^n(\Omega)$ is continuous.*

Here $\mathrm{L}^{n,1}(\Omega) \subset L^n(\Omega)$ is the Lorentz space (see Section 2). We may relate Lorentz spaces to spaces determined by a family of Orlicz integrability conditions. Here we state this for simplicity for $\mathrm{L}^{n,1}$. In what follows, given a positive function φ on $(0, \infty)$, we write

$$F_\varphi(s) = \begin{cases} s\varphi^{\frac{1}{n}-1}(s), & s > 0, \\ 0, & s = 0. \end{cases} \tag{1.1}$$

Then g is in $\mathrm{L}^{n,1}(\Omega)$ if and only if there is a positive nonincreasing function $\varphi \in L^{1/n}((0,\infty))$ such that

$$\int_\Omega F_\varphi(|g|) < \infty.$$

Applying the results of [5], we get immediately the following consequences for Theorem A. Theorem B is due to Stein [12] and our result gives an alternative proof of this. We believe that Theorems C and D are new. Theorem D states that the mapping in question is *almost open* (cf. [8]).

Theorem B. *Suppose that* $u \in W^{1,1}_{\text{loc}}(\Omega)$ *is a function whose weak partial derivatives belong to* $\mathrm{L}^{n,1}(\Omega)$. *Then there is a representative of* u *that is continuous, and differentiable a.e.*

Theorem C. *Suppose that* $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^m)$ *is a continuous mapping whose weak partial derivatives belong to* $\mathrm{L}^{n,1}(\Omega)$. *Then* f *satisfies the* N*-condition.*

Theorem D. *Suppose that* $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$ *is a continuous mapping whose weak partial derivatives belong to* $\mathrm{L}^{n,1}(\Omega)$. *If* $G \subset \Omega$ *is open, then a.e.* $y \in f(G)$ *is an interior point of* $f(G)$.

We observe how Theorems A, C and D are sharp from the following equivalence.

Theorem E. *Suppose that* $n \geq 2$. *Let* φ *be a positive nonincreasing function on* $(0, \infty)$. *Then the following assertions are equivalent:*

(i) $\int_1^\infty \varphi^{1/n} < \infty$.

(ii) *Each* $f \in W^{1,n}(\Omega, \mathbf{R}^m)$ *with*

$$\int_\Omega F_\varphi(|\nabla f|) < \infty$$

has a representative that is locally n*-absolutely continuous.*

(iii) *Each continuous mapping* $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ *with*

$$\int_\Omega F_\varphi(|\nabla f|) < \infty$$

satisfies the N*-condition.*

(iv) *Given a continuous mapping* $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ *with*

$$\int_\Omega F_\varphi(|\nabla f|) < \infty$$

and an open set $G \subset \Omega$, *a.e.* $y \in f(G)$ *is an interior point of* $f(G)$.

Cianchi and Pick [2] showed that for a rearrangement invariant Banach space X of functions, there is a continuous embedding of the space $\{u : \nabla u \in X\}$ into L^∞, if and only if X is contained in $\mathrm{L}^{n,1}$. In combination with our results, we somewhat surprisingly conclude that this embedding into L^∞ is further equivalent with continuous embedding into AC^n.

Theorem F. *Let* X *be a rearrangement invariant Banach space* X *of functions on* $\mathbf{R}^n$. *Then the following assertions are equivalent:*

(i) *There is* $C < \infty$ *such that if* $\nabla u \in X$, *then* $\inf_{a \in \mathbf{R}} \|u - a\|_{L^\infty} \leq C\|\nabla u\|_X$.

(ii) $\{u : \nabla u \in X\}$ *is continuously embedded into* $AC^n(\mathbf{R}^n)$.

(iii) X *is continuously embedded into* $\mathrm{L}^{n,1}(\mathbf{R}^n)$.

We will use the following condition to verify n-absolute continuity in our main result. We say that a function u satisfies the RR (Rado-Reichelderfer) condition with the weight $\theta \in L^1(\Omega)$ if

$$\big(\operatorname{osc}_B u\big)^n \leq \int_B \theta(x)\,dx$$

for any ball $B \subset\subset \Omega$. It is easy to see (cf. [5]) that u belongs to $AC^n(\Omega)$ with $\|u\|^n_{AC^n} \leq \|\theta\|_{L^1}$ if the RR condition holds. Thus, for proving n-absolutely continuity, it suffices to establish the RR condition.

The cubical version of the RR condition was already known to Rado and Reichelderfer [10] as a sufficient condition for differentiability a.e. and for the area formula.

2 Characterization of $\mathrm{L}^{p,q}$

In this section, let (X, μ) be a measure space and $1 \leq q < p < \infty$. We denote $A = \mu(X)$. If f is a measurable function on X, we define its *distribution function* $\omega(\cdot, f)$ by

$$\omega(\alpha, f) = \mu\big(\{x \in X : |f(x)| > \alpha\}\big), \qquad \alpha \geq 0,$$

and the *nonincreasing rearrangement* f^* of f by

$$f^*(t) = \inf\{\alpha \geq 0 : \omega(\alpha, f) \leq t\}.$$

Then we have

$$\mu(\{|f| > \alpha\}) = \mathcal{L}^1(\{f^* > \alpha\}) \quad \text{for each } \alpha \geq 0.$$

The Lorentz space $\mathrm{L}^{p,q}(X)$ is defined as the class of all measurable functions on X for which the norm

$$\|f\|_{\mathrm{L}^{p,q}} := \left(\int_0^A (t^{1/p} f^*(t))^q \,\frac{dt}{t}\right)^{1/q}$$

is finite. For an introduction to Lorentz spaces see e.g. [13].

Proposition 2.1. *Let f be a nonnegative measurable function on X, ω be the distribution function of f and $A = \mu(X)$. Then*

$$\begin{aligned} \int_0^A \big(t^{\frac{1}{p}} f^*(t)\big)^q \,\frac{dt}{t} &= p\int_0^\infty s^{q-1}\omega^{q/p}(s)\,ds \\ &= q\int_{\{[t,s]:0<t<\omega(s)\}} s^{q-1}t^{\frac{q}{p}-1}\,dt\,ds. \end{aligned} \tag{2.1}$$

Proof. For $[t, s]$ in $(0, A)\times(0,\infty)$ we see that $t < \omega(s)$ if and only if $s < f^*(t)$. Hence the formula is a direct output of application of the Fubini theorem to the last integral.

Theorem 2.2. *Let $f \in \mathrm{L}^{p,q}(X)$ be a nonnegative function on X. Then there is a nonnegative nonincreasing function φ on $(0,\infty)$ such that*

$$\int_0^\infty s^{q-1}\varphi^{q/p}(s)\,ds \le \frac{1}{p}\|f\|^q_{\mathrm{L}^{p,q}} \tag{2.2}$$

and

$$\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x) \le \|f\|^q_{\mathrm{L}^{p,q}}. \tag{2.3}$$

Proof. Let ω be the distribution function of f and $A = \mu(X)$. If we take

$$\varphi(s) = \inf_{s'<s} \omega(s'), \tag{2.4}$$

then (2.2) holds by Proposition 2.1. Since

$$\varphi(f^*(t)) \ge t \quad \text{for } 0 < t < A,$$

we obtain (2.3) from

$$\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x) \le \int_0^A (f^*(t))^q\,t^{\frac{q}{p}-1}\,dt.$$

Theorem 2.3. *Let f be a nonnegative measurable function on X and φ be a nonnegative nonincreasing function on $(0,\infty)$ such that $\varphi(f(x)) > 0$ a.e. $x \in \{f > 0\}$. Then*

$$\begin{aligned}\|f\|^q_{\mathrm{L}^{p,q}} \le C_{p,q}&\left(\int_0^\infty s^{q-1}\varphi^{q/p}(s)\,ds\right)^{1-\frac{q}{p}}\\ &\left(\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x)\right)^{\frac{q}{p}}.\end{aligned} \tag{2.5}$$

Proof. We may assume that $f \ne 0$. Consider $\lambda > 0$ to be specified later. Let

$$\begin{aligned} E &= \{[t,s] \in (0,A)\times(0,\infty) :\ t < \omega(s)\},\\ E' &= \{[t,s] \in E : t < \lambda\varphi(s)\},\\ E'' &= E \setminus E'. \end{aligned}$$

We estimate the double integral in (2.1) by splitting it into two parts. We have

$$\begin{aligned} q\int_{E'} s^{q-1}t^{\frac{q}{p}-1}\,dt\,ds &\le q\int_0^\infty\left(\int_0^{\lambda\varphi(s)} s^{q-1}t^{\frac{q}{p}-1}\,dt\right)ds\\ &= p\lambda^{q/p}\int_0^\infty s^{q-1}\varphi^{q/p}(s)\,ds. \end{aligned} \tag{2.6}$$

Consider $[t,s] \in E''$. Then, since $E'' \subset E$,

$$s < f^*(t),$$

and hence by monotonicity

$$t \geq \lambda\varphi(s) \geq \lambda\varphi(f^*(t)).$$

It follows that

$$\begin{aligned}
q\int_{E''} s^{q-1} t^{\frac{q}{p}-1}\,dt\,ds &\leq q\lambda^{\frac{q}{p}-1}\int_{E''} s^{q-1}\varphi^{\frac{q}{p}-1}(f^*(t))\,dt\,ds \\
&\leq q\lambda^{\frac{q}{p}-1}\int_0^A\left(\int_0^{f^*(t)} s^{q-1}\varphi^{\frac{q}{p}-1}(f^*(t))\,ds\right)dt \\
&= \lambda^{\frac{q}{p}-1}\int_0^A (f^*(t))^q\varphi^{\frac{q}{p}-1}(f^*(t))\,dt \\
&= \lambda^{\frac{q}{p}-1}\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x).
\end{aligned} \tag{2.7}$$

Setting

$$\lambda = \frac{p-q}{pq}\;\frac{\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x)}{\int_0^\infty s^{q-1}\varphi^{q/p}(s)\,ds}\,,$$

we obtain from (2.6), (2.7) and (2.1) the desired inequality.

Corollary 2.4. *Let f be a nonnegative measurable function on X. Then the following properties are equivalent:*

(i) $f \in \mathrm{L}^{p,q}(X)$.

(ii) *There is a nonnegative nonincreasing function φ on $(0,\infty)$ such that $\varphi(f(x)) > 0$ a.e. $x \in \{f > 0\}$ and*

$$\int_0^\infty s^{q-1}\varphi^{q/p}(s)\,ds < \infty$$

and

$$\int_{\{f>0\}} f^q(x)\,\varphi^{\frac{q}{p}-1}(f(x))\,d\mu(x) < \infty.$$

3 Verifying the n-absolute continuity

In what follows, $\Omega \subset \mathbf{R}^n$ will be a fixed open set. We denote by $\mathcal{L}^n$ the n-dimensional Lebesgue measure and by $\boldsymbol{\alpha}_n$ the measure of the unit ball in $\mathbf{R}^n$. If $B \subset \mathbf{R}^n$ is a ball and u is an integrable function on B, we write

$$u_B = (\mathcal{L}^n(B))^{-1}\int_B u,$$

this is the mean value of u on B. For the definition of F_φ we refer to (1.1).

We begin with a crucial estimate on a Riesz potential.

Theorem 3.1. *Let g be a nonnegative measurable function on Ω and φ be a nonincreasing positive function on $(0,\infty)$. Then for any $z \in \Omega$ and any measurable set $E \subset \Omega$ we have the inequality*

$$\begin{aligned} &\left(\int_E |x-z|^{1-n} g(x)\,dx\right)^n \\ &\qquad \le 2^n \left(n\boldsymbol{\alpha}_n \int_0^\infty \varphi^{1/n}(t)\,dt\right)^{n-1} \int_E F_\varphi(g(x))\,dx. \end{aligned} \tag{3.1}$$

Proof. Let $E \subset \Omega$ be a measurable set and $z \in \Omega$. We may assume that $z = 0$. If the integral on the left of (3.1) vanishes the proof is over. Otherwise we choose $0 < h < \infty$ such that

$$h \le \int_E |x|^{1-n}\, g(x)\,dx.$$

We consider a constant $\lambda > 0$ to be specified later and write

$$\begin{aligned} J_1 &= \int_E F_\varphi(g(x))\,dx, \\ J_2 &= n\boldsymbol{\alpha}_n \int_0^\infty \varphi^{1/n}(t)\,dt, \\ P &= \{[x,t] \in E \times \mathbf{R} : 0 < t < g(x)\}, \\ P' &= \left\{[x,t] \in P : \varphi(t) < \left(\frac{|x|}{\lambda h}\right)^n\right\}, \\ P'' &= P \setminus P'. \end{aligned}$$

We have

$$\int_E |x|^{1-n}\, g(x)\,dx = \iint_P |x|^{1-n}\,dx\,dt.$$

We split the integration into the P' part and P'' part. If $[x,t] \in P'$, then

$$|x|^n \ge \lambda^n h^n \varphi(t) \ge \lambda^n h^n \varphi(g(x)).$$

Hence

$$\begin{aligned} &\iint_{P'} |x|^{1-n}\,dx\,dt \\ &\le \lambda^{1-n} h^{1-n} \int_{E\cap\{g>0\}} \left(\int_0^{g(x)} dt\right) \varphi^{(1-n)/n}(g(x))\,dx \\ &= \lambda^{1-n} h^{1-n} J_1. \end{aligned} \tag{3.2}$$

For the P''-part we have

$$\begin{aligned} \iint_{P''} |x|^{1-n}\,dx\,dt &\le \int_0^\infty \left(\int_{B(0,\lambda h \varphi^{1/n}(t))} |x|^{1-n}\,dx\right) dt \\ &\le n\boldsymbol{\alpha}_n \lambda h \int_0^\infty \varphi^{1/n}(t)\,dt \\ &= \lambda h J_2. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3) we have

$$\begin{aligned} h &\le \iint_{P'} |x|^{1-n}\,dx\,dt + \iint_{P''} |x|^{1-n}\,dx\,dt \\ &\le \lambda^{1-n} h^{1-n} J_1 + \lambda h J_2. \end{aligned}$$

If we choose

$$\lambda = \frac{J_1^{1/n}}{J_2^{1/n} h},$$

then we obtain

$$h \le 2 J_1^{1/n} J_2^{(n-1)/n}$$

so that

$$\begin{aligned} h^n &\le 2^n J_1 J_2^{n-1} \\ &= 2^n \left(n\boldsymbol{\alpha}_n \int_0^\infty \varphi^{1/n}(t)\,dt \right)^{n-1} \int_E F_\varphi(g(x))\,dx. \end{aligned}$$

This concludes the proof.

Using the previous theorem we now give a sufficient condition for RR.

Theorem 3.2. *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and φ be a nonincreasing positive function on $(0,\infty)$. Suppose that*

$$\int_\Omega F_\varphi(|\nabla u(x)|)\,dx < \infty \tag{3.4}$$

and

$$\int_0^\infty \varphi^{1/n}(t)\,dt < \infty. \tag{3.5}$$

Then u, properly represented, verifies the RR condition with the weight

$$\theta(x) = C_\theta \left(\int_0^\infty \varphi^{1/n}(t)\,dt \right)^{n-1} F_\varphi(|\nabla u(x)|), \tag{3.6}$$

where

$$C_\theta = \frac{2^{n(n+2)}}{n\boldsymbol{\alpha}_n}.$$

Proof. By Lemma 1.50 in [6], there is a set N with $\mathcal{L}^n(N) = 0$ such that all points in $\Omega \setminus N$ are Lebesgue points for u and

$$|u(z) - u_B| \le \frac{2^n}{n\boldsymbol{\alpha}_n} \int_B |x - z|^{1-n}\,|\nabla u(x)|\,dx \tag{3.7}$$

for each ball $B \subset\subset \Omega$ and $z \in B \setminus N$. Let us fix a ball $B = B(x_0, R) \subset\subset \Omega$. We write

$$g = |\nabla u|$$

and use notation J_1 and J_2 as in the proof of Theorem 3.1. We may assume that $\operatorname{osc}_{B\setminus N} u > 0$. Choosing

$$0 \le a < \operatorname{osc}_{B\setminus N} u,$$

we can find a point $z \in B \setminus N$ such that

$$a \le 2|u(z) - u_B|. \tag{3.8}$$

Applying Theorem 3.1 to $E = B$ and using (3.8) and (3.7) we obtain

$$\begin{aligned} a^n &\le \left(\frac{2^{n+1}}{n\boldsymbol{\alpha}_n} \int_B |x-z|^{1-n}\, g(x)\, dx \right)^n \\ &\le \left(\frac{2^{n+1}}{n\boldsymbol{\alpha}_n} \right)^n 2^n (J_2)^{n-1} J_1 \\ &= \int_B \theta(x)\, dx. \end{aligned}$$

Letting $a \to \operatorname{osc}_{B\setminus N} u$ we obtain

$$\big(\operatorname{osc}_{B\setminus N} u\big)^n \le \int_B \theta(x)\, dx.$$

It follows that u is locally uniformly continuous on $\Omega \setminus N$ and hence u has a continuous representative on Ω. This representative verifies the RR condition with the weight θ.

4 Almost open mappings

In this section we show that every n-absolutely continuous mapping is almost open.

Lemma 4.1. *Suppose that $f : \Omega \to \mathbf{R}^n$ is a continuous mapping differentiable at a point $x_0 \in \Omega$. Suppose that $Jf(x_0) \neq 0$. If G is an open set containing x_0, then the interior of $f(G)$ contains $f(x_0)$.*

Proof. This is an application of the Brouwer fixed point theorem, cf. [11], Lemma 7.23 and Theorem 7.24. For fixed point theorem see e.g. [3].

Theorem 4.2. *Suppose that $f : \Omega \to \mathbf{R}^n$ is an n-absolutely continuous mapping. Then f is almost open.*

Proof. Denote by A the set where f' does not exist and by B the set where $Jf = 0$. By Lemma 4.1 it suffices to show that A and B are mapped into the set of measure zero. Since $\mathcal{L}^n(A) = 0$ and f satisfies the N-condition we have $\mathcal{L}^n(f(A)) = 0$. On the other hand, the change of variables formula (see [5], Theorem 3.4) is valid for f, and hence $\mathcal{L}^n(f(B)) = 0$.

5 Examples

It was pointed out by Stein [12] that the condition $\nabla f \in \mathrm{L}^{n,1}(\Omega)$ is essentially sharp for continuity and for differentiability almost everywhere. We show below that this is also the case regarding the N-condition and almost openness for continuous mappings. Throughout this section we assume that $n \geq 2$.

Given an Orlicz function F_φ, we want to construct "wild" functions f with

$$\int_\Omega F_\varphi(|\nabla f|)\, dx < \infty.$$

There are two steps in our construction. The first one is to construct a radial function u with

$$\int_{B(0,1)} F_\varphi(|\nabla u(x)|)\, dx < \infty$$

and

$$\lim_{x\to 0} u(x) = \infty.$$

Although the existence of such a function follows from [2], we include here a direct proof for the convenience of the reader.

The second step is based on a refinement of a method due to Cesari [1], see also Malý and Martio [4]. Originally, Cesari constructed a continuous function in $W^{1,2}((0,1)^2, \mathbf{R}^2)$ whose Lebesgue area is zero but whose image is a square.

Lemma 5.1. *Let φ be a positive nonincreasing function on $(0,\infty)$ and*

$$\int_1^\infty \varphi^{1/n}(s)\, ds = \infty. \tag{5.1}$$

Then there is $u \in W^{1,n}(B(0,1))$ so that u is nonnegative, radial, continuous outside the origin, tends to infinity when $x \to 0$, and satisfies

$$\int_{B(0,1)} F_\varphi(|\nabla u(x)|)\, dx < \infty. \tag{5.2}$$

Proof. Consider for a while the positive nonincreasing function $\tilde\varphi(t) = \min\{\varphi(t), t^{-n}\}$. We claim that

$$\int_1^\infty \tilde\varphi^{1/n}(s)\, ds = \infty. \tag{5.3}$$

Indeed, assuming the contrary, there is an $a > 0$ such that

$$\int_a^\infty \tilde\varphi^{1/n}(s)\, ds \leq \frac{1}{3}.$$

Then, using the monotonicity, for $t > 3a$,

$$\tilde{\varphi}^{1/n}(t) \leq \frac{1}{t-a} \int_a^t \tilde{\varphi}^{1/n}(s)\, ds \leq \frac{1}{3t-3a} \leq \frac{1}{2t}$$

and thus $\tilde{\varphi}(t) = \varphi(t)$, which implies (5.3). Hence we may suppose that $\varphi(s) \leq s^{-n}$, otherwise we would replace φ by $\tilde{\varphi}$. It follows that $F_\varphi(s) \geq s^n$ and once proving (5.2), the integrability of $|\nabla u|^n$ is also verified.

We define a sequence h_k of real functions on $(0,\infty)$ by

$$h_k(t) = \inf\{s > 0 : \varphi(2s) \leq (2^k t)^n\}.$$

Find $\sigma_k > 0$ such that $\varphi(2\sigma_k) < 2^{kn}$. Then

$$\begin{aligned} &\{[t,s] : 0 < t < 1, 0 < s \leq h_k(t)\} \\ &\quad \supset \{[t,s] : s > \sigma_k,\ 0 < t < 2^{-k}\varphi^{1/n}(2s)\}. \end{aligned}$$

Hence using Fubini's theorem we obtain

$$\begin{aligned} \int_0^1 h_k(t)\, dt &= \int_{\{[t,s]:0<t<1,0<s\leq h_k(t)\}} dt\, ds \\ &\geq \int_{\{[t,s]:s>\sigma_k,\ 0<t<2^{-k}\varphi^{1/n}(2s)\}} dt\, ds \\ &= 2^{-k} \int_{\sigma_k}^\infty \varphi^{1/n}(2s)\, ds = \infty. \end{aligned}$$

It follows that we may define a decreasing sequence $\{a_k\}$ of positive real numbers such that $a_1 = 1$ and

$$\int_{a_{k+1}}^{a_k} h_k = 1, \quad k = 1, 2, \dots.$$

Since $h_k(a_{k+1}) \geq 1$, we have $\varphi(1) > (2^k a_{k+1})^n$ and thus $a_k \to 0$. Set

$$u(x) = k + \int_{|x|}^{a_k} h_k, \quad a_{k+1} \leq |x| < a_k.$$

Then obviously

$$\lim_{|x|\to 0} u(x) = \infty.$$

Since

$$\varphi(h_k(t)) \geq (2^k t)^n$$

we have

$$F_\varphi(h_k(t)) \leq (2^k t)^{1-n} h_k(t)$$

and thus

$$\begin{aligned}\int_{\{a_{k+1}\le|x|\le a_k\}} F_\varphi(|\nabla u(x)|)\,dx &= \int_{\{a_{k+1}\le|x|\le a_k\}} F_\varphi(h_k(|x|))\,dx\\ &= n\boldsymbol{\alpha}_n\int_{a_{k+1}}^{a_k} t^{n-1}F_\varphi(h_k(t))\,dt\\ &\le C\int_{a_{k+1}}^{a_k} t^{n-1}(2^k t)^{1-n}h_k(t)\,dt\\ &= C2^{-k(n-1)}.\end{aligned}$$

It follows that

$$\int_{B(0,1)} F_\varphi(|\nabla u(x)|)\,dx<\infty$$

as required.

Theorem 5.2. *Let φ be as in Lemma 5.1. Suppose that a sequence $\{\mathcal{K}_m\}_{m=0}^{\infty}$ of finite families of closed cubes is given such that*

$$\mathcal{K}_0=\{\,[-1,1]^n\,\},\tag{5.4}$$

$$\text{for each } K\in\mathcal{K}_{m+1} \text{ there is } K'\in\mathcal{K}_m \text{ such that } K\subset K', \text{ and}\tag{5.5}$$

$$\lim_{m\to\infty}\sup_{K\in\mathcal{K}_m}\operatorname{diam}K=0.\tag{5.6}$$

Let L be a line segment in $\mathbf{R}^n$. Then there exists a continuous mapping $f\in W^{1,n}(\mathbf{R}^n,\mathbf{R}^n)$ and a set $S\subset\mathbf{R}^n$ which is a countable union of line segments such that

$$\int_{\mathbf{R}^n} F_\varphi(|\nabla f|)<\infty,\tag{5.7}$$

$$\det\nabla f=0 \ \text{a.e.},\tag{5.8}$$

$$f(\mathbf{R}^n)=f(L)=S\cup\bigcap_{m=0}^{\infty}\bigcup_{K\in\mathcal{K}_m}K.\tag{5.9}$$

Proof. We denote by y_K the center of a cube K. We will define recursively a sequence $\{f_m\}$ of Lipschitz continuous mappings in $W^{1,n}(\mathbf{R}^n,\mathbf{R}^n)$ such that for every m and $K\in\mathcal{K}_m$ there is a point $z_K\in L$ and a radius $r_K\in(0,2^{-m})$

such that the balls $B(z_K, r_K)$, $K \in \mathcal{K}_m$, are pairwise disjoint and

$$f_m(x) = y_K \text{ for each } x \in B(z_K, r_K), \tag{5.10}$$

$$f_j(x) \in K \text{ for each } x \in B(z_K, r_K) \text{ and } j \geq m, \tag{5.11}$$

$$f_j(x) = f_m(x) \text{ for each } x \notin \bigcup_{K \in \mathcal{K}_m} B(z_K, r_K) \text{ and } j \geq m, \tag{5.12}$$

$$\begin{aligned} &f_m(\mathbf{R}^n) = f_m(L) \text{ is a finite family of line segments and} \\ &f_m(\mathbf{R}^n) \subset f_j(\mathbf{R}^n) \text{ for each } j \geq m, \end{aligned} \tag{5.13}$$

$$\int_{\mathbf{R}^n} \tilde{F}_\varphi(|\nabla f_0|) < \infty \text{ and } \int_{\bigcup_{K \in \mathcal{K}_m} B(z_K, r_K)} \tilde{F}_\varphi(|\nabla f_{m+1}|) \leq 2^{-m-1}, \tag{5.14}$$

where

$$\tilde{F}_\varphi(s) := F_\varphi(s) + s^n, \quad s > 0.$$

By (5.6), (5.11) and (5.12) such a sequence converges uniformly to a continuous mapping f. By (5.10), (5.12) and (5.14) the sequence converges also in $W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$, and in particular the limit belongs to the same space. From (5.10) and (5.12) we infer that

$$|\nabla f_0| \leq |\nabla f_1| \leq \dots$$

and that $|\nabla f_m|$ converges to $|\nabla f|$ a.e. Then using Levi's monotone convergence theorem and (5.14) we obtain (5.7). Since the image is one dimensional, the rank of ∇f_m is 1 and thus $\det \nabla f_m = 0$ a.e. Passing to the limit we obtain (5.8). From (5.10)–(5.13) we easily derive (5.9).

It remains to present details of the construction. The family $\mathcal{K}_0$ contains only the cube $K_0 = [-1, 1]^n$. We start with a constant mapping $f_0 = y_{K_0}$. We also choose a point $z_{K_0} \in L$ and a radius $r_{K_0} \in (0, 1)$ such that $L \not\subset B(z_{K_0}, r_{K_0})$. Let us assume that the construction is accomplished for $f_0, \dots, f_m$. For all $K' \in \mathcal{K}_m$ with every $K \in \mathcal{K}_{m+1}$ such that $K \subset K'$ we associate a point $z_K \in L \cap B(z_{K'}, r_{K'})$ and a radius $R_K > 0$ such that $B(z_K, R_K) \subset B(z_{K'}, r_{K'})$ and the balls $B(z_K, R_K)$, $K \in \mathcal{K}_{m+1}$, are pairwise disjoint. Let N_{m+1} be the cardinality of $\mathcal{K}_{m+1}$. For every $K \in \mathcal{K}_{m+1}$ find a radius $\rho_K > 0$ such that $\rho_K \leq \min(2^{-m-1}, R_K)$ and

$$\int_{B(0,\rho_K)} \tilde{F}_\varphi(|\nabla u(x)|)\, dx < \frac{1}{2^{m+1} N_{m+1}}$$

where u is as in Lemma 5.1. We find $r_K \in (0, \rho_K]$ such that

$$u(r_K e_1) - u(\rho_K e_1) = |y_K - y_{K'}|$$

and define

$$f_{m+1}(x) = y_{K'} + (u(x - z_K) - u(\rho_K e_1)) \frac{y_K - y_{K'}}{|y_K - y_{K'}|}$$

if $r_K < |x - z_K| < \rho_K$ and

$$f_{m+1}(x) = y_K$$

if $|x - z_K| \leq r_k$. Outside the balls $B(z_K, \rho_K)$ we set $f_{m+1} = f_m$. It is easy to verify the properties (5.10)–(5.14) so that the proof is completed.

Example 5.3. *Let $u \in W^{1,n}(\Omega)$. The condition*

$$\int_\Omega |\nabla u|^n \log^\alpha(e + |\nabla u|) < \infty$$

guarantees the n-absolute continuity of a representative of u (and thus also the N-property and almost openness if u is vector valued) if $\alpha > n-1$ but not if $\alpha \leq n-1$.

6 Proofs of Theorems A–E

In this section we give the proofs of Theorems A–E. Note that $f = (f_1, \dots, f_m)$ satisfies RR (or is n-absolutely continuous) if and only if each coordinate function f_j does.

Proof of Theorem A. By Theorems 2.4 and 3.2 there is a representative of u that verifies the RR condition. According to Theorem 3.1 in [5], the RR condition implies that u belongs to $AC^n(\Omega)$. Let ω be the distribution function for ∇u and choose φ as in (2.4). Using Theorem 3.2 and Theorem 2.2, we obtain the estimate

$$\begin{aligned}\|u\|_{AC^n}^n &\leq C\left(\int_0^\infty \varphi^{1/n}(t)\,dt\right)^{n-1} \int_\Omega F_\varphi(|\nabla u(x)|)\,dx \\ &\leq C\|\nabla u\|_{\mathrm{L}^{n,1}}^n,\end{aligned}$$

which proves continuity of the embedding.

Proof of Theorem B. The n-absolutely continuous representative of u given by Theorem A is clearly continuous, and moreover differentiable a.e. by Theorem 3.3 in [5].

Proof of Theorem C. By Theorem A, f is n-absolutely continuous and hence satisfies the N-condition.

Proof of Theorem D. This follows form Theorems A and 4.2.

Proof of Theorem E. (i) $\Longrightarrow$ (ii) We have verified all the assumptions of Theorem 3.2 except that

$$\int_0^1 \varphi^{1/n} < \infty.$$

For this purpose we modify φ by changing $\varphi(s)$ to $\varphi(1)$ for $0 < s < 1$. Now the integrability of $F_\varphi(|\nabla f|)$ over the set $\{|\nabla f| \leq 1\}$ may break, but only if $|\Omega| = \infty$. Thus we have guaranteed n-absolute continuity at least locally.

(ii) $\Longrightarrow$ (iii) f is locally n-absolutely continuous and thus verifies the N-condition.

(ii) $\Longrightarrow$ (iv) Since f is locally n-absolutely continuous, the claim follows from Theorem 4.2.

(iii) $\Longrightarrow$ (i) Suppose that $\int_1^\infty \varphi^{1/n} = \infty$. Theorem 5.2 applied to the families $\mathcal{K}_m$ such that

$$\bigcap_{m=0}^{\infty} \bigcup_{K \in \mathcal{K}_m} K = [-1,1]^n$$

gives a continuous mapping $f \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} F_\varphi(|\nabla f|) < \infty$$

and $f(L) \supset [-1,1]^n$, in particular f does not satisfy the N-condition.

(iv) $\Longrightarrow$ (i) Suppose that $\int_1^\infty \varphi^{1/n} = \infty$. Theorem 5.2 applied to the families $\mathcal{K}_m$ such that

$$\bigcap_{m=0}^{\infty} \bigcup_{K \in \mathcal{K}_m} K$$

is a nowhere dense Cantor set gives a continuous mapping $f \in W^{1,n}(\mathbf{R}^n, \mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} F_\varphi(|\nabla f|) < \infty$$

and $f(\mathbf{R}^n)$ is uncountable with no interior points (by Baire category theorem), in particular f is not almost open.

Remark. We would like to mention an alternate proof of Theorem D. In [8], it was shown that $\int_\Omega |Jf| = \int_{\mathbf{R}^n} M(f,y)\,dy$ if $f \in W^{1,n}(\Omega, \mathbf{R}^n)$ is continuous. Here, $M(f,y)$ is the multiplicity function defined in [8]. It follows immediately from its definition that if $M(f,y) > 0$ and $y \in f(G)$, then y is in the interior of $f(G)$ whenever G is open. Using Theorem C, it follows that $\int_\Omega |Jf| = \int_{\mathbf{R}^n} N(f,y)\,dy$ if $\nabla f \in L^{n,1}(\Omega)$, where $N(f,y)$ is the number of points in $f^{-1}(y)$. Since $M(f,y) \leq N(f,y)$, it follows that $M(f,y) = N(f,y)$ for a.e. y, and therefore that f is almost open, as desired.

Acknowledgements

J. Kauhanen is supported in part by the Academy of Finland grant 41933, by the foundation Magnus Ehrnroothin säätiö and by the foundation Vilho, Yrjö ja Kalle Väisälän rahasto. P. Koskela is supported in part by the Academy of Finland grant 41933. J. Malý is supported by the grants GAČR 201/97/1161 and GAUK 186/96.

We thank Eero Saksman, Luboš Pick, Joan Cerdà and Javier Soria for important suggestions. We also wish to thank the referee for his comments and for pointing out to us the alternate proof of Theorem D given in the last remark.

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