REGULARITY OF DISTANCE MEASURES AND SETS

PERTTI MATTILA AND PER SJÖLIN

Abstract. Let μ be a Radon measure with compact support in $\mathbb{R}^n$ such that $\iint |x-y|^{-\alpha}\, d\mu x\, d\mu y < \infty$ for some α, $(n+1)/2 \le \alpha < n$. We show that the image of $\mu \times \mu$ under the distance map $(x,y) \mapsto |x-y|$ is an absolutely continuous measure with density of class $C^{\alpha-(n+1)/2}$. As a corollary we get that if $A \subset \mathbb{R}^n$ is a Suslin set with Hausdorff dimension greater than $(n+1)/2$, then the distance set $\{|x-y| : x, y \in A\}$ has non-empty interior.

1. Introduction

Let μ be a Radon measure in $\mathbb{R}^n$, $n \ge 2$, with compact support. We shall study regularity properties of the "distance measure" $\delta(\mu)$ defined for Borel sets $A \subset \mathbb{R}$ by

$$\delta(\mu)(A) = \int \mu\{y : |x-y| \in A\}\, d\mu x.$$

That is, $\delta(\mu)$ is the image of the product measure $\mu \times \mu$ under the distance map $(x,y) \mapsto |x-y|$. It was shown in [**M2**] that if the α-energy

$$I_\alpha(\mu) = \iint |x-y|^{-\alpha}\, d\mu x\, d\mu y = c \int |x|^{\alpha-n}\, |\widehat{\mu}(x)|^2\, dx$$

is finite for $\alpha = (n+1)/2$, then $\delta(\mu)$ is absolutely continuous with respect to the Lebesgue measure with bounded density. It follows from Example 2.2 in [**M1**] that this is not true for $\alpha < (n+1)/2$. In Theorem 2.4 we shall show that the density of $\delta(\mu)$ is continuous if $I_\alpha(\mu) < \infty$ for $\alpha = (n+1)/2$ and we shall derive stronger regularity properties when $\alpha > (n+1)/2$.

Falconer showed in [**F2**] that if $A \subset \mathbb{R}^n$ is a Suslin set (all Borel sets are Suslin) with the Hausdorff dimension $\dim A > (n+1)/2$, then the distance set

$$D(A) = \{|x-y| : x, y \in A\}$$

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has positive Lebesgue measure. Any such A supports a Radon probability measure μ with $I_{(n+1)/2}(\mu) < \infty$, and the continuity of $\delta(\mu)$ implies the stronger statement that the interior of $D(A)$ is non-empty.

Bourgain improved Falconer's result in [**B**] in the dimensions $n = 2$ and $n = 3$. For example, for $n = 2$ he showed that $\dim A > 13/9$ implies that $D(A)$ has positive Lebesgue measure. We do not know if $D(A)$ has interior points under this, or some other, weaker assumption. In [**F2**] Falconer gave an example of A with $\dim A = n/2$ for which $D(A)$ has zero Lebesgue measure.

According to the well-known result of Steinhaus $D(A)$ contains some interval $[0, \varepsilon]$, $\varepsilon > 0$, if A has positive Lebesgue measure. For this it is not enough to assume even that $\dim A = n$, see [**F2**].

2. Regularity of the distance measure

In this section we shall always assume that $(n+1)/2 \le \alpha < n$ and that μ is a non-negative Radon measure in $\mathbb{R}^n$ with compact support and such that

$$I_\alpha(\mu) = c \int |x|^{\alpha - n} |\widehat{\mu}(x)|^2 \, dx < \infty.$$

By c we denote positive constants depending only on α and n. Introducing the spherical averages

$$\sigma(\mu)(r) = \int_{|\zeta|=1} |\widehat{\mu}(r\zeta)|^2 \, d\zeta, \quad r > 0,$$

we have then

$$I_\alpha(\mu) = c \int_0^\infty r^{\alpha-1} \, \sigma(\mu)(r) \, dr < \infty. \tag{2.1}$$

For information on the influence of the finiteness of $I_\alpha(\mu)$ on $\sigma(\mu)$ and related averages, see [**M2**], [**S1**] and [**S2**].

We shall need the following estimates for the Bessel function $J = J_{(n-2)/2}$ and its derivatives.

2.2. Lemma. *For all $i = 0, 1, 2, \ldots, [n/2]$,*

$$|J^{(i)}(t)| \le Ct^{-1/2} \quad \textit{for } t > 0.$$

Proof. We shall first consider the case $t \ge 1$. We shall prove that

$$|J_m^{(i)}(t)| \le C_{i,m} \, t^{-1/2}, \quad t \ge 1,$$

for $i = 0, 1, 2, \ldots$ and $m = j/2$, $j = 0, 1, 2, \ldots$. We shall use induction and use the induction assumption

$$|J_m^{(i)}(t)| \le C_{i,m}\, t^{-1/2}, \quad t \ge 1, \quad \text{for all } i = 0, 1, 2, \ldots, l \quad \text{and } m = j/2,\ j = 0, 1, 2, \ldots. \tag{I_l}$$

The statement (I_0) is well-known (see [**SW**, p. 158]). Assume that (I_l) holds. From [**W**, p. 45], we find that

$$tJ_m'(t) - mJ_m(t) = -tJ_{m+1}(t),$$

whence

$$J_m'(t) = \frac{m}{t} J_m(t) - J_{m+1}(t).$$

Differentiating this l times we get (I_{l+1}) from (I_l). Thus we have proved the inequality in the lemma for $t \ge 1$.

For $0 < t \le 1$ we use the formula

$$J(t) = ct^{n/2-1} \int_{-1}^{1} e^{its}(1 - s^2)^{n/2-3/2}\, ds,$$

see [**SW**, p. 154]. If n is even, this gives immediately the desired estimate. Suppose n is odd: $n = 2l + 1$. Then

$$J(t) = ct^{l-1/2}\, \varphi(t),$$

where φ is a C^∞-function with bounded derivatives. If $i \le [n/2] = (n-1)/2 = l$, then

$$|(d/dt)^i\, t^{l-1/2}| \le Ct^{-1/2}, \quad 0 < t \le 1,$$

and the lemma follows. □

For smooth functions f with compact support (identified with a measure) $\delta(f)$ is absolutely continuous with density

$$\begin{aligned}\delta(f)(s) &= \int \left(\int_{|\zeta - x| = r} f(\zeta)\, d\zeta \right) f(x)\, dx \\ &= cs^{n/2} \int_0^\infty r^{n/2} J(sr)\, \sigma(f)(r)\, dr, \quad s > 0,\end{aligned}$$

see [**M2**, Lemma 4.3]. Applying this to $f = \varphi_\varepsilon * \mu$, where $\{\varphi_\varepsilon\}$ is an approximate identity, and letting $\varepsilon \to 0$ we have

$$\delta(\mu)(s) = cs^{n/2} \int_0^\infty r^{n/2}\, J(sr)\, \sigma(\mu)(r)\, dr; \tag{2.3}$$

because $\alpha \ge (n+1)/2$, (2.1) and Lemma 2.2 imply that the integral converges absolutely. We use this formula to prove the following theorem. We denote by C^0 the continuous functions on $\mathbb{R}$ and by C^k the k times continuously differentiable functions.

2.4. Theorem. *Write $\alpha = (n+1)/2 + k + \varepsilon$ where k is a non-negative integer and $0 \leq \varepsilon < 1$.*

(1) *If $\varepsilon = 0$, then $\delta(\mu) \in C^k$.*

(2) *If $\varepsilon > 0$, then $(d/dt)^k \delta(\mu)$ is Hölder continuous with exponent ε on every interval $[a, b]$, $0 < a < b < \infty$. More precisely,*

$$\left|(d/dt)^k\left(t^{-n/2}\delta(\mu)(t)\right) - (d/dt)^k\left(s^{-n/2}\delta(\mu)(s)\right)\right| \leq cI_\alpha(\mu)\, s^{-1/2}(t-s)^\varepsilon$$

for $0 < s < t$.

Proof. Write

$$\delta_1(\mu)(s) = \int_0^\infty r^{n/2}\, J(sr)\, \sigma(\mu)(r)\, dr.$$

We have $\delta(\mu)(s) = cs^{n/2}\, \delta_1(\mu)(s)$ and

$$(d/ds)^k \delta_1(\mu)(s) = \int_0^\infty r^{n/2+k}\, J^{(k)}(sr)\, \sigma(\mu)(r)\, dr \quad \text{for } s > 0,$$

by (2.1), Lemma 2.2 and (2.3) because $k < (n-1)/2$,

$$\left|r^{n/2+k}\, J^{(i)}(sr)\, \sigma(\mu)(r)\right| \leq cr^{n/2+k-1/2}\, s^{-1/2}\, \sigma(\mu)(r)$$

for $i = 0, 1, \ldots, k+1$ and $n/2 + k - 1/2 \leq \alpha - 1$. Hence (1) follows.

Let $0 < s < t$ and $d = 1/(t-s)$. Then

$$\begin{aligned}\left|(d/ds)^k \delta_1\mu(s) - (d/dt)^k \delta_1(\mu)(t)\right| &\leq \int_0^\infty r^{n/2+k}\left|J^{(k)}(sr) - J^{(k)}(tr)\right| \sigma(\mu)(r)\, dr \\ &= A + B,\end{aligned}$$

where A is the integral from 0 to d and B the integral from d to ∞. Then by Lemma 2.2 and (2.1)

$$\begin{aligned}A &\leq cs^{-1/2}(t-s) \int_0^d r^{n/2+k+1/2}\, \sigma(\mu)(r)\, dr \\ &\leq cs^{-1/2}(t-s) \int_0^d r^{n/2+k+1/2}(d/r)^{1-\varepsilon}\, \sigma(\mu)(r)\, dr \\ &= cs^{-1/2}(t-s)^{1+\varepsilon-1} \int_0^d r^{n/2+k+1/2+\varepsilon-1}\, \sigma(\mu)(r)\, dr \\ &= cs^{-1/2}(t-s)^\varepsilon \int_0^d r^{\alpha-1}\, \sigma(\mu)(r)\, dr \\ &\leq cs^{-1/2}(t-s)^\varepsilon\, I_\alpha(\mu)\end{aligned}$$

and

$$\begin{aligned}
B &\le cs^{-1/2}\int_d^\infty r^{n/2+k-1/2}\,\sigma(\mu)(r)\,dr\\
&\le cs^{-1/2}\int_d^\infty r^{n/2+k-1/2}(r/d)^\varepsilon\,\sigma(\mu)(r)\,dr\\
&= cs^{-1/2}(t-s)^\varepsilon\int_d^\infty r^{n/2+k-1/2+\varepsilon}\,\sigma(\mu)(r)\,dr\\
&= cs^{-1/2}(t-s)^\varepsilon\int_d^\infty r^{\alpha-1}\,\sigma(\mu)(r)\,dr\\
&\le cs^{-1/2}(t-s)^\varepsilon\, I_\alpha(\mu).
\end{aligned}$$

This proves the theorem. □

Remark. We do not know whether some smaller values of α would suffice to imply the same regularity except for the earlier mentioned fact that if $\alpha < (n+1)/2$ and $I_\alpha(\mu) < \infty$, then $\delta(\mu)$ need not be bounded, not even on any given interval $[a,b]$, $0 < a < b < \infty$, whence it need not be continuous.

3. Distance sets and Suslin rings

For $A \subset \mathbb{R}^k$ let $C_\alpha(A)$ be the Riesz α-capacity of A;

$$C_\alpha(A) = \sup I_\alpha(\mu)^{-1}$$

where the supremum is taken over all Radon probability measures μ with compact support, $\operatorname{spt}\mu$, and with $\operatorname{spt}\mu \subset A$. It is well-known that $\dim A > \alpha$ implies $C_\alpha(A) > 0$ for Suslin sets A, see e.g. [**C**]. The following is an immediate consequence of Theorem 2.4.

3.1. Theorem. *If $A \subset \mathbb{R}^k$ is a Suslin set with $C_{(n+1)/2}(A) > 0$, then $D(A)$ has non-empty interior.*

Proof. By the definition of C_α there is a Radon probability measure μ with compact support such that $\operatorname{spt}\mu \subset A$ and $I_{(n+1)/2}(\mu) < \infty$. By Theorem 2.4, $\delta(\mu)$ is continuous. Since $\operatorname{spt}\delta(\mu) \subset D(A)$, this implies that $D(A)$ has interior points. □

We finish by stating a result on Suslin subrings of $\mathbb{R}$:

3.2. Theorem. *Let $R \subset \mathbb{R}$ be a Suslin set. If R is a subring of the ring of real numbers and* $\dim R > 1/2$, *then* $R = \mathbb{R}$.

Falconer proved in [**F1**], see also [**F2**], that a Suslin subring R of the reals has either $0 \le \dim R \le 1/2$ or $\dim R = 1$. So under our assumptions we have $\dim R = 1$. To get that $R = \mathbb{R}$ it is enough to show that R has interior points. Since $\dim R \times R = 2$, see e.g. [**M3**, Theorem 8.10], $D(R\times R)$ has interior points by Theorem 3.1, and hence also $D^2(R \times R) = \{|x-y|^2 : x, y \in R \times R\}$. But this set is contained in R, since R is a ring, and the theorem follows.

We do not actually have to use Theorem 2.4. Theorem 3.2 is in fact a rather direct consequence of Falconer's results and methods. Namely, it is enough to know that $D(R \times R)$, and hence also R, by the above argument, has positive Lebesgue measure. Then Steinhaus's theorem gives that $\{x - y : x, y \in R\} = R$ contains an interval.

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Department of Mathematics, University of Jyväskylä, P.O. Box 35, FIN-40351 Jyväskylä, Finland
E-mail address: `pmattila@jylk.jyu.fi`

Department of Mathematics, Royal Institute of Technology, S-100 44 Stockholm, Sweden
E-mail address: `pers@math.kth.se`