LINEAR MAPPINGS AND GENERALIZED UPPER SPECTRUM FOR DIMENSIONS

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Abstract. We prove that for $0 < q \leq 1$ the upper q-dimension of the image of a Borel probability measure is a constant under typical linear maps (or typical orthogonal projections). This constant may be smaller than the upper q-dimension of the original measure.

1. Introduction and notation

There are several concepts of dimension which have been studied both in mathematics and in physics. These concepts are related to either sets or measures. For example, in the theory of dynamical systems it is sometimes more useful to study the dimension of a probability measure. In fact, an attractor of a dynamical system may carry a natural invariant measure (so called Sinai-Ruelle-Bowen-measure) which contains more information than its support (which is the attractor). The value of the dimension of a set or a measure may vary for different definitions.

The most common definition of dimension is the Hausdorff dimension, $\dim_H$. For sets it is defined in terms of Hausdorff measures. For a Borel probability measure μ on $\mathbb{R}^n$ it can be defined either by means of Hausdorff dimensions of sets or, equivalently, in the following way:

$$(1.1) \quad \dim_H \mu = \sup \Big\{ s \geq 0 \mid \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n \Big\}.$$

Here $B(x,r)$ is the closed ball of centre $x \in \mathbb{R}^n$ and radius r with $0 < r < \infty$. Recently there has been much interest in the packing dimension, $\dim_p$, defined by replacing the lower limit with the upper limit in (1.1), that is,

$$(1.2) \quad \dim_p \mu = \sup \Big\{ s \geq 0 \mid \limsup_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s \text{ for } \mu\text{-almost all } x \in \mathbb{R}^n \Big\}.$$

The basic geometrical properties of Hausdorff dimension – the projectional properties of sets ([Mar], [Mat1],[Kau]) and measures ([HT]), the dimensional properties of intersections of sets ([Mar], [Mat1], [Mat2], [Kah]), and sections of measures

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([J2],[JM]) – are well-known (see also [F1], [F2], and [Mat3]). In particular, if μ is a Borel probability measure on $\mathbb{R}^n$, then for $\gamma_{n,m}$-almost all $V \in G_{n,m}$

$$\dim_H \mu_V = \min\{\dim_H \mu, m\}. \tag{1.3}$$

Here n and m are integers with $0 < m < n$, $\gamma_{n,m}$ is the unique rotationally invariant Borel probability measure on the Grassmann manifold $G_{n,m}$ consisting of all m-dimensional linear subspaces of $\mathbb{R}^n$, and $\mu_V = (\mathrm{proj}_V)_*\mu$ is the image of μ under the orthogonal projection $\mathrm{proj}_V : \mathbb{R}^n \to V$.

The geometry of packing dimension is less regular than that of Hausdorff dimension ([FH1], [FH2], [FJ], [FM],[JM], [J1], [J2]). Unlike the Hausdorff dimension, the packing dimension is not preserved under typical projections. Nevertheless, the packing dimensions of the typical projections of a Borel probability measure μ cannot be too small: Falconer and Mattila [FM] showed that for $\gamma_{n,m}$-almost all $V \in G_{n,m}$

$$\dim_p \mu_V \geq \frac{\dim_p \mu(1 - \dim_H \mu/n)}{1 + (1/m - 1/n)\dim_p \mu - \dim_H \mu/m} \tag{1.4}$$

provided $\dim_H \mu \leq m$. They also gave an example which shows that this lower bound is the best possible one. Continuing the work of Falconer and Mattila, Falconer and Howroyd [FH2] proved that for any Borel probability measure μ on $\mathbb{R}^n$ the packing dimensions of its projections onto $\gamma_{n,m}$-almost all $V \in G_{n,m}$ are equal. For this purpose they gave a new characterization of the packing dimension of μ in terms of quantities called dimension profiles.

This paper seeks to establish analogues of these projection results for upper q-dimensions. For $q > 0$ Hunt and Kaloshin [HK] proposed the following potential theoretic defnition for the lower q-dimension of a Borel probability measure μ with compact support:

$$\underline{D}_q(\mu) = \sup\Big\{s \geq 0 : \int\Big(\int \frac{d\mu(y)}{|x-y|^s}\Big)^q d\mu(x) < \infty\Big\}. \tag{1.5}$$

We give an analogue of this definition in the case of the upper q-dimension:

$$\overline{D}_q(\mu) = \sup\{s \geq 0 : J^n_{s,q}(\mu) < \infty\}, \tag{1.6}$$

where

$$J^n_{s,q}(\mu) = \liminf_{r\to 0} r^{-sq}\int\Big(\int \min\{1, r^n|x-y|^{-n} d\mu(y)\}\Big)^q d\mu(x). \tag{1.7}$$

This definition allows us to apply the techniques from [Mat1], [FH2], [FM], and [HK] in order to study the behaviour of the upper q-dimension, $\overline{D}_q$, of a Borel probability measure under linear mappings. The corresponding questions for the lower q-dimension, $\underline{D}_q$, have been studied by Sauer and Yorke ([SY]) and by Hunt and Kaloshin ([HK]) (for definitions see Section 2). These quantities, which are also called the generalized lower and upper spectrum for dimensions, are one-parameter families of dimensions commonly used in the study of dynamical systems. They were introduced by Hentschel and Procaccia in [HP] as a generalization of the lower and upper correlation dimensions (which are $\underline{D}_1$ and $\overline{D}_1$ in our notation) and independently by Grassberger in [G]. By the generalized spectrum for dimensions we do not mean the multifractal spectrum of a measure (see for example [O]) although in some cases these concepts are related to each other by the Legendre transformation. For further information on these dimensions see [P].

Sauer and Yorke proved in [SY] that a part of the generalized lower spectrum, namely, the lower correlation dimension, $\underline{D}_1$, is preserved under $\mathcal{L}^{nm}$-almost all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Here $\mathcal{L}^{nm}$ is the nm-dimensional Lebesgue measure. Using the potential-theoretic definition for the generalized lower spectrum (1.5), Hunt and Kaloshin ([HK]) extended this result for all $0 < q \leq 1$ (in their notation for all $1 < q \leq 2$). They proved that if μ is a Borel probability measure on $\mathbb{R}^n$ with compact support, then for $\mathcal{L}^{nm}$-almost all linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ the following analogue of (1.3) holds:

$$\underline{D}_q(L_*\mu) = \min\{\underline{D}_q(\mu), m\}. \tag{1.8}$$

Here $L_*\mu$ is the image of μ under L. They also gave examples showing that the lower q-dimension is not preserved under typical linear maps for $q > 1$.

In this paper we address the problem of finding out how the generalized upper spectrum behaves under linear mappings (or under orthogonal projections). It appears that, unlike the lower q-dimension, the upper one is not preserved under typical linear mappings (or typical projections) for $0 < q \leq 1$. We will prove that if n and m are integers with $0 < m < n$, μ is a Borel probability measure on $\mathbb{R}^n$ with compact support, and $0 < q \leq 1$, then for $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ we have

$$\overline{D}_q(L_*\mu) = \dim_q^m(\mu), \tag{1.9}$$

where $\dim_q^m(\mu)$ is a constant obtained by convolving the measure μ with a certain kernel and it can be strictly less that the upper q-dimension of μ. Thus for $0 < q \leq 1$ the upper q-dimension behaves like the packing dimension under orthogonal projections, while the lower one is preserved like the Hausdorff dimension.

Because of being quite convenient for the purposes of numerical calculations, the lower and upper correlation dimensions, $\underline{D}_1$ and $\overline{D}_1$, have received much attention. For the upper correlation dimension we study how small the constant $\dim_1^m(\mu)$ in (1.9) can be. We show that the following analogue of (1.4) holds. If $\underline{D}_1(\mu) \leq m$, then

$$\overline{D}_1(\mu_V) \geq \frac{\overline{D}_1(\mu)(1 - \underline{D}_1(\mu)/n)}{1 + (1/m - 1/n)\overline{D}_1(\mu) - \underline{D}_1(\mu)/m} \tag{1.10}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ and $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$. After this work was completed we learnt from a paper by Falconer and O'Neil [FO] relating multifractal properties of measures - including q-dimensions - to those of projections. Applying their methods we shortly indicate how (1.10) can be extended to all $0 < q \leq 1$. We will also discuss an example from [FM] which shows that the lower bound given in (1.10) is the best possible one. In Section 2 we give the definition of prevalence introduced by Hunt, Sauer, and Yorke in [HSY] and extend our results to "almost every" continuosly differentiable function.

Influenced by the methods from [FM],[FH2], and [HK], we use the new characterization (see (1.6) and (1.7)) of the upper q-dimension of a Borel probability measure μ for $q > 0$. This characterization, which is given by means of quantities defined by convolving μ with a certain kernel, is a modification of the dimension profile approach introduced by Falconer and Howroyd in [FH2].

We have organized this paper in the following way. In Section 2 we give the definition of the upper q-dimension, consider an example showing that the lower bound in (1.10) is the best possible one, establish the new characterization (1.6)-(1.7) of the upper q-dimension, and prove the result (1.9). In Section 3 we prove (1.10) and finally in Section 4 we prove technical results needed throughout the paper.

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2. The upper q-dimension and orthogonal projections

In this section we consider for $0 < q \leq 1$ the behaviour of the upper q-dimension of a compactly supported Borel probability measure μ on $\mathbb{R}^n$ under projections and general linear maps.

Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. Let $r > 0$. For every $q \neq 0$ we define

$$K_q(\mu, r) = \int \mu(B(x,r))^q d\mu(x).$$

The lower and upper q-dimensions of μ are defined by

$$\underline{D}_q(\mu) = \liminf_{r \to 0} \frac{\log K_q(\mu, r)}{q \log r}$$

and

$$\overline{D}_q(\mu) = \limsup_{r \to 0} \frac{\log K_q(\mu, r)}{q \log r}.$$

Note that in our setting in the definition of $K_q(\mu, r)$ the integrand is raised to the power q. In many other papers, including [HK], the corresponding power is $q-1$. For $q = 0$ the corresponding definitions are given in [HK, Definition 4.2] (case $q = 1$ in their notation). These quantities $\underline{D}_0$ and $\overline{D}_0$ are called the lower and upper information dimension. In this paper we will consider only the case $q > 0$.

The following example constructed in [FM, Example 5.1] shows that the behaviour of the upper q-dimension under orthogonal projections and linear maps is different from that of the lower q-dimension, that is, for $0 < q \leq 1$ the upper q-dimension, unlike the lower one ([HK, Theorem 1.1]), is not preserved under projections or linear maps.

Example 1. *Let n and m be integers with $0 < m < n$. Let $0 < \underline{d} < \overline{d} < n$ and $\underline{d} < m$. Then there exists a Borel probability measure μ on $\mathbb{R}^n$ with compact support such that the following properties hold:*

(1) *there is a positive constant c such that*

$$cr^{\overline{d}} \leq \mu(B(x,r)) \leq \frac{r^{\underline{d}}}{c}$$

for all $x \in \operatorname{spt}\mu$ and $0 < r \leq 1$,

(2) *there exist sequences (r_k) and (R_k) of positive real numbers tending to zero such that*

$$\mu(B(x,\sqrt{n}r_k)) = r_k^{\underline{d}} \quad \text{and} \quad \mu(B(x,R_k/2)) = R_k^{\overline{d}}$$

for all $x \in \operatorname{spt}\mu$, and

(3)
$$\overline{\dim}_B(\operatorname{proj}_V(\operatorname{spt}\mu)) \leq \frac{\overline{d}(1-\underline{d}/n)}{1+(1/m-1/n)\overline{d}-\underline{d}/m}$$

for all $V \in G_{n,m}$. Here spt is the support of a measure and $\overline{\dim}_B$ is the upper box-counting dimension defined for all bounded sets $E \subset \mathbb{R}^n$ by

$$\overline{\dim}_B E = \limsup_{\varepsilon \to 0} \frac{\log N(E,\varepsilon)}{-\log \varepsilon}$$

where $N(E,\varepsilon)$ is the smallest number of sets of diameter at most ε that cover E.

Proof. We give the basic ideas of the construction in [FM, Example 5.1]. We concentrate on the projection property (3) and refer to [FM, Example 5.1] as far as (1) and (2) are concerned.

Define recursively sequences (r_k) and (R_k) of positive real numbers tending to zero and a sequence (m_k) of positive integers tending to infinity. Let $R_1 = r_1 = m_0 = 1$. Having defined r_k and R_k, choose a positive integer m_k in such a way that $m_k^{1/n}$ is also an integer with

$$m_k^{1/n} - 1 \leq r_k^{(\underline{d}-\overline{d})/(n-\overline{d})} \leq m_k^{1/n}. \tag{2.1}$$

Next define r_{k+1} and R_{k+1} such that

$$m_k r_{k+1}^{\underline{d}} = r_k^{\underline{d}} \quad \text{and} \quad m_k R_{k+1}^{\overline{d}} = R_k^{\overline{d}}. \tag{2.2}$$

Set $n_k = m_1 \cdot \ldots \cdot m_k$. By (2.2) we have

$$r_k^{\underline{d}} = R_k^{\overline{d}} \quad \text{and} \quad n_k R_{k+1}^{\overline{d}} = n_k r_{k+1}^{\underline{d}} = 1. \tag{2.3}$$

Further, (2.2),(2.3), and (2.1) imply that

$$R_{k+1} = R_k m_k^{-1/\overline{d}} = r_k^{\underline{d}/\overline{d}} m_k^{-1/\overline{d}} < r_k m_k^{-1/n}. \tag{2.4}$$

Define a hierarchy of cubes $Q_{k,j}$, $j = 1, \ldots, n_{k-1}$ of side-length r_k in the following way. First take $Q_{1,1} = [0,1]^n$. Divide $Q_{1,1}$ into m_1 subcubes with side-length $r_1 m_1^{-1/n}$. Using (2.4) we see that each of these $r_1 m_1^{-1/n}$-cubes contains a concentric cube of side-length R_2. Name these R_2-cubes $P_{2,i}$, $i = 1, \ldots, m_1$. Next take from inside of each cube $P_{2,i}$ a concentric cube of side-length r_2 and call these cubes $Q_{2,j}$, $j = 1, \ldots, n_1$. Continue by dividing each cube $Q_{2,j}$ into m_2 subcubes of side-length $r_2 m_2^{-1/n}$. Again by (2.4) each of these subcubes contains a concentric cube of side-length R_3. Let these R_3-cubes be $P_{3,i}$, $i = 1, \ldots, n_2$. Choose a concentric r_3-cube $Q_{3,j}$, $j = 1, \ldots, n_2$, from inside of each of the cubes $P_{3,i}$, $i = 1, \ldots, n_2$, and continue the construction. The measure μ is defined on the limiting set

$$\bigcap_{k=1}^{\infty} \bigcup_{j=1}^{n_{k-1}} Q_{k,j}$$

by a repeated subdivision such that

$$\mu(P_{k,j}) = \mu(Q_{k,j}) = 1/n_{k-1} = R_k^{\overline{d}} = r_k^{\underline{d}} \tag{2.5}$$

for all $k = 1, 2, \ldots$ and $j = 1, \ldots, n_{k-1}$. Translating (2.5) from cubes to balls gives (1) and (2) (for the details see [FM, Example 5.1].

Let $V \in G_{n,m}$. Fix a real number α with $0 \le \alpha \le m$ for the time being. In order to prove that $\overline{\dim}_B(\mathrm{proj}_V(\mathrm{spt}\,\mu)) \le \alpha$ we need to show that $N(\mathrm{proj}_V(\mathrm{spt}\,\mu), \varepsilon)\varepsilon^\alpha$ is bounded from above as ε tends to zero. Next we will find out what values of α are possible for the validity of this condition.

Let $\varepsilon > 0$. Consider k such that $r_{k+1} < \varepsilon \le r_k$. For all $j = 1, \ldots, n_{k-1}$ the projection $\mathrm{proj}_V(Q_{k,j})$ can be covered with $c(r_k/\varepsilon)^m$ cubes of side-length ε where $c = (2\sqrt{n})^m$. Thus the projection $\mathrm{proj}_V(\mathrm{spt}\,\mu)$ can be covered with $cn_{k-1}(r_k/\varepsilon)^m$ ε-cubes which gives by (2.3)

$$N(\mathrm{spt}\,\mu, \sqrt{n}\varepsilon)\varepsilon^\alpha \le cn_{k-1} r_k^m \varepsilon^{\alpha-m} = c r_k^{m-\underline{d}} \varepsilon^{\alpha-m}.$$

Hence $N(\mathrm{spt}\,\mu, \sqrt{n}\varepsilon)\varepsilon^\alpha \le c$ if

$$\varepsilon \ge r_k^{(m-\underline{d})/(m-\alpha)}. \tag{2.6}$$

On the other hand, for all $j = 1, \ldots, n_{k-1}$ the projection $\mathrm{proj}_V(\mathrm{spt}\,\mu \cap Q_{k,j})$ can be covered with the projections of the m_k cubes $Q_{k+1,l}$ with side-length $r_{k+1} < \varepsilon$ contained in $Q_{k,j}$. Using (2.3) and (2.2) we have

$$N(\mathrm{spt}\,\mu, \sqrt{n}\varepsilon)\varepsilon^\alpha \le cn_k \varepsilon^\alpha = c r_{k+1}^{-\underline{d}} \varepsilon^\alpha = c r_k^{-\underline{d}} m_k \varepsilon^\alpha.$$

If k is large enough then $m_k^{1/n} \le 2(m_k^{1/n} - 1)$ giving

$$N(\mathrm{spt}\,\mu, \sqrt{n}\varepsilon)\varepsilon^\alpha \le 2^n c r_k^{n(\underline{d}-\overline{d})/(n-\overline{d})-\underline{d}} \varepsilon^\alpha$$

by (2.1). This implies that $N(\mathrm{spt}\,\mu, \sqrt{n}\varepsilon)\varepsilon^\alpha \le 2^n c$ if

$$\varepsilon \le r_k^{\overline{d}(n-\underline{d})/(\alpha(n-\underline{d}))}. \tag{2.7}$$

Using (2.6) and (2.7) we have

$$N(\operatorname{spt}\mu, \sqrt{n}\varepsilon)\varepsilon^{\alpha} \le 2^n c$$

for all small ε when we make the choice $(m-\underline{d})/(m-\alpha) = (\overline{d}(n-\underline{d}))/(\alpha(n-\overline{d}))$ giving

$$\alpha = \frac{\overline{d}(1-\underline{d}/n)}{1+(1/m-1/n)\overline{d}-\underline{d}/m}.$$

This completes the proof. □

Remarks. 1) Let $q \ge -1$. Clearly by (1) and (2) $\underline{D}_q(\mu) = \underline{d}$ and $\overline{D}_q(\mu) = \overline{d}$. Further, by (3) we have

$$\overline{D}_q(\mu_V) \le \overline{\dim}_B(\operatorname{spt}\mu_V) \le \frac{\overline{D}_q(\mu)(1-\underline{D}_q(\mu)/n)}{1+(1/m-1/n)\overline{D}_q(\mu)-\underline{D}_q(\mu)/m} < \overline{D}_q(\mu)$$

for all $V \in G_{n,m}$. Here we used the fact that $\overline{D}_q(\mu) \le \overline{\dim}_B(\operatorname{spt}\mu)$ for all $q \ge -1$. This is easy to see using the original definition for the upper q-dimension given by Grassberger in [G]. This definition gives a nonincreasing function of q (see [B, (15)]) which equals $\overline{\dim}_B(\operatorname{spt}\mu)$ for $q=-1$.

Note that the relation between $\overline{D}_q$ and $\overline{\dim}_B$ gives that $\overline{D}_q(\mu) \le n$ for all $q \ge -1$.

2) The inequality (3) in Example 1 is also true if we replace proj_V by a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$, since the construction in [FM, Example 5.1] uses only the properties that the range of the map is m-dimensional and that the image of a cube of side length r is inside a cube of side length cr for some constant c.

3) In [FM] this example was constructed for showing that the lower bound for packing dimensions of projections obtained by Falconer and Mattila is the best possible one. We will later use this example for the same purpose in the upper correlation dimension case.

Now we give a useful characterization of the upper q-dimension of μ in terms of the convolutions mentioned in the introduction. For all $k = 1, 2, \ldots$ we define

$$\dim_q^k(\mu) = \sup\{s \ge 0 \mid J_{s,q}^k(\mu) < \infty\} = \inf\{s \ge 0 \mid J_{s,q}^k(\mu) = \infty\},$$

where for all $s \ge 0$

$$J_{s,q}^k(\mu) = \liminf_{r\to 0} r^{-sq} \int \Big(\int \min\{1, r^k|x-y|^{-k}\} d\mu(y) \Big)^q d\mu(x).$$

The following proposition is an analogue of [FH2, Corollary 3] and [HK, Proposition 2.1] for the upper q-dimension.

Proposition 2. *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. Then for all $q > 0$*

$$\overline{D}_q(\mu) = \dim_q^n(\mu).$$

Proof. Let $s > t > \overline{D}_q(\mu)$. Then $K_q(\mu, r) > r^{tq}$ for all small $r > 0$. Thus

$$\begin{aligned} J_{s,q}^n(\mu) &= \liminf_{r\to 0} r^{-sq} \int \Big(\int \min\{1, r^n|x-y|^{-n}\} d\mu(y) \Big)^q d\mu(x) \\ &\ge \liminf_{r\to 0} r^{-sq} K_q(\mu, r) \ge \liminf_{r\to 0} r^{q(t-s)} = \infty, \end{aligned}$$

giving $\overline{D}_q(\mu) \geq \dim_q^n(\mu)$.

On the other hand, let $\overline{D}_q(\mu) > t > s$. Then there is a sequence (r_k) tending to zero such that $K_q(\mu, r_k) \leq r_k^{qt}$ for all k. Let $0 < \varepsilon < 1$, $0 < a < 1$, and $0 < \delta < 1$ be such that

$$t - s - (1-a)n(1+\varepsilon) > \delta \text{ and } t - s - n\varepsilon > \delta.$$

Let k be large enough such that $r_k < \rho_1$ and $\log(1/r_k) \leq r_k^{-\delta}$, where ρ_1 and C_1 are as in Lemma 14. Let us first assume that $0 < q \leq 1$. Using Lemma 14, (4.2), and the fact that $(a+b)^q \leq a^q + b^q$ for all $a, b \geq 0$ and $0 < q \leq 1$, we obtain

$$\begin{aligned}
&\int \Big(\int \min\{1, r_k^n |x-y|^{-n}\} d\mu(y) \Big)^q d\mu(x) \leq 2C_1 (\log \tfrac{1}{r_k})^q \int_{E_{r_k}} g_{r_k}(x)^q d\mu(x) \\
&\leq 2C_1 C^q (\log \tfrac{1}{r_k})^q \int \big(\mu(B(x, r_k))^q r_k^{q(a-1)n(1+\varepsilon)} + \mu(B(x, r_k))^q r_k^{-qn\varepsilon} \\
&\qquad\qquad + r_k^{qn} \big) d\mu(x) \\
&\leq 2C_1 C^q (\log \tfrac{1}{r_k})^q \big(K_q(\mu, r_k) r_k^{q(a-1)n(1+\varepsilon)} + K_q(\mu, r_k) r_k^{-qn\varepsilon} + r_k^{qn} \big) \\
&\leq 2C_1 C^q (r_k^{q(t-(1-a)n(1+\varepsilon)-\delta)} + r_k^{q(t-n\varepsilon-\delta)} + r_k^{q(n-\delta)}).
\end{aligned}$$

Thus the choice of ε, a, and δ implies that

$$J_{s,q}^n(\mu) = \liminf_{r \to 0} r^{-sq} \int \Big(\int \min\{1, r^n |x-y|^{-n}\} d\mu(y) \Big)^q d\mu(x) = 0$$

which completes the proof in the case $0 < q \leq 1$.

Now let $q > 1$. Using the triangle inequality for the L^q-norm of the function $g_{r_k}|_{E_{r_k}}$, we obtain like above that

$$\liminf_{r \to 0} r^{-s} \Big(\int \Big(\int \min\{1, r^n |x-y|^{-n}\} d\mu(y) \Big)^q d\mu(x) \Big)^{\frac{1}{q}} = 0.$$

Thus $J_{s,q}^n(\mu) = 0$ also in this case. $\square$

Using Proposition 2 we obtain:

Theorem 3. *Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$. Let $0 < q \leq 1$ and let m be an integer with $0 < m < n$. Then for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ we have*

$$\overline{D}_q(\mu_V) = \dim_q^m(\mu).$$

Proof. Consider $V \in G_{n,m}$. Let $s \geq 0$. Since

$$\begin{aligned}
J_{s,q}^m(\mu) &= \liminf_{r \to 0} r^{-sq} \int \Big(\int \min\{1, r^m |x-y|^{-m}\} d\mu(y) \Big)^q d\mu(x) \\
&\leq \liminf_{r \to 0} r^{-sq} \int \Big(\int \min\{1, r^m |x-y|^{-m}\} d\mu_V(y) \Big)^q d\mu_V(x) = J_{s,q}^m(\mu_V),
\end{aligned}$$

we have by Proposition 2 that $\overline{D}_q(\mu_V) = \dim_q^m(\mu_V) \leq \dim_q^m(\mu)$.

Let $s < \dim_q^m(\mu)$ and $r > 0$. Using Jensen's inequality (see for example [Si, Theorem I.3.4]) for the concave function $x \mapsto x^q$, Fubini's theorem, and [Mat3, Lemma 3.11], we obtain

$$\begin{aligned}
&\iint \mu_V(B(\mathrm{proj}_V(x), r))^q d\gamma_{n,m}(V) d\mu(x) \\
&\qquad \le \int \Big(\int \mu(\{y \in \mathbb{R}^n \mid |\mathrm{proj}_V(x-y)| \le r\}) d\gamma_{n,m}(V) \Big)^q d\mu(x) \\
&\qquad = \int \Big(\int \gamma_{n,m}(\{V \in G_{n,m} \mid |\mathrm{proj}_V(x-y)| \le r\}) d\mu(y) \Big)^q d\mu(x) \\
&\qquad \le c \int \Big(\int \min\{1, r^m |x-y|^{-m}\} d\mu(y) \Big)^q d\mu(x)
\end{aligned}$$

where c does not depend on r. By Fubini's theorem

$$\liminf_{r \to 0} r^{-sq} \int K_q(\mu_V, r) d\gamma_{n,m}(V) < \infty.$$

Thus Fatou's lemma implies that for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ we have

$$\liminf_{r \to 0} r^{-sq} K_q(\mu_V, r) < \infty$$

giving $\overline{D}_q(\mu_V) \ge s$. The claim follows by taking a sequence (s_k) tending to $\dim_q^m(\mu)$ from below. □

The result of Theorem 3 can be generalized to hold for almost all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. This follows directly from Theorem 3 using the following arguments which were pointed out to us by a referee. It is clearly enough to study linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ with full rank. Such a map L can be uniquely decomposed as the projection onto the m-dimensional orthogonal complement of the kernel of L and a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^m$ with full rank. Since linear isomorphisms do not change the dimension, we obtain by Fubini's theorem that $\overline{D}_q(L_*\mu) = \dim_q^m(\mu)$ for $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$. We give also an alternative proof (see Theorem 5) which is useful when we extend this resut further to continuously differentiable functions (see Theorem 6). For this purpose we need the following technical lemma. For an analogous result in the lower q-dimension case see [SY, Lemma 2.6].

Lemma 4. *Let k and m be integers with $0 < m \le k$. Let $b \in \mathbb{R}^m$ and $A : \mathbb{R}^k \to \mathbb{R}^m$ be a linear map such that $A(Q_k(1)) \supset Q_m(\delta)$ for some $\delta > 0$ where $Q_k(1) = \{z \in \mathbb{R}^k \mid |z_i| \le 1$ for all $i = 1, \ldots, k\}$ and $Q_m(\delta) = \{z \in \mathbb{R}^m \mid |z_i| \le \delta$ for all $i = 1, \ldots, m\}$. Then for $\varepsilon > 0$ there exists a constant M depending on ε such that for all $0 < r < 1$*

$$\int_{Q_k(1)} \min\{1, r^m |A(z) + b|^{-m}\} d\mathcal{L}^k(z) \le M r^{-\varepsilon} \min\{1, r^m \delta^{-m}\}.$$

Proof. It is enough to prove the case $b = 0$ since

$$\int_{Q_k(1)} \min\{1, r^m |A(z) + b|^{-m}\} d\mathcal{L}^k(z) \le \int_{Q_k(1)} \min\{1, r^m |A(z)|^{-m}\} d\mathcal{L}^k(z)$$

for all $b \in \mathbb{R}^m$. We may assume that $r < \delta$, since for all $r \geq \delta$ we have

$$\int_{Q_k(1)} \min\{1, r^m |A(z)|^{-m}\} d\mathcal{L}^k(z) \leq \int_{Q_k(1)} 1 \, d\mathcal{L}^k(z) = 2^k \leq 2^k r^{-\varepsilon} \min\{1, r^m \delta^{-m}\}.$$

We use the notations $\mathcal{O}(k)$ and $\mathcal{O}(m)$ for the orthogonal groups of $\mathbb{R}^k$ and $\mathbb{R}^m$. Let $A = V\Sigma U$ be the singular value decomposition of A (see [St, Theorem 6.6.1]). Here $U \in \mathcal{O}(k)$, $\Sigma = \operatorname{diag}(a_1, a_2, \dots, a_m, 0, \dots, 0)$ is a diagonal $k \times k$-matrix with $a_1 \geq a_2 \geq \dots \geq a_m \geq 0$, and $V = (V_1, V_2)$ is an $m \times k$-matrix such that $V_1 \in \mathcal{O}(m)$ and V_2 is chosen such that V is unitary, that is, $V^T V = \mathrm{Id}_k$ and $VV^T = \mathrm{Id}_m$. Since $A(Q_k(1)) \supset Q_m(\delta)$, we have that $a_i \geq \delta/\sqrt{k}$ for all $i = 1, \dots, m$.

Let $\{e_1, \dots, e_k\}$ be an orthonormal basis of $\mathbb{R}^k$ such that the m-plane spanned by $\{e_1, \dots, e_m\}$ is mapped by U onto the m-plane spanned by the first m vectors of the standard basis of $\mathbb{R}^k$. We divide $Q_k(1)$ (which is defined using the standard basis) into two (not necessarily disjoint) parts S_1 and S_2 where

$$S_1 = \Big\{ z \in Q_k(1) \mid z = \sum_{i=1}^k z_i e_i \text{ and } |z_i| \leq \frac{r}{\delta} \text{ for all } i = 1, \dots, m \Big\}$$

and

$$S_2 = \Big\{ z \in Q_k(1) \mid z = \sum_{i=1}^k z_i e_i \text{ and } \sum_{i=1}^m z_i^2 \geq \frac{1}{k} \Big(\frac{r}{a_1}\Big)^2 \Big\}.$$

Note that there are no restrictions (other than those implied by the fact that $z \in Q_k(1)$) for the coordinates $z_{m+1}, \dots, z_k$ in either of these sets. Now

$$\int_{Q_k(1)} \min\{1, r^m |A(z)|^{-m}\} d\mathcal{L}^k(z) \leq \int_{S_1} 1 \, d\mathcal{L}^k(z) + \int_{S_2} r^m |A(z)|^{-m} d\mathcal{L}^k(z) = I_1 + I_2.$$

Further,

$$I_1 \leq (2r)^m \delta^{-m} (2\sqrt{k})^{k-m} \leq (2\sqrt{k})^k r^{-\varepsilon} r^m \delta^{-m}.$$

Since $|A(z)| \geq \delta \sqrt{\sum_{i=1}^m z_i^2} / \sqrt{k}$, we have

$$\begin{aligned} I_2 &\leq 2^{-m} (2\sqrt{k})^k \alpha(m) r^m \delta^{-m} \int_{\frac{r}{\sqrt{k} a_1}}^{\sqrt{k}} \frac{du}{u} \\ &= 2^{-m} (2\sqrt{k})^k \alpha(m) r^m \delta^{-m} (2 \log \sqrt{k} + \log a_1 - \log r) \leq M' r^{-\varepsilon} r^m \delta^{-m}, \end{aligned}$$

where $\alpha(m)$ is the volume of the $(m-1)$-dimensional unit sphere. □

Theorem 5. *Let $0 < q \leq 1$ and let n and m be integers with $0 < m < n$. Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. Then for $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$ we have*

$$\overline{D}_q(L_* \mu) = \dim_q^m(\mu).$$

Proof. The fact that $|L(x)| \leq \|L\| |x|$ implies that the inequality $\overline{D}_q(L_* \mu) \leq \dim_q^m(\mu)$ holds for all L (see the proof of Theorem 3).

Since $\overline{D}_q(L_*\mu) = \overline{D}_q((\alpha L)_*\mu)$ for all $\alpha > 0$ it is enough to prove the claim for $\mathcal{L}^{nm}$-almost all linear maps with $|L_{ij}| \leq 1$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ where L_{ij} are the elements of the matrix representing L (using some fixed orthonormal bases in $\mathbb{R}^n$ and $\mathbb{R}^m$).

Like in the proof of [HK, Proposition 3.2] we will apply Lemma 4 to the linear map $A_x : \mathbb{R}^{nm} \to \mathbb{R}^m$ defined by $A_x(L) = L(x)$ for some $x \in \operatorname{spt}\mu$. Note that the largest singular value of A_x depends on the choice of x. However, since $A_{\alpha x+\beta y} = \alpha A_x + \beta A_y$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ and the support of μ is bounded, we have that $\sup_{x\in\operatorname{spt}\mu} \|A_x\| < \infty$, and so the largest singular value of A_x is uniformly bounded for $x \in \operatorname{spt}\mu$. Thus the constant M in Lemma 4 (which depends on the largest singular value) can be chosen to be independent of x. Further, for each $x \in \mathbb{R}^n$ there exists $1 \leq i \leq n$ such that $|x_i| \geq |x|/\sqrt{n}$, which implies that the A_x-image of those $L \in Q_{nm}(1)$ for which $L_{jk} = 0$ for all (j, k) with $k \neq i$ contains the cube $Q_m(|x|/\sqrt{n})$.

Let $s < t < \dim_q^m(\mu)$ and let $\varepsilon = t - s$. Using Fatou's lemma, Fubini's theorem, Jensen's inequality, and Lemma 4, we obtain

$$
\begin{aligned}
&\int_{Q_{nm}(1)} J^m_{s,q}(L_*\mu)\,d\mathcal{L}^{nm}(L) \\
&\leq \liminf_{r\to 0} r^{-sq} \int \left(\int \int_{Q_{nm}(1)} \min\{1, r^m |L(x) - L(y)|^{-m}\}\,d\mathcal{L}^{nm}(L)\,d\mu(y) \right)^q d\mu(x) \\
&\leq M^q \liminf_{r\to 0} r^{-(s+\varepsilon)q} \int \left(\int \min\{1, (\sqrt{n}r)^m |x - y|^{-m}\}\,d\mu(y) \right)^q d\mu(x) \\
&= cJ^m_{t,q}(\mu).
\end{aligned}
\tag{2.8}
$$

Thus by Proposition 2 we have $\overline{D}_q(L_*\mu) = \dim_q^m(L_*\mu) \geq s$ for $\mathcal{L}^{nm}$-almost all $L \in Q_{nm}(1)$. Taking a sequence (s_k) tending to $\dim_q^m(\mu)$ from below implies the claim. $\square$

In order to prove the analogue of Theorem 5 for continuously differentiable functions we use the notation of prevalence from [HSY]. Fix two bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Each linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ can be given by an $m \times n$-matrix. Let $\mathcal{L}_1(\mathbb{R}^n, \mathbb{R}^m)$ be the set of linear transformations with the absolute values of the entries not greater than one. Denote by $\mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$. A set $P \subset \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ is called *prevalent* if for all $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ we have $f + L \in P$ for $\mathcal{L}^{nm}$-almost all $L \in \mathcal{L}_1(\mathbb{R}^n, \mathbb{R}^m)$.

Theorem 6. *Under the assumptions of Theorem 5 we have*

$$\overline{D}_q(f_*\mu) = \dim_q^m(\mu)$$

for a prevalent set of functions $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$.

Proof. For fixed $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ the proof of Theorem 5 gives that

$$\overline{D}_q((f + L)_*\mu) = \dim_q^m(\mu)$$

for $\mathcal{L}^{nm}$-amost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$. Indeed, the inequality

$$|L(x) + f(x) - L(y) - f(y)| \leq (\|L\| + \sup_z |f'(z)|)|x - y|$$

and the facts that μ has compact support and $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ imply that $\overline{D}_q((f+L)_*\mu) \le \dim_q^m(\mu)$ for all L. For the opposite inequality we use in (2.8) Lemma 4 with $b = f(x) - f(y)$. $\square$

Remarks. 1) The analogy of Theorem 6 for the lower q-dimension can be found from [HK, Theorem 3.1].

2) For $q > 1$ we do not know whether the upper q-dimensions of projections onto almost all planes are equal. However, Example 1 shows that the upper q-dimension is not preserved under projections or linear maps.

3. The upper correlation dimension

By Theorem 3 (Theorem 5) the upper correlation dimension $\overline{D}_1$ of projections (images under linear maps) of a Borel probability measure is constant "almost surely". In this section we address the problem of finding out how small this constant can be. For this purpose we use the methods from [FM].

Let m and n be integers with $0 < m < n$. Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. Let δ, $\underline{d}$, and t be real numbers with $0 < \delta < \underline{d} < \min\{t, m\}$ and $t < n$. Consider real numbers a and ε such that $(n-t)/(n-\underline{d}) < a < 1$, $0 < \varepsilon < \delta/n$, and

$$n(1+\varepsilon)(1-a) < \delta. \tag{3.1}$$

For all r with $0 < r < 1$ we set

$$h_r := r^{(t+(m-\underline{d})(n-t)/(n-\underline{d}))/m} \le r \le r^a \le r^{(n-t)/(n-\underline{d})} =: R_r. \tag{3.2}$$

We proceed as in Section 4. For all $0 < r < 1$ and $x \in \mathbb{R}^n$ define

$$G_r(x) := \int_{\{y \mid |x-y| \le R_r\}} \min\{1, h_r^m |x-y|^{-m}\} d\mu(y).$$

Let $K = 4\max\{4^{n(1+\varepsilon)}, m4^{n(1+\varepsilon)}/(n(1+\varepsilon)-m)\}$. A point $x \in \operatorname{spt}\mu$ is called (r,K,m)-regular with respect to μ if

$$G_r(x) \le K r^{-\delta} \mu(B(x,r)). \tag{3.3}$$

If (3.3) is not satisfied, we say that $x \in \operatorname{spt}\mu$ is (r,K,m)-irregular with respect to μ. The $(r,K/2,m)$-regularity and the $(r,K/2,m)$-irregularity with respect to μ are defined similarly. Note that these definitions depend also on δ, $\underline{d}$, t, a, and ε, but below these numbers are fixed.

Lemma 7. *Let δ, $\underline{d}$, t, a, and ε be as above. Then there exists a number c_0 depending only on a, ε, and n such that for all Borel probability measures μ on $\mathbb{R}^n$ and for all $r_0 \le 1/2$ we have*

$$\mu(\{x \in \operatorname{spt}\mu \mid x \textit{ is } (r,K/2,m)\textit{-irregular with respect to } \mu\}) \le 2c_0 r_0^{n\varepsilon(1-a)}$$

for all r with $0 < r < r_0$.

Proof. Let c_0 be as in Lemma 12. If $x \in \operatorname{spt}\mu$ is $(r,K/2,m)$-irregular with respect to μ, then there is u with $r^a \le u \le 1$ such that

$$\mu(B(x,u)) > \frac{1}{2}\Big(\frac{4u}{r}\Big)^{n(1+\varepsilon)} \mu(B(x,r)). \tag{3.4}$$

In fact, if this is not the case, then integrating by parts and using (3.1) and (3.2) we have

$$
\begin{aligned}
G_r(x) &= mh_r^m \int_{h_r}^{R_r} u^{-m-1}\mu(B(x,u))du + h_r^m R_r^{-m}\mu(B(x,R_r)) \\
&\le h_r^m \mu(B(x,r))\Big(m\int_{h_r}^{r} u^{-m-1}du + \frac{1}{2}4^{n(1+\varepsilon)}mr^{n(1+\varepsilon)(a-1)}\int_r^{r^a} u^{-m-1}du \\
&\quad + \frac{1}{2}4^{n(1+\varepsilon)}mr^{-n(1+\varepsilon)}\int_{r^a}^{R_r} u^{n(1+\varepsilon)-m-1}du + \frac{1}{2}4^{n(1+\varepsilon)}r^{-n(1+\varepsilon)}R_r^{n(1+\varepsilon)-m}\Big) \\
&\le h_r^m \mu(B(x,r))\Big(h_r^{-m} + \frac{1}{2}4^{n(1+\varepsilon)}r^{n(1+\varepsilon)(a-1)-m} \\
&\quad + \frac{1}{2}4^{n(1+\varepsilon)}m(n(1+\varepsilon)-m)^{-1}r^{-n(1+\varepsilon)}R_r^{n(1+\varepsilon)-m} \\
&\quad + \frac{1}{2}4^{n(1+\varepsilon)}r^{-n(1+\varepsilon)}R_r^{n(1+\varepsilon)-m}\Big) \\
&\le \mu(B(x,r))\big(1 + \frac{K}{8}h_r^m r^{-m-\delta} + \frac{K}{4}h_r^m r^{-n(1+\varepsilon)}R_r^{n(1+\varepsilon)-m}\big) \\
&\le \frac{K}{2}r^{-\delta}\mu(B(x,r)), \qquad (3.5)
\end{aligned}
$$

which is a contradiction. Using (3.4) the claim follows by Lemma 12. □

Lemma 8. *Let δ, $\underline{d}$, t, a, and ε be as above. Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. Then there are $\rho_2 > 0$ and $C_2 > 0$ such that for all $0 < r < \rho_2$ we have*

$$
\int_{\widetilde{F}_r} G_r(x)d\mu(x) \le C_2 \int_{\widetilde{E}_r} \mu(B(x,r))d\mu(x),
$$

where $\widetilde{E}_r = \{x \in \operatorname{spt}\mu \mid x$ is (r,K,m)-regular with respect to $\mu\}$ and $\widetilde{F}_r = \operatorname{spt}\mu \setminus \widetilde{E}_r$.

Proof. This can be proved as Lemma 14 using Lemma 7. Instead of (4.3) we use the following inequality for the function G_r. For $r > 0$, let L_{R_r} be the smallest integer such that $L_{R_r} > R_r/r$. In the same way as in (4.3) we obtain using (3.2)

$$
\begin{aligned}
G_r(x) &\le c' \sum_{l=1}^{L_{R_r}} \Big(\frac{lr}{r}\Big)^{n-1}\Big(\frac{h_r}{lr}\Big)^m s_r = c' s_r r^{t-m+(m-\underline{d})(n-t)/(n-\underline{d})} \sum_{l=1}^{L_{R_r}} l^{n-1-m} \\
&\le c' s_r r^{t-n+(m-\underline{d})(n-t)/(n-\underline{d})} R_r^{n-m} = c' s_r. \qquad (3.6)
\end{aligned}
$$

Here we do not have the logarithmic correction as in (4.3) since we consider the truncated kernel.

For all $l = 1,2,\dots$ and $j = 1,2,\dots$ we define the sets $\widetilde{E}_r^l$ and $\widetilde{F}_{rj}^l$ by replacing (r,C)-regularity and (r,C)-irregularity with (r,K,m)-regularity and (r,K,m)-irregularity and the function g_r with G_r in the definitions of E_r^l and F_{rj}^l (see Section 4). Similarly as in (4.7) we see that for all $l = 1,2\dots$

$$
\int_{\widetilde{F}_r^l} G_r(x)d\mu(x) \le 4c'\Big(\sum_{k=1}^{l} 2^{k-l}\int_{\widetilde{E}_r^k} \mu(B(x,r))d\mu(x)\Big),
$$

where $\widetilde{F}_r^l = \cup_{j=1}^\infty \widetilde{F}_{rj}^l$. The claim follows as in Lemma 14. □

For analogues of the two following results in the packing dimension case see [FM, Proposition 2.6 and Theorem 3.3].

Proposition 9. *Let m and n be integers with $0 < m < n$. Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. Assume that $\underline{d}$ and $\overline{d}$ are real numbers such that either $0 < \underline{d} < \underline{D}_1(\mu)$ or $\underline{d} = \underline{D}_1(\mu) = 0 < \overline{d}$, and $\underline{d} \le \overline{d} < \overline{D}_1(\mu)$. If $\underline{d} < m$, then there is a positive real number C_3 and a sequence (h_k) of positive real numbers tending to zero such that for all k*

$$h_k^m \int \int_{h_k}^\infty u^{-m-1} \mu(B(x,u)) du d\mu(x) \le C_3 h_k^{m\overline{d}/(\overline{d}+(m-\underline{d})(n-\overline{d})/(n-\underline{d}))}.$$

Proof. Let $0 < \delta < \underline{d}$ and $\overline{d} < t < \overline{D}_1(\mu)$. Since $\underline{d} < \underline{D}_1(\mu)$ or $\underline{d} = 0$, there is a constant c such that for all $r > 0$

$$K_1(\mu, r) \le c r^{\underline{d}}. \tag{3.7}$$

Further, since $t < \overline{D}_1(\mu)$, there is a sequence (r_k) of positive real numbers tending to zero such that for all k

$$K_1(\mu, r_k) \le r_k^t. \tag{3.8}$$

Let a and ε be real numbers such that $(n-t)/(n-\underline{d}) < a < 1$, $0 < \varepsilon < \delta/n$, and (3.1) holds. Let k be large enough such that $r_k < \rho_2$, where ρ_2 is as in Lemma 8. Let $h_k := h_{r_k}$ and $R_k := R_{r_k}$ be as in (3.2). Using Fubini's theorem, Lemma 8, (3.3), (3.7), and (3.8), we have

$$\begin{aligned} m h_k^m \int \int_{h_k}^\infty & u^{-m-1} \mu(B(x,u)) du d\mu(x) \\ & \le \int G_{r_k}(x) d\mu(x) + m h_k^m \int_{R_k}^\infty u^{-m-1} K_1(\mu,u) du \\ & \le 2K r_k^{-\delta} K_1(\mu, r_k) + c m h_k^m \int_{R_k}^\infty u^{\underline{d}-m-1} du \\ & \le 2K r_k^{t-\delta} + cm(m-\underline{d})^{-1} h_k^m R_k^{\underline{d}-m} \\ & \le C_3 r_k^{t-\delta} \le C_3 h_k^{m(t-\delta)/(t+(m-\underline{d})(n-t)/(n-\underline{d}))}, \end{aligned} \tag{3.9}$$

which implies the claim when we choose δ small enough. □

Theorem 10. *Let μ be a Borel probability measure on $\mathbb{R}^n$. If $\underline{D}_1(\mu) \ge m$, then*

$$\overline{D}_1(\mu_V) = m \tag{3.10}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. If $\underline{D}_1(\mu) < m$, then

$$\overline{D}_1(\mu_V) \ge \frac{\overline{D}_1(\mu)(1 - \underline{D}_1(\mu)/n)}{1 + (1/m - 1/n)\overline{D}_1(\mu) - \underline{D}_1(\mu)/m} \tag{3.11}$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$. The formulae (3.10) *and* (3.11) *are also true for $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$, when we replace μ_V by $L_*\mu$.*

Proof. If $\underline{D}_1(\mu) \geq m$ or $\overline{D}_1(\mu) = \underline{D}_1(\mu) \leq m$, the claim follows from the results for the lower correlation dimension (see [HK, Theorem 1.1]).

Assume that $\underline{D}_1(\mu) < \min\{m, \overline{D}_1(\mu)\}$ and $\overline{D}_1(\mu) > 0$. Let $\underline{d}$ and $\overline{d}$ be such that $0 < \underline{d} < \underline{D}_1(\mu)$ (if $\underline{D}_1(\mu) = 0$ we take $\underline{d} = 0$) and $\underline{d} < \overline{d} < \overline{D}_1(\mu)$. By Fatou's lemma, Fubini's theorem, [FM, Lemma 3.2], and Proposition 9 we have

$$\int \liminf_{h\to 0} h^{-m\overline{d}/(\overline{d}+(m-\underline{d})(n-\overline{d})/(n-\underline{d}))} \int \mu_V(B(x,h))d\mu_V(x)d\gamma_{n,m}(V)$$
$$\leq \liminf_{h\to 0} h^{-m\overline{d}/(\overline{d}+(m-\underline{d})(n-\overline{d})/(n-\underline{d}))} ch^m \int\int_h^\infty u^{-m-1}\mu(B(x,u))dud\mu(x) < \infty.$$

Thus for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ we have

$$\liminf_{h\to 0} h^{-m\overline{d}/(\overline{d}+(m-\underline{d})(n-\overline{d})/(n-\underline{d}))} K_1(\mu_V, h) < \infty,$$

giving

$$\overline{D}_1(\mu_V) \geq \frac{\overline{d}(1-\underline{d}/n)}{1+(1/m-1/n)\overline{d}-\underline{d}/m}.$$

The claim follows by taking sequences $(\underline{d}_i)$ tending to $\underline{D}_1(\mu)$ and $(\overline{d}_i)$ tending to $\overline{D}_1(\mu)$.

By Theorems 3 and 5 the claim is also true for $\mathcal{L}^{nm}$-almost all linear maps $L : \mathbb{R}^n \to \mathbb{R}^m$. $\square$

Corollary 11. *Let μ be a Borel probability measure on $\mathbb{R}^n$. If $\underline{D}_1(\mu) \geq m$, then*

$$\overline{D}_1(f_*\mu) = m \tag{3.12}$$

for a prevalent set of functions $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$. If $\underline{D}_1(\mu) < m$, then

$$\overline{D}_1(f_*\mu) \geq \frac{\overline{D}_1(\mu)(1-\underline{D}_1(\mu)/n)}{1+(1/m-1/n)\overline{D}_1(\mu)-\underline{D}_1(\mu)/m} \tag{3.13}$$

for a prevalent set of functions $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$.

Remarks. 1) The lower bound given in Theorems 10 and 11 is the best possible one. This follows from Example 1.

2) After this work was completed, we learnt about a paper by Falconer and O'Neil [FO], where they study, among other things, the projectional properties of generalized q-dimension using appropriately defined convolution kernels. In particular, they proved the inequality

$$\left(\int_r^\infty u^{-m-1}f(u)du\right)^q \leq cr^{-\varepsilon}\int_r^\infty u^{-mq-1}f(u)^q du,$$

where f is non-decreasing and constant for $u > R$ for some R. Using this in (3.9), we can show that the analogues of Theorems 10 and 11 hold also for $0 < q < 1$. Again Example 1 shows that the corresponding lower bound is the best possible one.

3) The examples of [HK, Section 5.2] show that (3.11) is not true for $q > 1$.

4. Appendix: Measures of balls and convolutions

In this section we prove a technical lemma concerning convolutions which we use to characterize the upper q-dimension. For this purpose we need the following result on the behaviour of measures of balls from [FM].

Lemma 12. *Let $0 < a < 1$ and $\varepsilon > 0$. There exists a number c_0, depending only on a, ε, and n, such that for every Borel probability measure μ on $\mathbb{R}^n$ and for all $r_0 \leq 1/2$ we have*

$$\mu\Big(\Big\{x \in \mathbb{R}^n \mid \text{there are } r \text{ and } u \text{ with } 0 < r < r_0 \text{ and } r^a \leq u \leq 1 \text{ such that}$$
$$\mu(B(x,u)) > \frac{1}{2}\Big(\frac{4u}{r}\Big)^{n(1+\varepsilon)}\mu(B(x,r))\Big\}\Big) \leq 2c_0 r_0^{n\varepsilon(1-a)}.$$

Proof. See [FM, Lemma 2.2]. □

Let μ be a Borel probability measure on $\mathbb{R}^n$. Let $0 < a < 1$ and $0 < \varepsilon < 1$. Consider $x \in \mathbb{R}^n$ and $r > 0$ such that

$$\mu(B(x,u)) \leq \Big(\frac{4u}{r}\Big)^{n(1+\varepsilon)}\mu(B(x,r)) \tag{4.1}$$

for all u with $r^a \leq u \leq 1$. Integrating by parts we have

$$\begin{aligned} g_r(x) &:= \int \min\{1, r^n|x-y|^{-n}\}d\mu(y) = nr^n\int_r^\infty u^{-n-1}\mu(B(x,u))du \\ &= nr^n\Big(\int_r^{r^a} u^{-n-1}\mu(B(x,u))du + \int_{r^a}^1 u^{-n-1}\mu(B(x,u))du \\ &\qquad + \int_1^\infty u^{-n-1}\mu(B(x,u))du\Big) \\ &\leq \mu(B(x,r^a)) + 4^{n(1+\varepsilon)}\varepsilon^{-1}\mu(B(x,r))r^{-n\varepsilon} + \mu(\mathbb{R}^n)r^n \\ &\leq C\big(\mu(B(x,r))r^{(a-1)n(1+\varepsilon)} + \mu(B(x,r))r^{-n\varepsilon} + r^n\big) \qquad (4.2) \end{aligned}$$

for $C = 4^{n(1+\varepsilon)}/\varepsilon$.

Note that by [FM, Corollary 2.3] for μ-almost all $x \in \mathbb{R}^n$ there is r_x such that for all $0 < r \leq r_x$ the inequality (4.1) is satisfied for all u with $r^a \leq u \leq 1$. However, we can not use this result because of the difficulties caused by the fact that r_x depends on x. In order to avoid these difficulties we prove two lemmas for which we need the following notation.

Let $0 < a < 1$ and $0 < \varepsilon < 1$. Let $r > 0$ and $C = 4^{n(1+\varepsilon)}/\varepsilon$. We denote by $\operatorname{spt}\mu$ the support of a Borel probability measure μ. A point $x \in \operatorname{spt}\mu$ is called (r, C)-regular with respect to μ if

$$\begin{aligned} g_r(x) &= \int \min\{1, r^n|x-y|^{-n}\}d\mu(y) \\ &\qquad \leq C\big(\mu(B(x,r))r^{(a-1)n(1+\varepsilon)} + \mu(B(x,r))r^{-n\varepsilon} + r^n\big). \end{aligned}$$

If $x \in \operatorname{spt}\mu$ is not (r, C)-regular with respect to μ we say that it is (r, C)-irregular with respect to μ. The $(r, C/2)$-regularity and the $(r, C/2)$-irregularity with respect to μ are defined in the same way. Note that these definitions depend also on a and ε. However, we will use them only for fixed a and ε.

Lemma 13. *Let $0 < a < 1$ and $0 < \varepsilon < 1$. There exists a number c_0, depending only on a, ε, and n, such that for every Borel probability measure μ on $\mathbb{R}^n$ and for all $r_0 \le 1/2$ we have*

$$\mu(\{x \in \operatorname{spt}\mu \mid x \text{ is } (r, C/2)\text{-irregular with respect to } \mu\}) \le 2c_0 r_0^{n\varepsilon(1-a)}$$

for all r with $0 < r < r_0$.

Proof. Let c_0 be as in Lemma 12. If $x \in \operatorname{spt}\mu$ is $(r, C/2)$-irregular with respect to μ, then integrating by parts as in (4.2) we see that

$$\mu(B(x,u)) > \frac{1}{2}\Big(\frac{4u}{r}\Big)^{n(1+\varepsilon)} \mu(B(x,r))$$

for some $r^a \le u \le 1$. This gives the claim by Lemma 12. □

Lemma 14. *Let $0 < a < 1$, $0 < \varepsilon < 1$, and $q > 0$. Let μ be a Borel probability measure on $\mathbb{R}^n$ with compact support. There exist $\rho_1 > 0$ and $C_1 > 0$ such that for all r with $0 < r < \rho_1$ we have*

$$\int_{F_r} g_r(x)^q d\mu(x) \le C_1(\log \tfrac{1}{r})^q \int_{E_r} g_r(x)^q d\mu(x),$$

where $E_r = \{x \in \operatorname{spt}\mu \mid x \text{ is } (r, C)\text{-regular with respect to } \mu\}$ and $F_r = \operatorname{spt}\mu \setminus E_r$.

Proof. The basic idea behind this proof is that the points $x \in \operatorname{spt}\mu$ for which $\mu(B(x,r))$ is "big" (for fixed r) are regular and a point $y \in \operatorname{spt}\mu$ can be irregular only if there are points $x \in \operatorname{spt}\mu$ such that $\mu(B(x,r)) > \mu(B(y,r))$. We will divide $\operatorname{spt}\mu$ into parts such that we can control those points which "mainly" cause the irregularity of a given point $y \in \operatorname{spt}\mu$.

Let $0 < r_0 \le 1/2$ be such that $2c_0 r_0^{n\varepsilon(1-a)} \le 1/2$, where c_0 is as in Lemma 13. Let $0 < r < r_0$. Define

$$s_r = \sup_{x \in \operatorname{spt}\mu} \mu(B(x,r)).$$

For each $x \in \operatorname{spt}\mu$ we can estimate $g_r(x)$ in the following way: First we cover $\operatorname{spt}\mu$ by balls with radius r and with centres at the distance lr from x, where $l = 1, \dots, L_r$ and L_r is the smallest integer with $L_r > \operatorname{diam}(\operatorname{spt}\mu)/r$. Then we cover each of these balls B with balls of radius r and centres in $B \cap \operatorname{spt}\mu$. Estimating the integrand in each of these balls, we obtain

$$g_r(x) \le c' \sum_{l=1}^{L_r} \Big(\frac{lr}{r}\Big)^{n-1} \Big(\frac{r}{lr}\Big)^n s_r \le c'' s_r \log \tfrac{1}{r}. \tag{4.3}$$

Here c' is a constant depending only on n and c'' is a constant depending only on n and the diameter of $\operatorname{spt}\mu$ denoted by $\operatorname{diam}(\operatorname{spt}\mu)$. Note that the second step is not necessary here but will be needed in the later applications of (4.3).

For all Borel sets $X \subset \mathbb{R}^n$ define

$$J(X) = 2\int_X \min\{1, r^n |x-y|^{-n}\} d\mu(y).$$

Set

$$E_r^1 = \Big\{x \in \operatorname{spt}\mu \mid x \text{ is } (r,C)\text{-regular with respect to } \mu \text{ and } \mu(B(x,r)) \geq \frac{s_r}{2}\Big\}$$

and

$$F_{r1}^1 = \{x \in \operatorname{spt}\mu \mid x \text{ is } (r,C)\text{-irregular with respect to } \mu \text{ and } g_r(x) \leq J(E_r^1)\}.$$

Further, for all $j = 2, 3, \dots$ we define inductively sets F_{rj}^1 by

$$F_{rj}^1 = \Big\{x \in \operatorname{spt}\mu \mid x \text{ is } (r,C)\text{-irregular with respect to } \mu \text{ and } g_r(x) \leq J\Big(E_r^1 \cup \Big(\bigcup_{i=1}^{j-1} F_{ri}^1\Big)\Big)\Big\}.$$

Finally, we define

$$F_r^1 = \bigcup_{j=1}^{\infty} F_{rj}^1$$

and

$$A_r^1 = E_r^1 \cup F_r^1.$$

Since $x \mapsto \mu(B(x,r))$ is upper semicontinuous and $x \mapsto g_r(x)$ is continuous, these sets are Borel sets.

We first show that

$$\int_{F_r^1} g_r(x)^q d\mu(x) \leq (2c'' \log \tfrac{1}{r})^q \int_{E_r^1} g_r(x)^q d\mu(x). \tag{4.4}$$

We may assume that $\mu(A_r^1) > 0$. Let $x \in F_r^1$. Then $g_r(x) \leq J(A_r^1)$ which implies that x is $(r, C/2)$-irregular with respect to $\mu(A_r^1)^{-1}\mu|_{A_r^1}$. Here $\mu|_{A_r^1}$ is the restriction of μ to the set A_r^1. Using Lemma 13 we obtain that $\mu(F_r^1) \leq \mu(E_r^1)$ which implies together with (4.3) and with the fact that $g_r(x) \geq \mu(B(x,r)) \geq s_r/2$ for all $x \in E_r^1$ that

$$\int_{F_r^1} g_r(x)^q d\mu(x) \leq (c'' s_r \log \tfrac{1}{r})^q \mu(F_r^1) \leq (2c'' \log \tfrac{1}{r})^q \int_{E_r^1} g_r(x)^q d\mu(x).$$

Thus (4.4) holds.

For all $l = 2, 3, \dots$ we define

$$E_r^l = \{x \in \operatorname{spt}\mu \mid x \text{ is } (r,C)\text{-regular with respect to } \mu \text{ and } 2^{-l}s_r \leq \mu(B(x,r)) < 2^{-l+1}s_r\}$$

and

$$F_{r1}^l = \Big\{x \in \operatorname{spt}\mu \setminus \bigcup_{k=1}^{l-1} A_r^k \mid x \text{ is } (r,C)\text{-irregular with respect to } \mu \text{ and } g_r(x) \leq J\Big(\Big(\bigcup_{k=1}^{l-1} A_r^k\Big) \cup E_r^l\Big)\Big\}.$$

Further, for all $j = 2, 3, \ldots$ we define

$$F^l_{rj} = \Big\{x \in \operatorname{spt}\mu \setminus \bigcup_{k=1}^{l-1} A^k_r \mid x \text{ is } (r, C)\text{-irregular with respect to } \mu$$
$$\text{and } g_r(x) \le J\Big(\Big(\bigcup_{k=1}^{l-1} A^k_r\Big) \cup E^l_r \cup \Big(\bigcup_{i=1}^{j-1} F^l_{ri}\Big)\Big)\Big\}.$$

Let

$$F^l_r = \bigcup_{j=1}^{\infty} F^l_{rj}$$

and

$$A^l_r = E^l_r \cup F^l_r.$$

As before, these sets are Borel sets.

We now prove that for all $l = 1, 2, \ldots$ we have

$$\mu(B(x, r)) < 2^{-l} s_r \tag{4.5}$$

for all $x \in \operatorname{spt}\mu \setminus \cup_{k=1}^{l} A^k_r$. In order to prove this we proceed by induction on l. We may assume that $4c'' \log(1/r) \le Cr^{-n\varepsilon}$ by making r smaller if necessary. In the case $l = 1$, we obtain from (4.3) that if $x \in \operatorname{spt}\mu$ and $\mu(B(x, r)) \ge s_r/2$, then x is (r, C)-regular with respect to μ. Thus $x \in E^1_r \subset A^1_r$, and so (4.5) holds for $l = 1$.

We now assume that $\mu(B(x, r)) < 2^{-(l-1)} s_r$ for all $x \in \operatorname{spt}\mu \setminus \cup_{k=1}^{l-1} A^k_r$. Let $x \in \mathbb{R}^n$ with $\mu(B(x, r)) \ge 2^{-l} s_r$. We will prove that $x \in (\mathbb{R}^n \setminus \operatorname{spt}\mu) \cup (\cup_{k=1}^{l} A^k_r)$. For this purpose we may assume that $x \in \operatorname{spt}\mu \setminus \cup_{k=1}^{l-1} A^k_r$. If x is (r, C)-regular with respect to μ, then $x \in E^l_r \subset A^l_r$. If x is (r, C)-irregular with respect to μ, then $g_r(x) \le J(\operatorname{spt}\mu \setminus \cup_{k=1}^{l-1} A^k_r)$, since we may assume that $x \notin F^l_r$. Further, by the induction hypothesis $\operatorname{spt}\mu \setminus \cup_{k=1}^{l-1} A^k_r \subset \{y \in \operatorname{spt}\mu \mid \mu(B(y, r)) < 2^{-(l-1)} s_r\}$ and the same calculation as in (4.3) gives

$$\begin{aligned} g_r(x) &\le 2 \int_{\{y \in \operatorname{spt}\mu \mid \mu(B(y,r)) < 2^{-(l-1)} s_r\}} \min\{1, r^n |x - y|^{-n}\} d\mu(y) \\ &\le 4c'' 2^{-l} s_r \log \tfrac{1}{r} \le 4c'' \log \tfrac{1}{r} \mu(B(x, r)) \le Cr^{-n\varepsilon} \mu(B(x, r)), \end{aligned}$$

which is a contradiction since x is (r, C)-irregular with respect to μ. Thus (4.5) holds.

By (4.5) we have

$$F_r = \bigcup_{l=1}^{\infty} F^l_r, \tag{4.6}$$

since if $x \in F_r \setminus \cup_{l=1}^{\infty} F^l_r$, then (4.5) implies that $\mu(B(x, r)) = 0$, which is a contradiction since $x \in \operatorname{spt}\mu$.

It suffices to prove that for all $l = 1, 2, \ldots$

$$\int_{F^l_r} g_r(x)^q d\mu(x) \le (4c'' \log \tfrac{1}{r})^q \Big(\sum_{k=1}^{l} 2^{q(k-l)} \int_{E^k_r} g_r(x)^q d\mu(x)\Big). \tag{4.7}$$

In fact, since $E_r = \cup_{l=1}^{\infty} E_r^l$ where the Borel sets E_r^l are disjoint, we obtain the claim using (4.6) and (4.7) since

$$\int_{F_r} g_r(x)^q d\mu(x) \le (4c'' \log \tfrac{1}{r})^q \Big(\sum_{k=0}^{\infty} 2^{-qk}\Big)\Big(\sum_{l=1}^{\infty} \int_{E_r^l} g_r(x)^q d\mu(x)\Big)$$
$$\le C_1 (\log \tfrac{1}{r})^q \int_{E_r} g_r(x)^q d\mu(x).$$

If $l = 1$, then (4.7) follows from (4.4). Let $l \ge 2$. In order to prove (4.7) we may assume that $\mu(\cup_{k=1}^{l} A_r^k) > 0$. Let $x \in F_r^l$. Then for all $j = 1, 2, \ldots$ we have

$$g_r(x) \ge J\Big(\Big(\bigcup_{k=1}^{l-2} A_r^k\Big) \cup E_r^{l-1} \cup \Big(\bigcup_{i=1}^{j-1} F_{ri}^{l-1}\Big)\Big)$$

giving $g_r(x) \ge J(\cup_{k=1}^{l-1} A_r^k)$. Thus $g_r(x) \le J(\operatorname{spt} \mu \setminus \cup_{k=1}^{l-1} A_r^k)$. Using (4.5), the same calculation as in (4.3) gives that for all $x \in F_r^l$

$$g_r(x) \le 4c'' 2^{-l} s_r \log \tfrac{1}{r}. \tag{4.8}$$

Further, if $x \in \cup_{k=1}^{l} F_r^k$, then $g_r(x) \le J(\cup_{k=1}^{l} A_r^k)$ which implies that x is $(r, C/2)$-irregular with respect to $\nu = \mu(\cup_{k=1}^{l} A_r^k)^{-1} \mu|_{\cup_{k=1}^{l} A_r^k}$. Thus Lemma 13 gives that $\nu(\cup_{k=1}^{l} F_r^k) \le 1/2$, which implies

$$\nu(\bigcup_{k=1}^{l} E_r^k) = \nu(\bigcup_{k=1}^{l} A_r^k) - \nu(\bigcup_{k=1}^{l} F_r^k) \ge 1/2.$$

Thus $\mu(F_r^l) \le \mu(\cup_{k=1}^{l} E_r^k)$. Since $s_r \le 2^k g_r(x)$ for all $x \in E_r^k$ and $k = 1, \ldots, l$, we have by (4.8)

$$\int_{F_r^l} g_r(x)^q d\mu(x) \le (4c'' 2^{-l} \log \tfrac{1}{r})^q \int_{\cup_{k=1}^{l} E_r^k} s_r^q d\mu(x)$$
$$\le (4c'' \log \tfrac{1}{r})^q \sum_{k=1}^{l} 2^{q(k-l)} \int_{E_r^k} g_r(x)^q d\mu(x),$$

which completes the proof of (4.7). □

References

[B] Beck, C., *Upper and lower bounds on the Renyi dimensions and the uniformity of multifractals*, Phys. D **41** (1990), 67-78.

[F1] Falconer, K. J., *Fractal Geometry: Mathematical foundations and applications*, John Wiley & Sons, Chichester, 1990.

[F2] Falconer, K. J., *Techniques in Fractal Geometry*, John Wiley & Sons, Chichester, 1997.

[FH1] Falconer, K. J. and Howroyd, J. D., *Projection theorems for box and packing dimensions*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 287–295.

[FH2] Falconer, K. J. and Howroyd, J. D., *Packing dimensions of projections and dimension profiles*, Math. Proc. Cambridge Philos. Soc. **121** (1997), 269–286.

[FJ] Falconer, K. J. and Järvenpää, M., *Packing dimensions of sections of sets*, Math. Proc. Cambridge Philos. Soc. (to appear).

[FM] Falconer, K. J. and Mattila, P., *The packing dimension of projections and sections of measures*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 695–713.

[FO] Falconer, K. J. and O'Neil, T., *Convolutions and the geometry of multifractal measures*, Math. Nachr. (to appear).

[G] Grassberger, P., *Generalized dimensions of strange attractors*, Phys. Lett. A **97** (1983), 227–230.

[HP] Hentschel, H. G. E. and Procaccia, I., *The infinite number of generalized dimensions of fractals and strange attractors*, Phys. D **8** (1983), 435–444.

[HT] Hu, X. and Taylor, S. J., *Fractal properties of products and projections of measures in* $\mathbb{R}^d$, Math. Proc. Cambridge Philos. Soc. **115** (1994), 527–544.

[HK] Hunt, B. R. and Kaloshin, V. Yu., *How projections affect the dimension spectrum of fractal measures*, Nonlinearity **10** (1997), 1031–1046.

[HSY] Hunt, B. R., Sauer T., and Yorke J. A., *Prevalence: a translation-invariant almost every for infinite dimensional spaces*, Bull. Am. Math. Soc. **27** (1992), 217–238.

[J1] Järvenpää, M., *On the upper Minkowski dimension, the packing dimension, and orthogonal projections*, Ann. Acad. Sci. Fenn. Math. Diss. **99** (1994), 1–34.

[J2] Järvenpää, M., *Hausdorff and packing dimensions, intersection measures, and similarities*, Ann. Acad. Sci. Fenn. Math. (to appear).

[JM] Järvenpää, M. and Mattila P., *Hausdorff and packing dimension, and sections of measures*, Mathematika (to appear).

[Kah] Kahane J.-P., *Sur la dimension des intersections*, in Aspects of Mathematics and its applications (1986), Elsevier Science Publishers B. V., 419–430.

[Kau] Kaufmann, R., *On Hausdorff dimension of projections*, Mathematika **15** (1968), 153–155.

[Mar] Marstrand, M., *Some fundamental geometrical properties of plane sets of fractional dimension*, Proc. London Math. Soc. (3) **4** (1954), 257–302.

[Mat1] Mattila, P., *Hausdorff dimension, orthogonal projections and intersections with planes*, Ann. Acad. Sci. Fenn. Math. **1** (1975), 227–244.

[Mat2] Mattila, P., *Hausdorff dimension and capacities of intersection of sets in n-space*, Acta Math. **152** (1984), 77–105.

[Mat3] Mattila, P., *Geometry of Sets and Measures in Euclidean Spaces: Fractals and rectifiability*, Cambridge University Press, Cambridge, 1995.

[O] Olsen, L., *A multifractal formalism*, Adv. Math. **116** (1995), 82–195.

[P] Pesin, Ya., *On rigorous mathematical definitions of correlation dimension and generalized spectrum for dimensions*, J. Statist. Phys. **71** (1993), 529–547.

[SY] Sauer, T.D. and Yorke, J.A., *Are the dimensions of a set and its image equal under typical smooth functions?*, Ergodic Theory Dynam. Systems **17** (1997), 941–956.

[Si] Simon, B., *The Statistical Mechanics of Lattice Gases, Vol 1*, Princeton University Press, Princeton, New Jersey, 1993.

[St] Stewart, G.W., *Introduction to Matrix Computations*, Academic Press, New York, 1973.