REMOVABLE SETS FOR LIPSCHITZ HARMONIC FUNCTIONS
IN THE PLANE

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1. INTRODUCTION

The main motivation for this work comes from the century-old Painlevé problem: try to characterize geometrically removable sets for bounded analytic functions in $\mathbb{C}$. A compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if whenever U is an open set containing E and $f: U \setminus E \to \mathbb{C}$ is bounded and analytic, then f has an analytic extension to U. For other formulations and relations to analytic capacity, see, e.g., [G], [C1] or [M]. Painlevé proved that if $H^1(E) = 0$ then E is removable. Here H^1 is the one-dimensional Hausdorff (length) measure.

The following conjecture, stated first by Vitushkin, seems plausible.

1.1. Conjecture. *If $E \subset \mathbb{C}$ is compact and $H^1(E) < \infty$, then E is removable for bounded analytic functions if and only if E is purely unrectifiable, that is, $H^1(E \cap \Gamma) = 0$ for every rectifiable curve Γ.*

Although a positive solution of this conjecture would not give a complete characterization, it would leave out only sets of infinite length and Hausdorff dimension 1, because sets of Hausdorff dimension bigger than 1 are not removable, see, e.g., [G].

We have not been able to prove this conjecture, but we have proved the corresponding result for a closely related removability question concerning Lipschitz harmonic functions. A compact set $E \subset \mathbb{C}$ is removable for Lipschitz harmonic functions if whenever U is an open set containing E and $u: U \to \mathbb{R}$ is a Lipschitz function which is harmonic in $U \setminus E$, then u is harmonic in U. Every E which is removable for bounded analytic functions is also removable for Lipschitz harmonic functions (use the fact that if u is Lipschitz harmonic, then $\partial_z u$ is bounded analytic). But it is not known if these two classes are the same. Again sets of Hausdorff dimension bigger than 1 are not removable.

In this paper we shall prove

1.2. Theorem. *Let $E \subset \mathbb{C}$ be compact with $H^1(E) < \infty$. Then E is removable for Lipschitz harmonic functions if and only if E is purely unrectifiable.*

Both for this theorem and Conjecture 1.1 only one direction needs to be proven because of the result due to Calderón and others, see, e.g., [C1]: if E is a compact subset of a rectifiable curve with $H^1(E) > 0$, then E is not removable (for either class). Hence the removable sets are purely unrectifiable.

Conjecture 1.1, and hence also Theorem 1.2, were proven in [MMV] for compact sets E which are regular in the sense that there is a constant C such that

$$r/C \leq H^1\big(E \cap B(x,r)\big) \leq Cr \quad \text{for } x \in E,\ 0 < r \leq 1.$$

Here $B(x,r)$ is the closed disc with centre x and radius r. Our proof of Theorem 1.2 will rely on some of the main ingredients of the method of [MMV], so we discuss it briefly here.

The proof in [MMV] was based on a relation between the Cauchy kernel $1/z$ and the Menger curvature $c(z_1, z_2, z_3)$ of a triple $z_1, z_2, z_3 \in \mathbb{C}$. By definition, $c(z_1, z_2, z_3)$ is the reciprocal of the radius of the circle passing through z_1, z_2 and z_3. A formula, found by Melnikov in [Me], says that

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{\left(z_{\sigma(1)} - z_{\sigma(3)}\right) \overline{\left(z_{\sigma(2)} - z_{\sigma(3)}\right)}}$$

where the summation is over all permutations of $\{1, 2, 3\}$. From this it is not difficult to show, see [MV] or [MMV], that if ν is a (positive) Borel measure on $\mathbb{C}$ with compact support such that

$$(1.3) \qquad\qquad \nu\big(B(x,r)\big) \leq r \quad \text{for } x \in \mathbb{C} \text{ and } r > 0,$$

then

$$(1.4) \qquad\qquad \sup_{\varepsilon > 0} \int \left| \int_{\mathbb{C} \setminus B(x,\varepsilon)} \frac{d\nu(y)}{x - y} \right|^2 d\nu(x) < \infty$$

if and only if

$$(1.5) \qquad\qquad \iiint c(x, y, z)^2\, d\nu(x)\, d\nu(y)\, d\nu(z) < \infty.$$

On the other hand, it follows from a result of David and Léger, see [L], that if ν satisfies (1.5) and there is a Borel set $F \subset E$ such that $\mathcal{H}^1(F \cap B) \leq \nu(B) \leq C\mathcal{H}^1(F \cap B) < \infty$ for $B \subset F$, then there are rectifiable curves $\Gamma_1, \Gamma_2, \ldots$ such that

$$(1.6) \qquad\qquad \nu\left(F \setminus \bigcup_i \Gamma_i\right) = 0.$$

The proof of Theorem 1.2 will be finished if we can use the non-removability of E to find ν satisfying (1.3) and (1.4) and F as above such that $\nu(F) > 0$. Remember that we only have to show that if E is not removable, then $H^1(E \cap \Gamma) > 0$ for some rectifiable curve Γ. But (1.6) and $\nu(F) > 0$ give that $\nu(F \cap \Gamma_i) > 0$ for some i while (1.3) gives that $\nu(B) \leq H^1(B)$ for all $B \subset F$.

To find such a ν we proceed as follows. Suppose E is not removable for Lipschitz harmonic functions. Then there is a non-constant Lipschitz function $u \colon \mathbb{C} \to \mathbb{R}$ which is harmonic in $\mathbb{C} \setminus E$ such that $\langle \Delta u, 1 \rangle \neq 0$ and $\nabla u(\infty) = 0$, see [MP, Proposition 2.2]. Such a function u has a representation, see, for example, [MP, Lemma 5.3],

$$u(x) = \int \log|x - y|\, d\sigma(y) \quad \text{for } x \in \mathbb{C} \setminus E,$$

where σ is a signed Borel measure with support in E. Hence

$$\nabla u(x) = \int \frac{x-y}{|x-y|^2} \, d\sigma(y) \quad \text{for } x \in \mathbb{C} \setminus E.$$

Moreover, one can show, see [MP, Lemma 5.3], that σ is absolutely continuous with respect to H^1 with bounded Radon–Nikodym derivative f. Since u is Lipschitz, ∇u is bounded. Thus

$$\int_E \frac{x-y}{|x-y|^2} \, f(y) \, dH^1(y)$$

is bounded in $\mathbb{C} \setminus E$. This is the same as to say that the Cauchy transform

$$Tf(x) = \int_E \frac{1}{x-y} \, f(y) \, dH^1(y)$$

is bounded in $\mathbb{C} \setminus E$.

Moreover, $\int_E f dH^1 = c\langle \Delta u, 1 \rangle \neq 0$, and we may assume that $\int_E f dH^1 > 0$. Almost all of the paper then consists of a modification of the measure $f dH^1 | E$ to a desired measure ν. The construction of ν will be given in Section 4 and the proof of (1.4) will be completed in Section 5. The modification uses stopping time arguments somewhat similar to those in [C2]. For that we need a system of dyadic cubes in E with some good properties. They will be constructed in Section 3. Earlier similar systems were built in [D1], see also [D2], on regular subsets of $\mathbb{R}^n$ and in [C2] on spaces of homogeneous type, that is, when the measure is doubling. But now we do not have any doubling property which causes considerable complications in our applications to the Cauchy transform. The construction of the cubes will be done for an arbitrary locally finite Borel measure in $\mathbb{R}^n$.

In our modification result, which is stated in Section 2, we shall consider complex valued functions f. Then we do not get a positive measure, but we get a complex measure $g d\nu$ where ν is as above and g is bounded and accretive in the sense that $\operatorname{Re} g \geq c > 0$. This does not require much more work and our hope is that it would be useful in connection with Conjecture 1.1.

The main obstacle for extending Theorem 1.2 to higher dimensions is the lack of the curvature method. The associated kernel in $\mathbb{R}^n$ is $|x|^{-n} x$. We can still form the sum of permutations as above using inner products, but when $n \geq 3$ the resulting function in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ takes both positive and negative values which seems to make it useless.

Some results and examples on this question in $\mathbb{R}^n$ can be found in [U1], [U2] and [MP].

2. Statement of the main technical proposition

We are given a compact set $E \subset \mathbb{C}$ such that

$$(2.1) \qquad\qquad 0 < H^1(E) < +\infty,$$

and a bounded H^1 measurable complex valued function f on E, with the following property. Denote by μ the restriction of H^1 to E (i.e., the measure on $\mathbb{C}$ defined by $\mu(A) = H^1(A \cap E)$ for all Borel sets $A \subset \mathbb{C}$). We assume that

$$(2.2) \qquad\qquad \|f\|_\infty \leq 1 \quad \text{and} \quad \int_E f d\mu = a > 0,$$

and that the Cauchy integral of $f d\mu$, defined on $C \setminus E$ by

$$(2.3) \qquad T(fd\mu)(x) = \int_E \frac{f(y)\,d\mu(y)}{x-y},$$

is bounded. Our main technical result is as follows.

2.4. Theorem. *Let E, μ and f be as above. Then there exist a positive Borel measure ν and a bounded Borel function g such that*

$$(2.5) \qquad \nu\big(B(x,r)\big) \leq Cr \quad \text{for all } x \in \mathbb{C} \text{ and } r > 0,$$

$$(2.6) \qquad |g(x)| \leq C \quad \text{and} \quad \operatorname{Re} g(x) \geq C^{-1} \quad \text{for all } x \in \mathbb{C},$$

$$(2.7) \qquad \int g d\nu = \int f d\mu = a,$$

$$(2.8) \qquad \begin{cases} \text{there is a Borel set } F \subset E \text{ such that} \\ C^{-1}\mu \leq \nu \leq \mu \text{ on } F \text{ and } \nu(F) \geq \dfrac{a}{2}, \end{cases}$$

and

$$(2.9) \qquad \int T^*(gd\nu)^2 \, d\nu < +\infty,$$

where T^* is the maximal Cauchy operator defined by

$$(2.10) \qquad T^*(gd\nu)(x) = \sup_{\varepsilon>0} \left| \int_{\mathbb{C}\setminus B(x,\varepsilon)} \frac{g(y)\,d\nu(y)}{x-y} \right|.$$

In this statement, the constant C in (2.5) may depend on many things, including the rate at which the densities $r^{-1}\mu\big(B(x,r)\big)$ stop being too large when r gets small. If we know already that $\mu\big(B(x,r)\big) \leq Mr$ for some M and all $x \in \mathbb{C}$, $r > 0$, and if we normalized our statement by assuming that $\operatorname{diam} E = 1$, the proof will give a constant in (2.5) which is less than $C_1 M$, with C_1 an absolute constant. See the comment to this effect a little above Lemma 4.56.

The constant C in (2.6) and (2.8) can be taken to be $\leq C_1 a^{-1} \mu(E)$, where C_1 is an absolute constant. See the remark near the end of Section 4.2.

Our estimate for (2.9), just like (2.5), can depend somewhat wildly on E, but again if we assume that $\mu\big(B(x,r)\big) \leq Mr$ as above, our proof will give an estimate on $\int T^*(gd\nu)^2 \, d\nu$ that depends only on M, $\operatorname{diam} E$, $a^{-1} \mu(E)$, and $\|T(fd\mu)\|_\infty$.

If we start with a function f which is real-valued, then we get a function g which is real-valued as well, and the accretivity condition (2.6) simply says that

$$(2.11) \qquad C^{-1} \leq g(z) \leq C.$$

This is the case that we need for our application to Lipschitz harmonic functions, but the statement for complex valued functions f is not much harder to get, and may be interesting as well.

Note that Theorem 1.2 follows from Theorem 2.4 and the discussion in Chapter 1. Because of (2.11), the measure $gd\nu$ satisfies (1.3) because of (2.5) and (1.4) by (2.9), while the condition on F (for a multiple of $gd\nu$) comes from (2.8).

It is possible that our proof will give slightly more precise, BMO-like, properties, of $T^*(gd\nu)$ than (2.9), but it is not too clear how to formulate this in a useful way.

The rest of this paper is a proof of Theorem 2.4.

3. Dyadic cubes with small boundaries associated to a measure in $\mathbb{R}^n$

In this chapter we want to construct partitions of the support of a given measure μ into analogues of the dyadic cubes in $\mathbb{R}^n$. We shall do the construction with very little information on the measure, even if some extra information on μ will be useful when we try to use our cubes.

Let μ be a locally finite (positive Borel) measure on $\mathbb{R}^n$, and let E denote its support.

Let C_0 and A be two large constants. For the construction and the statement of the theorem, we shall only require that

$$(3.1) \qquad\qquad C_0 > 1 \quad \text{and} \quad A > 5000C_0,$$

say, but our small boundary condition will only become useful if A is larger than some high power of C_0. In our application to Theorem 2.4, we shall take $A \approx C_0^{100}$ and C_0 very large, for instance.

3.2. Theorem. *Let μ, E, C_0 and A be as above. Then there exists a sequence of partitions of E into Borel subsets Q, $Q \in \Delta_k$, with the following properties.*
For each integer $k \geq 0$,

$$(3.3) \qquad\qquad E \text{ is the disjoint union of the "cubes" } Q, Q \in \Delta_k,$$

and

$$(3.4) \qquad \text{if } k < l, Q \in \Delta_k \text{ and } R \in \Delta_l, \text{ then either } Q \cap R = \emptyset \text{ or else } R \subset Q.$$

The general position of the cubes Q can be described as follows. For each $k \geq 0$ and each cube $Q \in \Delta_k$, there is a ball

$$(3.5) \qquad B(Q) = B\big(x(Q), r(Q)\big)$$

such that

$$(3.6) \qquad x(Q) \in E,$$
$$(3.7) \qquad A^{-k} \leq r(Q) \leq C_0 A^{-k},$$
$$(3.8) \qquad E \cap B(Q) \subset Q \subset E \cap 28B(Q) = E \cap B\big(x(Q), 28r(Q)\big),$$

and

$$(3.9) \qquad\qquad \text{the balls } 5B(Q), Q \in \Delta_k, \text{ are disjoint.}$$

Also, the cubes $Q \in \Delta_k$ have "small boundaries". For each cube $Q \in \Delta_k$ and each integer $l \geq 0$, set

$$(3.10) \qquad\qquad N_l^{\text{ext}}(Q) = \{x \in E \setminus Q : \text{dist}(x, Q) < A^{-k-l}\},$$
$$(3.11) \qquad\qquad N_l^{\text{int}}(Q) = \{x \in Q : \text{dist}(x, E \setminus Q) < A^{-k-l}\},$$

and

(3.12) $$N_l(Q) = N_l^{\text{ext}}(Q) \cup N_l^{\text{int}}(Q).$$

Then

(3.13) $$\mu\big(N_l(Q)\big) \le [C^{-1}C_0^{-3n-1}A]^{-l}\mu\big(90B(Q)\big).$$

Finally let $\mathcal{G}_k$ denote the set of good cubes $Q \in \Delta_k$ for which

(3.14) $$\mu\big(100B(Q)\big) \le C_0\mu\big(B(Q)\big),$$

and set $\mathcal{B}_k = \Delta_k \setminus \mathcal{G}_k$. We have that

(3.15) $$r(Q) = A^{-k} \quad \text{when } Q \in \mathcal{B}_k,$$

and also

(3.16) $$\begin{cases} \mu\big(100B(Q)\big) \le C_0^{-l}\mu\big(100^{l+1}B(Q)\big) \text{ for all} \\ l \ge 1 \text{ such that } 100^l \le C_0 \text{ when } Q \in \mathcal{B}_k. \end{cases}$$

This completes the statement of our theorem. The constant C in (3.13) may depend on n, but nothing else. Of course the constants 28, 90, 100 and $3n+1$ are not optimal.

The last condition (3.16) will be useful. When $Q \in \mathcal{B}_k$, it is not so easy to use (3.13), and we cannot really prevent situations where Q is a cube with very small mass, very close to other cubes with very large masses. In these situations it is hard to get estimates on $N_l(Q)$ that depend on $\mu(Q)$ alone, and we shall fight this back by saying that $\mu(Q)$ was actually very tiny. (The constant C_0^{-l}, where $100^l \simeq C_0$, is much smaller than any given negative power of C_0 when C_0 is large.)

Now we can start the construction of cubes. The rest of this chapter will not interfere with the other parts of the paper.

Let μ, E, A, and C_0 be given, as in the statement. We start our construction with an approximation to Δ_0 (i.e., we shall first work at scale 1). A somewhat similar construction of "cubes of one generation" has been given by O'Neil in [O]. We first associate a ball $B(x) = B\big(x, r(x)\big)$ to each $x \in E$, as follows.

Denote by $\mathcal{G}$ the good set of points $x \in E$ such that we can find a radius $r(x)$ such that

(3.17) $$1 \le r(x) \le C_0$$

and

(3.18) $$\mu\big(B(x, 100r(x))\big) \le C_0\mu\big(B(x, r(x))\big).$$

For each $x \in \mathcal{G}$, choose $r(x)$ as in (3.17), (3.18), and set $B(x) = B\big(x, r(x)\big)$. Set $\mathcal{B} = E \setminus \mathcal{G}$, and simply take

(3.19) $$r(x) = 1 \quad \text{and} \quad B(x) = B(x, 1) \quad \text{for } x \in \mathcal{B}.$$

By the definition of $\mathcal{G}$, we have that

$$(3.20) \qquad \mu\big(100^l B(x)\big) \leq C_0^{-1}\mu\big(100^{l+1}B(x)\big) \text{ for all } l \geq 0 \text{ such that } 100^l \leq C_0$$

when $x \in \mathcal{B}$.

Next we want to choose, for each $x \in E$, two auxiliary radii $r_1(x)$ and $r_2(x)$ such that

$$(3.21) \qquad \frac{11}{10}r(x) < r_1(x) < \frac{12}{10}r(x),$$
$$(3.22) \qquad 25r(x) < r_2(x) < 26r(x),$$

and that we have the first "small boundary conditions"

$$(3.23) \qquad \mu\big(\{z \in \mathbb{R}^n : \big||z - x| - r_1(x)\big| \leq \tau r(x)\}\big) \leq C\tau\mu\Big(\frac{13}{10}B(x)\Big) \quad \text{for } 0 < \tau < \frac{1}{10}$$

and

$$(3.24) \qquad \mu\big(\{z \in \mathbb{R}^n : \big||z - x| - r_2(x)\big| \leq \tau r(x)\}\big) \leq C\tau\mu\big(27B(x)\big) \quad \text{for } 0 < \tau < 1.$$

The existence of $r_1(x)$ and $r_2(x)$ is easy to derive from the standard weak-L^1 estimate for the maximal function of the measure on $\mathbb{R}_+$ which is the image under the radial projection $z \to |x - z|$ of $\mu\lfloor\frac{13}{10}B(x)$ or $\mu\lfloor 27B(x)$. Here the constant C does not even depend on n.

Choose $r_i(x)$, $i = 1, 2$, as above, and set $B_i(x) = B\big(x, r_i(x)\big)$. Next we use the standard "Vitali-type" covering lemma (as in [M, Theorem 2.1] or see the proof on p. 9 of [S]) to select a (discrete) subset I of E such that

$$(3.25) \qquad\qquad\qquad \text{the balls } 5B(x), \, x \in I, \text{ are disjoint}$$

and

$$(3.26) \qquad\qquad\qquad E \subset \bigcup_{x \in I} 25B(x)$$

(so that the $B_2(x)$, $x \in I$, also cover E).

For each $x \in I$, denote by $J(x)$ the set of points $y \in I \setminus \{x\}$ such that $B_1(y)$ meets $B_2(x)$. Because (3.17) holds for all $y \in I$, (3.25) yields that

$$(3.27) \qquad\qquad\qquad J(x) \text{ has less than } CC_0^n \text{ elements.}$$

Set

$$(3.28) \qquad B_3(x) = B_2(x) \setminus \Big\{\bigcup_{y \in I \setminus \{x\}} B_1(y)\Big\} = B_2(x) \setminus \Big\{\bigcup_{y \in J(x)} B_1(y)\Big\}.$$

Let us check that

$$(3.29) \qquad\qquad \frac{13}{10}B(y) \subset 90B(x) \quad \text{for all } y \in J(x).$$

If $y \in J(x)$, then $\frac{13}{10}B(y)$ meets $26B(x)$ because it meets $B_2(x)$, and so $|x-y| \leq 26r(x) + \frac{13}{10}r(y)$. If (3.29) did not hold for y, we would also have that $|x-y| > 90r(x) - \frac{13}{10}r(y)$, and so $\frac{26}{10}r(y) > 64r(x)$. In particular, $r(y) > 20r(x)$, $|x-y| \leq 26r(x) + \frac{13}{10}r(y) < 3r(y)$, x lies on $5B(y)$, and this contradicts (3.25). This proves (3.29).

Hence we may apply (3.24) to x and (3.23) to each $y \in J(x)$ to get that

$$(3.30) \qquad \mu\big(\{z \in \mathbb{R}^n : \operatorname{dist}(z, \partial B_3(x)) \leq \tau\}\big) \leq CC_0^n \tau \mu\big(90B(x)\big)$$

for all $x \in I$ and $0 < \tau < \frac{1}{10}$. (Of course this uses (3.27), and the fact that you cannot be close to $\partial B_3(x)$ without being close to $\partial B_2(x)$ or some $\partial B_1(y)$, $y \in J(x)$.)

Next put an order on I such that

$$(3.31) \qquad y < x \quad \text{in } I \text{ whenever } \mu\big(90B(y)\big) < \mu\big(90B(x)\big).$$

(When $\mu\big(90B(x)\big) = \mu\big(90B(y)\big)$, we can decide whether $x < y$ or not at random.) We define new sets $B_4(x)$, $x \in I$, by

$$(3.32) \qquad B_4(x) = B_3(x) \setminus \left\{ \bigcup_{\substack{y \in I \\ y < x}} B_3(y) \right\}.$$

Here again, the union looks infinite, but we are only interested in the $y \in I$ such that $B_3(y)$ meets $B_3(x)$, and there are at most CC_0^n of those.

3.33. Lemma. *The sets $B_4(x)$, $x \in I$, are disjoint and cover E.*

This is easy. The $B_4(x)$, $x \in I$, are disjoint because of the formula (3.32) alone. To prove the rest, first observe that

$$(3.34) \qquad 5B(x) \subset B_3(x) \quad \text{for all } x \in I.$$

This is because $5B(x) \subset B_2(x)$ by the definition of $r_2(x)$, and $5B(x)$ does not meet any of the $B_1(y)$, $y \neq x$, because it does not even meet the larger balls $5B(y)$ (by (3.25)).

Because of (3.34),

$$(3.35) \qquad \text{the sets } B_3(x), \ x \in I, \ \text{cover } E.$$

Indeed, if $z \in E$, then $z \in 25B(x)$ for some $x \in I$ (by (3.26)), and if it does not lie in the corresponding $B_3(x)$, it must be in $B_1(y)$ for some $y \in J(x)$; (3.34) then says that $z \in B_3(y)$.

The rest of Lemma 3.33 follows easily from (3.35) and the formula (3.32).

Once again, the set $J'(x)$ of all $y \in I$ such that $y < x$ and $B_3(y)$ meets $B_3(x)$ has at most CC_0^n elements. We may apply (3.30) to x and each $y \in J'(x)$, and we get that

$$(3.36) \qquad \mu\big(\{z \in \mathbb{R}^n : \operatorname{dist}(z, \partial B_4(x)) \leq \tau\}\big)$$
$$\leq CC_0^{2n} \tau \mu\big(90B(x)\big) \quad \text{for all } x \in I \text{ and } 0 < \tau < \frac{1}{10}.$$

(We are also using the definition of $<$ on I.)

Notice that for each $x \in I$, $B_1(x) \subset B_3(x)$ (for instance by (3.34)), and since $B_1(x)$ does not meet any of the $B_3(y)$, $y \neq x$ (by the definition (3.28) of $B_3(y)$), it is also contained in $B_4(x)$. Thus

$$(3.37) \qquad \frac{11}{10}B(x) \subset B_1(x) \subset B_4(x) \subset 26B(x)$$

(because of (3.21) and (3.22)).

The sets $E \cap B_4(x)$, $x \in I$, are our first approximations to the cubes Q, $Q \in \Delta_0$. Now we want to do a similar construction at each scale A^{-k}, $k \geq 0$, and then modify our sets so that the nesting property (3.4) holds.

Let us first apply the preceding construction at each scale A^{-k}. For each $k \geq 0$, we get a set $I_k \subset E$ of centers and, for each $x \in I_k$, a ball $B(x) = B^k(x)$ (if we want to be explicit) and a set $B_4(x) = B_4^k(x)$ with the following properties:

$(3.38) \qquad B(x)$ is centered on x, and has a radius $r(x) \in [A^{-k}, C_0 A^{-k}]$;

$(3.39) \qquad$ the balls $5B(x)$, $x \in I_k$, are disjoint;

$$(3.40) \qquad \frac{11}{10}B(x) \subset B_4(x) \subset 26B(x);$$

$(3.41) \qquad$ the sets $B_4(x)$, $x \in I_k$, cover E and are disjoint;

$$(3.42) \qquad \mu\left(\left\{z \in \mathbb{R}^n : \operatorname{dist}(z, \partial B_4(x)) \leq \tau A^{-k}\right\}\right) \leq CC_0^{2n}\tau\mu\left(90B(x)\right)$$

for all $k \geq 0$, all $x \in I_k$, and $0 < \tau < \frac{1}{10}$.

Moreover, if $x \in I_k$, then either

$$(3.43) \qquad \mu\left(100B(x)\right) \leq C_0\mu\left(B(x)\right),$$

or else $r(x) = A^{-k}$ and

$$(3.44) \qquad \mu\left(100^l B(x)\right) \leq C_0^{-1}\mu\left(100^{l+1}B(x)\right) \quad \text{for all } l \geq 0 \text{ such that } 100^l \leq C_0.$$

Next we want to replace each set $B_4(x) = B_4^k(x)$, $x \in I_k$, by finer versions obtained by taking unions of sets $B_4(y)$ from smaller generations. We have to do this in a coherent way.

For each $k \geq 1$ and $y \in I_k$, let $h(y)$ denote the point $x \in I_{k-1}$ such that $y \in B_4^{k-1}(x)$. There is nothing very special about this choice; any x such that $E \cap B_4^{k-1}(x) \cap B_4^k(y) \neq \emptyset$, for instance, would work as well. It will be just as convenient to choose $h(y)$ so that

$$(3.45) \qquad y \in B_4^{k-1}\left(h(y)\right),$$

though.

Also set $h^l(y) = h \circ \cdots \circ h(y)$ (l times) when $k \geq l$. Thus $h^l(y) \in I_{k-l}$. Define, for each $x \in I_k$, new sets $D_l^k(x)$, $l \geq 0$, by

$$(3.46) \qquad D_l^k(x) = \bigcup_{\substack{y \in I_{k+l} \\ h^l(y)=x}} B_4^{k+l}(y).$$

(In particular, use the convention that $D_0^k(x) = B_4(x)$.) Our intention is to take the limits of the $D_l^k(x)$, with k and x fixed and $l \to +\infty$, as our almost final version of Δ_k.

For k and l fixed, the sets $h^{-l}(x)$, $x \in I_k$, form a partition of I_{k+l}, and hence

$(3.47) \qquad$ the sets $D_l^k(x)$, $x \in I_k$, are disjoint and cover E,

because of the corresponding properties (3.41) of the sets $B_4^{k+l}(y)$, $y \in I_{k+l}$.

3.48. Lemma. *We have that*

$$\text{(3.49)} \qquad\qquad \text{dist}\left(z, E \cap D_l^k(x)\right) \le 26C_0 A^{-k-l-1}$$

for all $z \in D_{l+1}^k(x)$ (and all choices of $k \ge 0$, $x \in I_k$, and $l \ge 0$).

Indeed, let $z \in D_{l+1}^k(x)$ be given. By definitions, there is an $y \in I_{k+l+1}$ such that $h^{l+1}(y) = x$ and $z \in B_4^{k+l+1}(y)$. Then $|z - y| \le 26r(y) \le 26C_0 A^{-k-l-1}$ by (3.40) and (3.38). On the other hand, $y \in B_4^{k+l}\left(h(y)\right)$ by the definition of h (see (3.45)), and so $y \in D_l^k(x)$ because $h^l\left(h(y)\right) = x$. This proves the lemma.

Since $A \ge 10$, say, by (3.1), we can iterate the estimate from Lemma 3.48 and get that

$$\text{(3.50)} \qquad\qquad \text{dist}\left(z, E \cap D_{l_0}^k(x)\right) \le 50C_0 A^{-k-l_0-1}$$

for all $z \in \bigcup_{l \ge l_0} D_l^k(x)$ and all $l_0 \ge 0$. (Connect z to $D_{l_0}^k(x)$ by a sequence of points in the $D_m^k(x)$, $l_0 < m < l$, and sum a geometric series.)

Let us now prove an estimate in the other direction. We claim that

$$\text{(3.51)} \qquad\qquad \text{dist}\left(z, E \cap D_{l+1}^k(x)\right) \le 26C_0 A^{-k-l} \quad \text{for all } z \in D_l^k(x).$$

Remember that $D_l^k(x)$ is the union of the $B_4^{k+l}(y)$, $y \in h^{-l}(x)$. Thus, by (3.38) and (3.40), it will be enough to show that each $y \in h^{-l}(x)$ lies on $D_{l+1}^k(x)$. Let $u \in I_{k+l+1}$ be such that $y \in B_4^{k+l+1}(u)$. It is enough to show that $h^{l+1}(u) = x$, or even that $h(u) = y$ (because we already know that $h^l(y) = x$). Since $y \in B_4^{k+l+1}(u)$, we know that $|y - u| \le 26C_0 A^{-k-l-1}$. Since by (3.1) $A > 26C_0$, this implies that $u \in B(y)$ (by (3.38)), and then that $u \in B_4(y)$ (by (3.40)) and $h(u) = y$ (by the definition of $h(u)$, see (3.45)). This proves our claim (3.51).

¿From Lemma 3.48 and (3.51) we deduce that the Hausdorff distance between the sets $D_l^k(x)$ and $D_{l+1}^k(x)$ is at most $26C_0 A^{-k-l}$. Thus the closures of these sets converge (for the Hausdorff metric on compact sets) to a set

$$\text{(3.52)} \qquad\qquad D(x) = D^k(x) = \lim_{l \to +\infty} \overline{D_l^k(x)}.$$

The sets $D(x)$, $x \in I_k$, are almost our cubes Q, $Q \in \Delta_k$, but we may have lost the property that they are disjoint, and so we want to fix that. Before, let us collect a few of the properties of the $D^k(x)$'s. Let us first check that

$$\text{(3.53)} \qquad\qquad D^k(x) \subset E \cap 28B(x) \quad \text{for all } x \in I_k.$$

Indeed, let $z \in D^k(x)$ be given. Then $z = \lim_{l \to +\infty} z_l$ for some sequence (z_l) such that $z_l \in D_l^k(x)$. Because of Lemma 3.48 (or (3.51)), $\text{dist}(z_l, E) \le 26C_0 A^{-k-l}$, and so $z \in E$. From (3.40), the fact that $D_0^k(x) = B_4^k(x)$ and (3.50) (with $l_0 = 0$), we deduce that $|x - z_l| \le 26r(x) + 50C_0 A^{-k-1}$ for all l; (3.53) follows because $A > 50C_0$.

Next we check that

$$\text{(3.54)} \qquad\qquad E = \bigcup_{x \in I_k} D^k(x).$$

We already know that $D^k(x) \subset E$. If $z \in E$, then for each l there is an $x_l \in I_k$ such that $z \in D_l^k(x_l)$ (because of (3.47)). Moreover, there is only a finite set of points $x \in I_k$ for which $28B(x)$ contains z (because the $5B(x)$ are disjoint and have comparable diameters). Since (3.53) tells us that all the x_l's must be picked in this finite set, we see that there is an $x \in I_k$ such that $z \in D_l^k(x)$ for infinitely many values of l. Then $z \in D^k(x)$, and (3.54) holds.

3.55. Lemma. *For each* $x \in I_k$, $E \cap B(x) \subset D^k(x)$, *and* $E \cap B(x)$ *does not meet any of the* $D^k(x')$, $x' \in I_k \setminus \{x\}$.

Because we already know that the $D^k(x')$, $x' \in I_k$, cover E, it is enough to prove the second half of the statement. Let $z \in E \cap B(x)$ be given, and suppose that $z \in D^k(x')$ for some $x' \in I_k$. Then we can find a point z_l such that $z_l \in D_l^k(x')$ for some $l > 1$ and $|z - z_l| < \frac{r(x)}{100}$, say. From (3.50) (with $l_0 = 0$) we deduce that dist $\left(z_l, E \cap B_4^k(x')\right) \leq 50 C_0 A^{-k-1}$. Since $A > 5000 C_0$ by (3.1), we get that dist $\left(z, E \cap B_4^k(x')\right) \leq \frac{r(x)}{50}$. Then $E \cap B_4^k(x')$ meets $\frac{11}{10} B(x)$, and since (3.40) and (3.41) tell us that $B_4^k(x')$ is disjoint from $\frac{11}{10} B(x)$ when $x' \neq x$, we get that $x' = x$. This proves the lemma.

3.56. Lemma. *For each* $x \in I_k$ *and each* $l \geq 0$,

$$D^k(x) = \bigcup_{\substack{y \in I_{k+l} \\ h^l(y)=x}} D^{k+l}(y).$$

Indeed for each $m \geq 0$, by (3.46),

$$(3.57) \qquad D_{m+l}^k(x) = \bigcup_{z \in h^{-m-l}(x)} B_4^{k+m+l}(z)$$

$$= \bigcup_{y \in h^{-l}(x)} \left\{ \bigcup_{z \in h^{-m}(y)} B_4^{k+m+l}(z) \right\}$$

$$= \bigcup_{y \in h^{-l}(x)} D_m^{k+l}(y).$$

The lemma follows by taking closures in (3.57), and then passing to the limit.

Now we want to modify our sets $D^k(x)$ to make them disjoint. We need to be a little careful because we want some coherence between the different values of k, but nothing too interesting will be happening here. Our future estimates will show that the various intersections of the $D^k(x)$, $x \in I_k$, have measure 0, so we do not really need to worry about the effect of our choices now on singular integrals, for instance.

First put any order on I_0. This does not need to be the same one as in the definition of $B_3(x)$, but we really do not care. Then put orders on I_1, then I_2, then I_3, etc., with the only constraint that

$$(3.58) \qquad y_1 < y_2 \quad \text{on } I_{k+1} \text{ whenever } h(y_1) < h(y_2) \text{ on } I_k.$$

In other words, when we define the order on I_{k+1}, we decompose I_{k+1} into the classes $h^{-1}(x)$, $x \in I_k$, and make sure that they come out in the same order on I_{k+1} as the $x \in I_k$. The order in each class is irrelevant.

For each $k \geq 0$ and each $x \in I_k$, set

$$(3.59) \qquad Q(x) = Q^k(x) = D^k(x) \setminus \left\{ \bigcup_{\substack{y \in I_k \\ y < x}} D^k(y) \right\}.$$

These are the cubes we wanted. To get back to the notations of Theorem 3.1, we set $\Delta_k = \{Q^k(x), x \in I_k\}$. We should now start verifying the various conclusions of Theorem 3.2.

The fact that for each $k \geq 0$, E is the disjoint union of the $Q^k(x)$, $x \in I_k$, is an easy consequence of (3.54) and (3.59). This takes care of (3.3).

Let us check now that if $x \in I_k$ and $y \in I_{k+1}$ is such that $h(y) = x$, then $Q(y) \subset Q(x)$. We already know from Lemma 3.56 that $Q(y) \subset D^{k+1}(y) \subset D^k(x)$, so it is enough to check that $Q(y)$ does not meet any of the $D^k(x')$, $x' < x$ (the sets that were removed from $D^k(x)$ to get $Q^k(x)$). If it did, then $Q(y)$ would also meet $D^{k+1}(y')$ for some $y' \in I_{k+1}$ such that $h(y') = x'$ (because of Lemma 3.56). This is not possible, because $y' < y$ (since $x' < x$). Thus $Q(y) \subset Q(x)$, as promised.

Now if $y \in I_{k+1}$ is such that $h(y) \neq x$, then $Q(y) \subset Q\big(h(y)\big)$, and so $Q(y)$ does not meet $Q(x)$ (by (3.3)). This implies the nesting property (3.4). We even have that $Q^{k+l}(y) \subset Q^k(x)$ if and only if $h^l(y) = x$.

The properties (3.5), (3.6) and (3.7) are just the same as (3.38) (if we set $B\big(Q(x)\big) = B(x)$ for $x \in I_k$), while (3.8) follows from (3.53) and Lemma 3.55. The balls $5B(Q)$, $Q \in \Delta_k$ are indeed disjoint (as needed for (3.9)) by (3.39). The small boundary property (3.10–13) will be more interesting.

Fix $x \in I_k$, let $Q = Q(x)$ and let $N_l(Q)$ be as in (3.10–12). Also denote by $\mathcal{A}_l(Q)$ the set of the cubes $R \in \Delta_{k+l}$ such that $N_l(Q) \cap R \neq \emptyset$. We first consider the case $l = 1$.

3.60. Lemma.

$$\sum_{S \in \mathcal{A}_1(Q)} \mu\big(90B(S)\big) \leq CC_0^{3n+1} A^{-1} \mu\big(90B(Q)\big).$$

First we want to check that

$$(3.61) \qquad \operatorname{dist}\big(z, \partial B_4(x)\big) \leq 51C_0 A^{-k-1} \quad \text{for all } z \in N_1(Q).$$

Let $z \in N_1(Q)$ be given. Then there are points z_1 and z_2, with $z_1 \in Q(x)$ and $z_2 \in E \setminus Q(x)$, such that $z = z_1$ or z_2 and $|z_1 - z_2| < A^{-k-1}$. Let $x_2 \in I_k$ be such that $z_2 \in Q(x_2)$.

Since $z_1 \in Q(x) \subset D^k(x)$, we can find points of $\bigcup_{l \geq 0} D_l^k(x)$ that are as close as we want to z_1. Because of (3.50), applied with $l_0 = 0$, those points are all within $50C_0 A^{-k-1}$ of $E \cap D_0^k(x) = E \cap B_4(x)$. So $\operatorname{dist}\big(z_1, B_4(x)\big) \leq 50C_0 A^{-k-1}$. The same estimate with z_2 gives that $\operatorname{dist}\big(z_2, B_4(x_2)\big) \leq 50C_0 A^{-k-1}$, and since $B_4(x_2)$ does not meet $B_4(x)$ (by (3.41)), we get (3.61).

Now we can prove Lemma 3.60. Each $90B(S)$, $S \in \mathcal{A}_1(Q)$, meets $N_1(Q)$ by (3.8). Therefore their union lies entirely within $300C_0 A^{-k-1}$ of $\partial B_4(x)$ by (3.61). By (3.42) this union has a measure $\leq CC_0^{2n}(C_0 A^{-1})\mu\big(90B(x)\big)$. (Here we have applied (3.42) with $\tau = 300C_0 A^{-1}$; we have that $\tau < \frac{1}{10}$ because of (3.1).) Also, a given point of the union cannot lie in more than CC_0^n balls $90B(S)$, because all these balls have centers at distances $> A^{-k-1}$ from each other, and at the same time $\leq 90C_0 A^{-k-1}$ from the given point. The lemma follows from all this. This proves also (3.13) for $l = 1$.

Now we want to estimate $\mu\big(N_l(Q)\big)$ for $l > 1$. (When $l = 0$, (3.13) follows immediately from the fact that $N_0 \subset 90B(Q)$, by (3.8).) We shall prove the estimate

$$(3.62) \qquad \sum_{S \in \mathcal{A}_l(Q)} \mu\big(90B(S)\big) \leq (CC_0^{3n+1} A^{-1})^l \, \mu\big(90B(Q)\big),$$

which clearly is stronger than (3.13).

Let $S \in \mathcal{A}_l(Q)$ and choose $z \in S \cap N_l(Q)$. Then there are z_1 and z_2 such that $z = z_1$ or z_2, $z_1 \in Q$, $z_2 \in Q'$ for some $Q' \in \Delta_k$, $Q' \neq Q$, and $|z_1 - z_2| < A^{-k-l}$. Clearly, z_1, $z_2 \in N_l(Q)$. Let R be the cube of Δ_{k+l-1} which contains z. Then $R \in \mathcal{A}_{l-1}(Q)$, as $z \in N_l(Q) \subset N_{l-1}(Q)$, and $S \subset R$. Since $|z_1 - z_2| < A^{-k-l}$, both z_1 and z_2 belong to $N_1(R)$ and so $S \in \mathcal{A}_1(R)$. This shows that

$$\mathcal{A}_l(Q) \subset \bigcup_{R \in \mathcal{A}_{l-1}(Q)} \mathcal{A}_1(R).$$

Because of this we can deduce from Lemma 3.60 that

$$\sum_{S \in \mathcal{A}_l(Q)} \mu\big(90B(S)\big) \leq \sum_{R \in \mathcal{A}_{l-1}(Q)} \sum_{S \in \mathcal{A}_1(R)} \mu\big(90B(S)\big)$$

$$\leq CC_0^{3n+1} A^{-1} \sum_{R \in \mathcal{A}_{l-1}(Q)} \mu\big(90B(R)\big).$$

Iterating this we get (3.62).

Finally (3.14–3.16) is the same as (3.43–3.44). This completes the proof of Theorem 3.1.

4. CONSTRUCTION OF $gd\nu$

Let E, μ, and f be given, as in the statement of Theorem 2.4. In this chapter we construct a measure $gd\nu$, and check that it satisfies the requirements (2.5)–(2.8) of the theorem (i.e., all the requirements that do not concern the Cauchy integral). The construction itself will only use the hypothesis (2.1) and (2.2). It is only when we are finished with the construction that we shall start thinking about Cauchy kernels, except for the detail that we shall keep the right to choose the constant C_0 large enough for the arguments of Chapter 5 to work.

4.1. THE CONSTRUCTION ITSELF

The first thing we do is to apply the construction of Chapter 3 to find collections of cubes Q, $Q \in \Delta_k$, as in Theorem 3.2. As was explained before, we want to keep the right to choose C_0 enormous later (just after (5.29) and before (5.148)); on the other hand we can already decide to take

(4.1) $$A = CC_0^{100}, \quad \text{with } C \text{ as in (3.13)}.$$

Without loss of generality, we can assume that $\operatorname{diam} E = 1$, and then our construction gives that Δ_0 is composed of just one cube Q^0. We shall often use the following conventions about cubes. The generation of a cube Q is the integer $k(Q)$ such that $Q \in \Delta_{k(Q)}$. Note that we shall always consider Q as a subset of E, plus the information of its generation. Thus it could be that cubes from different generations correspond to the same set of E, but we shall still think of them as different cubes.

If $k(Q) \geq 1$, the parent of Q is the cube $\widehat{Q}$ of $\Delta_{k(Q)-1}$ such that $Q \subset \widehat{Q}$. The brothers of Q are the other cubes of $\Delta_{k(Q)}$ that are contained in $\widehat{Q}$. The children of Q are the cubes of $\Delta_{k(Q)+1}$ that are contained in Q. (This does not require $k(Q) \geq 1$.)

For each $Q \in \Delta = \bigcup_{k \geq 0} \Delta_k$, set

(4.2) $$\mathcal{C}(Q) = \partial\big(\tfrac{1}{2}B(Q)\big),$$

where $B(Q)$ is the ball associated to Q in the statement of Theorem 3.2. This circle has the advantage that

$$(4.3) \qquad\qquad \operatorname{dist}\big(\mathcal{C}(Q), E \setminus Q\big) \geq \frac{1}{2} A^{-k(Q)}$$

by (3.7) and (3.8).

Our first concern will be to get rid of the places where the density of μ is too large. For each $x \in E$, set

$$(4.4) \qquad\qquad \mathcal{M}_\mu(x) = \sup_{r>0} r^{-1} \mu\big(B(x,r)\big) = \sup_{r>0} r^{-1} H^1\big(E \cap B(x,r)\big).$$

Let M be a large constant, to be chosen soon, and set

$$(4.5) \qquad\qquad \mathcal{O}(M) = \{x \in E : \mathcal{M}_\mu(x) > M\}.$$

Because $H^1(E) < \infty$, $\mathcal{M}_\mu(x) < +\infty$ for μ-almost every $x \in E$ and so we can choose M so large that

$$(4.6) \qquad\qquad \mu\big(\mathcal{O}(M)\big) < \frac{a}{8},$$

where a is as in (2.2). With this choice of M done, let us start to describe a few categories of cubes which we want to use in a stopping-time construction.

Denote by HD (high density) the set of maximal cubes $Q \in \Delta$ that are contained in $\mathcal{O}(M)$. (When two cubes correspond to the same set of E, the "largest one" is by definition the one from the earliest (= lowest) generation.)

Let $0 < a_0 < a_1 < a$ be two small constants, to be chosen later (around (4.68)). Denote by MI (medium-sized integral) the set of maximal cubes Q with the property that

$$(4.7) \qquad\qquad a_0 \mu(Q) < \operatorname{Re} \int_Q f \, d\mu \leq a_1 \mu(Q).$$

Next denote by LI (low integral) the set of maximal cubes Q with the properties that

$$(4.8) \qquad\qquad Q \text{ is not strictly contained in any cube of } \mathrm{HD} \cup \mathrm{MI},$$

and

$$(4.9) \qquad\qquad \operatorname{Re} \int_Q f \, d\mu \leq a_0 \mu(Q).$$

(If $Q = Q'$ as sets but $k(Q') < k(Q)$, we still say that Q is strictly contained in Q'.)

Note that if $Q \in \mathrm{LI}$, then $k(Q)$ (the generation of Q) is ≥ 1, since the top cube $Q^0 = E$ does not satisfy (4.9).

Denote by PLI the set of parents of cubes in LI. We have that

$$(4.10) \qquad\qquad \operatorname{Re} \int_Q f \, d\mu > a_1 \mu(Q) \quad \text{when } Q \in \mathrm{PLI},$$

since we know that Q is not contained in any cube of MI (because some child of Q satisfies (4.8)), and also that Q does not satisfy (4.9) (because it satisfies (4.8)). We shall essentially want to replace all cubes of $\mathrm{HD} \cup \mathrm{MI} \cup \mathrm{LI}$ by circles $\mathcal{C}(Q)$, but we want to get organized first.

Let I_1 denote the set of maximal cubes of $\mathrm{HD} \cup \mathrm{MI}$ that are not contained in any cube of LI. Thus

$$(4.11) \qquad\qquad \text{the cubes of } I_1 \cup \mathrm{LI} \text{ are all disjoint from each other.}$$

(The cubes of LI are disjoint by maximality and are never strictly contained in cubes of I_1, by (4.8), and the cubes of I_1 are disjoint from each other by maximality and are never contained in cubes of LI, by definition.)

4.12. Remark. If Q is any cube such that $Q \subset \mathcal{O}(M)$ or $\mathrm{Re} \int_Q f d\mu \leq a_1 \mu(Q)$, then Q is contained in some cube of $I_1 \cup \mathrm{LI}$.

Indeed, if Q is contained in some cube of $\mathrm{HD} \cup \mathrm{MI}$, then it is either contained in a cube of I_1 or else the maximal cube of $\mathrm{HD} \cup \mathrm{MI}$ that contains it has been removed because it was contained in some cube of LI. In both cases Q is contained in a cube of $I_1 \cup \mathrm{LI}$. If Q satisfies one of the properties of the remark but is not contained in any cube of $\mathrm{HD} \cup \mathrm{MI}$, then it satisfies (4.9) because it does not satisfy (4.7). It is contained in a maximal cube R that satisfies (4.9); R satisfies (4.8) as well because Q already does. In this case R lies in LI, and we are happy.

By our choice of M, recall (4.6), E is not contained in $\mathcal{O}(M)$. Then the top cube Q^0 is not in $I_1 \cup \mathrm{LI}$. We shall use later the fact that

$$
(4.13) \qquad
\begin{cases}
\mu\big(100B(Q)\big) \leq C A^{-k(Q)} & \text{for all the cubes of } I_1 \cup \mathrm{LI} \text{ and} \\
\text{all the cubes that are not contained in any cube of } I_1 \cup \mathrm{LI},
\end{cases}
$$

which follows from the fact that if Q is as in (4.13), then either $Q = Q^0$ or else the parent of Q cannot be contained in a cube of HD, by Remark 4.12.

Next set $I = I_1 \cup \mathrm{PLI}$. Note that the cubes of I are not necessarily disjoint now: cubes of PLI can contain other cubes of I. For each cube $Q \in I$, define the "operating generation k_Q" by

$$
(4.14) \qquad
\begin{cases}
k_Q = k(Q) & \text{if } Q \in I_1 \\
k_Q = k(Q) + 1 & \text{if } Q \in \mathrm{PLI}.
\end{cases}
$$

We put an order on I such that Q comes before Q' if $k_Q \leq k_{Q'}$, and also, when $k_Q = k_{Q'}$, Q comes before Q' if $Q \in \mathrm{PLI}$ and $Q' \in I_1$. (A cube cannot be in PLI and I_1 at the same time, by (4.11).) Now denote by Q_n, $n \geq 1$, the nth cube of I for this order. Also set $k_n = k_{Q_n}$. Thus $k_{n+1} \geq k_n$, and if $k_{n+1} = k_n$ we cannot have that $Q_n \in I_1$ and $Q_{n+1} \in \mathrm{PLI}$. In what follows, we shall always act as if I was infinite. If this is not the case, simply truncate the construction described below.

We want to define a sequence of measures F_n. We shall do this by induction on n. Each F_n will be of the form

$$
(4.15) \qquad F_n = \rho_n f d\mu + \sum_{m \leq n} \alpha_m \, d\nu_m,
$$

where the α_m's will be complex numbers, each ν_m will be a positive constant multiple of the arclength measure on some circle $\mathcal{C}_m$, and ρ_n will be a Borel function on E such that

$$
(4.16) \qquad 0 \leq \rho_n \leq 1.
$$

Moreover, the function ρ_n will be constant on each of the cubes Q_m, $m > n$.

Now let us construct the measures F_n for good. We start with $F_0 = f d\mu$ (i.e., no measure ν_0 and $\rho_0 \equiv 1$). Now suppose that we already know F_{n-1}, and that it has the form described above, and let us construct F_n.

The first case is when $Q_n \in I_1$. In this case we simply want to replace Q_n with $\mathcal{C}_n = \mathcal{C}(Q_n)$. We want to do this without changing $\int F_n$, though. Denote by ρ_n^* the constant value of ρ_{n-1} on Q_n. We take

$$(4.17) \qquad d\nu_n = \rho_n^* \frac{\mu(Q_n)}{H^1(\mathcal{C}_n)} \, dH^1|\mathcal{C}_n$$

$$(4.18) \qquad \rho_n = \rho_{n-1} \mathbf{1}_{E \setminus Q_n},$$

$$(4.19) \qquad \alpha_n = \mu(Q_n)^{-1} \int_{Q_n} f d\mu,$$

and define F_n by the formula (4.15). Thus

$$(4.20) \qquad F_n = F_{n-1} - \mathbf{1}_{Q_n} \rho_{n-1} f d\mu + \alpha_n \, d\nu_n.$$

It is easy to see that F_n is of the type described above. In particular, if $m > n$, then ρ_{n-1} was constant on Q_m by induction hypothesis, and so it is enough to check that Q_n does not meet Q_m. If $Q_m \in I_1$, this follows from the disjointness of the cubes of I_1 (by (4.11)). Otherwise $Q_m \in \mathrm{PLI}$, and hence $k_m > k_n$ by definition of our order. Now if $Q_m \cap Q_n \neq \emptyset$, then $Q_m \subset Q_n$, and this is not possible because Q_m contains some cubes of LI, which are disjoint from Q_n by (4.11).

Let us collect a few easy facts for future reference:

$$(4.21) \qquad \|\nu_n\| = \rho_n^* \, \mu(Q_n) \le \mu(Q_n);$$

$$(4.22) \qquad \int F_n = \int F_{n-1}$$

(compare (4.17), (4.19) and (4.20));

$$(4.23) \qquad |\alpha_n| \le 1$$

(by (2.2)). Let us also check that

$$(4.24) \qquad \mathrm{Re} \, \alpha_n \ge a_0.$$

If $Q_n \in \mathrm{MI}$, this follows from (4.7). Otherwise, we are not worried either; we can use the fact that $Q_n \in I_1$, and hence is not contained in any cube of LI. Since Q_n is not strictly contained in any cube of $\mathrm{HD} \cup \mathrm{MI}$ (it is a maximal cube of $\mathrm{HD} \cup \mathrm{MI}$), it cannot satisfy (4.9) because otherwise it would be in LI. This proves (4.24).

Now let us define F_n when $Q_n \in \mathrm{PLI}$. Some amount of notation will be useful. Denote by $\mathcal{A}_n$ the set of the children of Q_n that lie in LI. These are the cubes Q such that $k(Q) = k_n$, $Q \subset Q_n$, and $Q \in \mathrm{LI}$. Also let $\mathcal{A}_n^*$ denote the set of the children of Q_n that are not in LI. Set

$$(4.25) \qquad H_n = \bigcup_{Q \in \mathcal{A}_n} Q \quad \text{and} \quad G_n = \bigcup_{Q \in \mathcal{A}_n^*} Q.$$

Thus Q_n is the disjoint union of H_n and G_n. We want to remove H_n and keep G_n.

Choose any cube $Q_n^* \in \mathcal{A}_n$, and take $\mathcal{C}_n = \mathcal{C}(Q_n^*)$. Also choose

$$(4.26) \qquad d\nu_n = \rho_n^* \frac{\mu(H_n)}{H^1(\mathcal{C}_n)} \, dH^1|\mathcal{C}_n,$$

where ρ_n^* still denotes the constant value of ρ_{n-1} on Q_n (which exists by induction hypothesis). We want to define ρ_n by

$$(4.27) \qquad \rho_n(x) = \begin{cases} \rho_{n-1}(x) & \text{when } x \in E \setminus Q_n \\ 0 & \text{when } x \in H_n \\ (1 - \theta_n)\,\rho_{n-1}(x) & \text{when } x \in G_n, \end{cases}$$

with a number θ_n such that

$$(4.28) \qquad \theta_n \operatorname{Re} \int_{G_n} f\,d\mu = a_0\mu(H_n) - \operatorname{Re} \int_{H_n} f\,d\mu.$$

Let us check that $\operatorname{Re} \int_{G_n} f\,d\mu \neq 0$, and at the same time derive some estimates on θ_n. First observe that H_n is a disjoint union of cubes of LI, and so (4.9) tells us that

$$(4.29) \qquad \operatorname{Re} \int_{H_n} f\,d\mu \leq a_0\mu(H_n).$$

Also

$$(4.30) \qquad \operatorname{Re} \int_{Q_n} f\,d\mu \geq a_1\mu(Q_n)$$

by (4.10). Then

$$(4.31) \qquad \begin{aligned} \operatorname{Re} \int_{G_n} f\,d\mu &= \operatorname{Re} \int_{Q_n} f\,d\mu - \operatorname{Re} \int_{H_n} f\,d\mu \\ &\geq a_1\mu(Q_n) - \operatorname{Re} \int_{H_n} f\,d\mu \\ &\geq a_1\mu(Q_n) - a_0\mu(H_n) \\ &> (a_1 - a_0)\,\mu(Q_n). \end{aligned}$$

¿From this we deduce that (4.28) defines a unique θ_n, and also that θ_n has the same sign as $a_0\mu(H_n) - \operatorname{Re} \int_{H_n} f\,d\mu$, i.e., is nonnegative by (4.29). From the second line of (4.31) we deduce that $\operatorname{Re} \int_{G_n} f\,d\mu$ is larger than the right-hand side of (4.28). Thus

$$(4.32) \qquad 0 \leq \theta_n < 1.$$

We also have the brutal estimate that

$$a_0\mu(H_n) - \operatorname{Re} \int_{H_n} f\,d\mu \leq (a_0 + 1)\,\mu(H_n)$$

(since $\|f\|_\infty \leq 1$), from which we deduce that

$$(4.33) \qquad \theta_n \leq \frac{a_0 + 1}{a_1 - a_0} \frac{\mu(H_n)}{\mu(Q_n)}$$

with the help of (4.31).

We may now return to the definition of F_n when $Q_n \in$ PLI. We already defined ν_n and ρ_n. Now take

(4.34) $$\operatorname{Re} \alpha_n = a_0,$$

(4.35) $$\operatorname{Im} \alpha_n = \mu(H_n)^{-1} \left\{ \operatorname{Im} \int_{H_n} f d\mu + \theta_n \operatorname{Im} \int_{G_n} f d\mu \right\}$$

and define F_n by

(4.36) $$F_n = F_{n-1} - \mathbf{1}_{H_n} \rho_{n-1} f d\mu - \theta_n \mathbf{1}_{G_n} \rho_{n-1} f d\mu + \alpha_n \, d\nu_n.$$

Then (4.15) follows from the definition (4.27) of ρ_n and our induction hypothesis.

Now we should check that F_n has the form described above. First, $0 \leq \rho_n \leq 1$ because $0 \leq \theta_n \leq 1$. Let us check that ρ_n is constant on each cube Q_m, $m > n$. If $k(Q_m) \geq k_n$, then this follows from the induction hypothesis, and the fact that the cubes that compose H_n and G_n are of generation k_n. Otherwise, $Q_m \in$ PLI and $k_m = k_n$, because our order on I is such that $k_m \geq k_n$. In this case Q_m and Q_n are disjoint because they are of the same generation, and $\rho_n = \rho_{n-1}$ or Q_m. Thus F_n has the form required for the continuation of the induction.

This completes our definition of the sequence F_n. If I is finite, we simply stop when there is no Q_n left and adapt the rest of this proof in the obvious way. Let us also collect a few estimates from the construction of F_n when $Q_n \in$ PLI. Obviously (4.26) gives

(4.37) $$\|\nu_n\| = \rho_n^* \mu(H_n) \leq \mu(H_n);$$

we deduce from (4.34), (4.35), (4.33) and the boundedness of f that

(4.38) $$|\alpha_n| \leq C.$$

Let us check that

(4.39) $$\int F_n = \int F_{n-1}$$

also when $Q_n \in$ PLI. Replace ρ_{n-1} by its value ρ_n^* on Q_n in (4.36), and also recall that $\|\nu_n\| = \rho_n^* \mu(H_n)$ by (4.37); we see that (4.39) is the same as

(4.40) $$\alpha_n \rho_n^* \mu(H_n) = \rho_n^* \int_{H_n} f d\mu + \rho_n^* \theta_n \int_{G_n} f d\mu.$$

Now (4.40) holds because of (4.34) and (4.28) for the real part, and (4.35) for the imaginary part.

Now we wish to define our function g and measure ν. Of course $g d\nu$ will be the limit of the measures F_n, but it will be better to spend some time and define g and ν quietly.

First observe that the functions ρ_n are a decreasing sequence of nonnegative functions on E. Denote by ρ_∞ the limit of this sequence. Obviously, $0 \leq \rho_\infty \leq 1$ on E. Also set $d\mu_\infty = \rho_\infty \, d\mu$. This is the first half of $d\nu$. The second half is the sum of the measures $d\nu_n$. In other words, we take

(4.41) $$d\nu = d\mu_\infty + \sum_n d\nu_n = \rho_\infty \, d\mu + \sum_n d\nu_n.$$

Set

(4.42) $$E_\infty = \{ x \in E : \rho_\infty(x) \neq 0 \}.$$

Our intention is to take $g(x) = f(x)$ on E_∞ and $g(x) = \alpha_n$ on $\mathcal{C}_n$, but this will be more pleasant to do once we know that those sets are disjoint. It will also be useful to know that they are reasonably far from each other.

4.43. Lemma. *We have that*

$$\text{(4.44)} \qquad \text{dist}(\mathcal{C}_n, E_\infty) \geq \frac{1}{2} A^{-k_n}$$

and

$$\text{(4.45)} \qquad \text{dist}(\mathcal{C}_n, \mathcal{C}_m) \geq \frac{1}{4} \operatorname{Max}(A^{-k_n}, A^{-k_m})$$

for all n and $m \neq n$.

First suppose that $Q_n \in I_1$. Then $\mathcal{C}_n = \partial\left(\frac{1}{2} B(Q_n)\right)$ and $k_n = k(Q_n)$. In this case,

$$\text{(4.46)} \qquad \text{dist}(\mathcal{C}_n, E \setminus Q_n) \geq \frac{1}{2} A^{-k_n}$$

by (4.3), and (4.44) follows from this (recall that $\rho(x) \equiv 0$ on Q_n by (4.18)).

When $Q_n \in \text{PLI}$, $\mathcal{C}_n = \mathcal{C}(Q_n^*)$ for some child Q_n^* of Q_n, and $k_n = k(Q_n^*)$ by (4.14). Since H_n contains Q_n^* because $Q_n^* \in \text{LI}$, (4.3) implies that

$$\text{(4.47)} \qquad \text{dist}(\mathcal{C}_n, E \setminus H_n) \geq \frac{1}{2} A^{-k_n};$$

(4.44) follows from this because $\rho(x) = 0$ on H_n, by (4.27). Thus (4.44) holds in all cases.

Clearly it is enough to prove (4.45) when $n < m$. By definition of the order on I, $k_m \geq k_n$ and the maximum in (4.45) is A^{-k_n}. Let Q and Q' be the cubes such that $\mathcal{C}_n = \mathcal{C}(Q)$ and $\mathcal{C}_m = \mathcal{C}(Q')$. Thus Q is Q_n itself, or a child Q_n^* if $Q_n \in \text{PLI}$, and similarly for Q'. A first case is when $k_m = k_n$, i.e., Q and Q' are of the same generation k_n. Then Q and Q' are disjoint (the only "dangerous" case would be when $Q_n \in \text{PLI}$ and $Q' \subset Q_n$, but then $Q' \in I_1$ and $Q \in \text{LI}$ so they are disjoint). In this case we may use the fact that $5B(Q)$ and $5B(Q')$ are disjoint (by (3.9)), and (4.45) follows from the lower bound (3.7) or the radii of $B(Q)$ and $B(Q')$.

When $k_m > k_n$, Q' is still disjoint from Q (because they are both cubes of $I_1 \cup \text{LI}$, and they are different because they are from different generations). Then $\text{dist}(Q', \mathcal{C}_n) \geq \frac{1}{2} A^{-k_n}$ by (4.3), and since the center of $\mathcal{C}_m$ belongs to Q' and $\operatorname{diam} \mathcal{C}_m \leq C_0 A^{-k_m} < \frac{1}{4} A^{-k_n}$ by (3.7), we get (4.45). This completes the proof of Lemma 4.43.

An easy consequence of this lemma is that the circles $\mathcal{C}_n$ are disjoint from each other and from E_∞. This allows us to define g on $E_\infty \cup \left(\bigcup_n \mathcal{C}_n \right)$ by

$$\text{(4.48)} \qquad \begin{cases} g(x) = \alpha_n & \text{on } \mathcal{C}_n, \text{ and} \\ g(x) = f(x) & \text{on } E_\infty. \end{cases}$$

Next we want to check that g and ν satisfy the properties required for Theorem 2.4, except for the last one which is related to the Cauchy kernel.

4.2. The easy properties of $gd\nu$

We shall start with the upper-regularity condition (2.5). ¿From now on throughout the rest of the paper C will denote constants which may change from line to line and which may depend only on C_0, A, M, a, a_0, a_1, $\mu(E)$ and M_0, defined in (4.87). (But recall that we can fix C_0 and A as large absolute constants and a_0 and a_1 will depend only on $a^{-1} \mu(E)$.)

4.49. Lemma. *There is a constant $C > 0$ such that*

$$(4.50) \qquad \mu_\infty\big(B(x,r)\big) \leq Cr \quad \text{for all } x \in \mathbb{C} \text{ and } r > 0.$$

Since E is compact (and even has diameter ≤ 1 by our earlier normalization), it is enough to prove (4.50) when $r \leq 1$. Cover $E \cap B(x,r)$ by cubes $Q \in \Delta_k$, where k is chosen so that $A^{-k} \sim r$. You need less than C cubes Q, because all the balls $5B(Q)$ are disjoint and have radii $\geq 5A^{-k}$, while the balls $28B(Q)$ all meet $B(x,r)$ (when Q meets it) and have radii $\leq 28C_0 A^{-k}$. We have that $\mu(Q) \leq CA^{-k} \leq Cr$ for all the cubes of our collection that are not contained in any cube of $I_1 \cup \mathrm{LI}$, by Remark 4.12. (If $\mu(A) > CA^{-k}$ with C large enough, then Q is contained in $\mathcal{O}(M)$.) So the total mass of these cubes is $\leq Cr$. On the other hand, if Q is contained in some cube of $I_1 \cup \mathrm{LI}$, then $\rho_\infty \equiv 0$ on Q (because $\rho_\infty(x) = 0$ on all cubes of $I_1 \cup \mathrm{LI}$, by (4.18) and (4.27)). Hence the total mass of these cubes for μ_∞ is zero. This proves the lemma.

Now we want to control the measures ν_n. We start with estimates on the position of the cubes Q_n. We claim that

$$(4.51) \qquad \sum_{\substack{Q \in I_1 \cup \mathrm{LI} \\ Q \text{ meets } B(x,r) \\ A^{-k(Q)} \leq r}} \mu(Q) \leq Cr$$

for all $x \in \mathbb{C}$ and $r > 0$.

Since $\operatorname{diam} E \leq 1$ by our earlier normalization, we can safely suppose that $r \leq 1$. If $Q \in I_1 \cup \mathrm{LI}$, then its parent $\widehat{Q}$ is not contained in $\mathcal{O}(M)$; this follows from Remark 4.12 and the disjointness property (4.11). If Q meets $B(x,r)$ and $A^{-k(Q)} \leq r$, then $\widehat{Q} \subset B(x,Cr)$ if C is large enough, and then $B(x,Cr)$ is not contained in $\mathcal{O}(M)$ either. Thus either the sum in (4.51) is empty, or else $\mu\big(B(x,r)\big) \leq \mu\big(B(x,Cr)\big) \leq MCr$, and (4.51) follows from the disjointness property (4.11).

Let us also record that

$$(4.52) \qquad \sum_{\substack{Q_n \in \mathrm{PLI} \\ Q_n \subset B(x,r)}} \mu(H_n) \leq Cr$$

for all $x \in \mathbb{C}$ and $r > 0$. This is an easy consequence of the proof of (4.51), because the cubes Q_n are not contained in $\mathcal{O}(M)$ and the sets H_n are obtained from the cubes of LI by regrouping brothers.

We are now ready to prove that ν satisfies (2.5). Recall from (4.41) that $\nu = \mu_\infty + \sum_n \nu_n$. Because of Lemma 4.49, it is enough to show that

$$(4.53) \qquad \sum_n \nu_n\big(B(x,r)\big) \leq Cr \quad \text{for all } x \in \mathbb{C} \text{ and } r > 0.$$

Since by construction all the circles $\mathcal{C}_n$ lie in a fixed ball of radius C (and even with $C = 1$), it is enough to prove (4.53) when $r \leq 1$. Let $B(x,r)$ be given, and start with the set $\mathcal{N}_1$ of integers n for which $Q_n \in I_1$, $\mathcal{C}_n$ meets $B(x,r)$ and $A^{-k(Q_n)} \leq r$. Then

$$(4.54) \qquad \sum_{n \in \mathcal{N}_1} \nu_n\big(B(x,r)\big) \leq \sum_{n \in \mathcal{N}_1} \|\nu_n\| \leq \sum_{n \in \mathcal{N}_1} \mu(Q_n) \leq Cr$$

by (4.21) and (4.51). Similarly, if $\mathcal{N}_2$ is the set of integers n for which $Q_n \in$ PLI and $Q_n \subset B(x, 2r)$, then

$$
(4.55) \qquad \sum_{n \in \mathcal{N}_2} \nu_n\big(B(x, r)\big) \leq \sum_{n \in \mathcal{N}_2} \|\nu_n\| \leq \sum_{n \in \mathcal{N}_2} \mu(H_n) \leq Cr
$$

by (4.37) and (4.52).

Thus we are left with the set $\mathcal{N}_3$ of integers n for which $\mathcal{C}_n$ meets $B(x, r)$ and either $Q_n \in I_1$ and $A^{-k(Q_n)} > r$ or $Q_n \in$ PLI and is not contained in $B(x, 2r)$. In this last case we also have that $A^{-k_n} \geq r/C$, and (4.45) tells us that $\mathcal{N}_3$ has at most C elements. Thus it is enough to prove that $\nu_n\big(B(x, r)\big) \leq Cr$ for each single $\mathcal{C}_n$. This is an immediate consequence of the fact that the densities $\rho_n^* \frac{\mu(Q_n)}{H^1(\mathcal{C}_n)}$ and $\rho_n^* \frac{\mu(H_n)}{H^1(\mathcal{C}_n)}$ in (4.17) and (4.26) are less than C, which follows from (4.13) applied to Q_n.

Thus ν satisfies the upper regularity condition (2.5).

The constant that we obtain in (2.5) may depend wildly on E, because it depends on our initial choice of M, as in (4.6). On the other hand, if we start from a measure μ that already satisfies (2.5) with a constant C_1, then we can take $M \leq CC_1$ and ν satisfies (2.5) with a constant $\leq C'C_1$, where C and C' are absolute constants. This follows from the proof above, together with the fact that the constants that come from the construction of dyadic cubes can be taken to be absolute constants.

Next we want to prove the accretivity condition (2.6). When $x \in \mathcal{C}_n$, $g(x) = \alpha_n$, and so $|\alpha_n| \leq C$ (by (4.23) or (4.38)) and $\operatorname{Re} \alpha_n \geq a_0$ (by (4.24) or (4.34)). When $x \in E_\infty$, $g(x) = f(x)$ and so $|g(x)| \leq 1$ by (2.2). We shall not be able to prove that $\operatorname{Re} f(x) \geq a_0$ everywhere on E_∞, but only μ-almost everywhere. Of course this is enough, because modifying g on a subset of μ-measure zero of E_∞ will not alter the other properties required for the theorem. We shall use a simple variant of the Lebesgue density theorem.

4.56. Lemma. *For each $x \in E$ and each $k \geq 0$, denote by $Q(x, k)$ the cube of Δ_k that contains x. Then*

$$
(4.57) \qquad f(x) = \lim_{k \to +\infty} \left\{ \mu\big(Q(x, k)\big)^{-1} \int_{Q(x, k)} f \, d\mu \right\}
$$

for μ-almost every $x \in E$.

Let us first say why (2.6) follows from this lemma. Let $x \in E$ be such that (4.57) holds. If $\operatorname{Re} f(x) < a_0$, we can find a small cube Q that contains x and for which $\operatorname{Re} \int_Q f \, d\mu < a_0 \mu(Q)$. Then Remark 4.12 says that Q is contained in a cube of $I_1 \cup$ LI. Since $\rho(x) = 0$ or all cubes of $I_1 \cup$ LI (by the definition of ρ_n on Q_n), x cannot lie in E_∞. This proves that $\operatorname{Re} f(x) \geq a_0$ (and hence $\operatorname{Re} g(x) \geq a_0$) μ-almost everywhere on E_∞. This completes the proof of (2.6), modulo the lemma.

The proof of Lemma 4.56 is standard. We certainly have (4.57) everywhere for all continuous functions f. Because of the standard approximation argument, it is enough to prove a weak-L^1 estimate for the associated maximal function. Let us make this more precise.

For each L^1-function h on E, define the maximal function Mh by

$$
(4.58) \qquad Mh(x) = \sup_{k \geq 0} \left\{ \mu\big(Q(x, k)\big)^{-1} \int_{Q(x, k)} |h| \, d\mu \right\}
$$

for all $x \in E$, and where $Q(x, k)$ is as in the statement of Lemma 4.56.

For each $\lambda > 0$, set $U(\lambda) = \{x \in E : Mh(x) > \lambda\}$, and denote by $\Delta(\lambda)$ the set of maximal cubes of $\Delta = \bigcup_k \Delta_k$ such $\int_Q |h|\,d\mu > \lambda\mu(Q)$. Each cube of $\Delta(\lambda)$ is clearly contained in $U(\lambda)$, and conversely every point of $U(\lambda)$ lies in a $Q \in \Delta(\lambda)$. Thus $U(\lambda)$ is the disjoint union of the cubes of $\Delta(\lambda)$, and

$$(4.59) \qquad \mu\big(U(\lambda)\big) = \sum_{Q \in \Delta(\lambda)} \mu(Q) \leq \lambda^{-1} \sum_{Q \in \Delta(\lambda)} \int_Q |h|\,d\mu \leq \lambda^{-1} \int_E |h|\,d\mu.$$

This is the weak-L^1 estimate that we needed to deduce (4.57) for all $f \in L^1$ from the case of continuous functions. This completes our proof of (2.6).

The next property (2.7) follows from (2.2), (4.22) and (4.39) by taking limits. The reader should not worry about convergence: it is easy to see that $\sum_n \|F_{n+1} - F_n\|$ converges by (4.15), the fact that μ is finite and the sequence $\{\rho_n\}$ is decreasing, and because $|\alpha_n| \leq C$ and

$$\sum \|\nu_n\| \leq \sum_{Q \in I_1} \mu(Q) + \sum_{Q_n \in \mathrm{PLI}} \mu(H_n) < +\infty$$

by (4.21), (4.37), (4.51) and (4.52) (or directly (4.11).)

Next we want to prove (2.8). Set

$$(4.60) \qquad F = \{x \in E_\infty : \rho_\infty(x) \geq \tau\},$$

where $\tau = \frac{a}{4\mu(E)}$. Note that $d\nu = \rho_\infty d\mu$ on F because $F \subset E_\infty$ and the circles C_n live away from E_∞. Clearly $\tau d\mu \leq d\nu \leq d\mu$ on F, and so it is enough to prove that $\nu(F) \geq \frac{a}{2}$.

We know from (2.7) that $\int g\,d\nu = a$. Also $\nu(F) \geq \int_F \mathrm{Re}\,f\,d\nu = \int_F \mathrm{Re}\,g\,d\nu$ because $|f| \leq 1$ and $f = g$ on E_∞. Thus it is enough to prove that

$$(4.61) \qquad \mathrm{Re} \int g\,d\nu - \mathrm{Re} \int_F g\,d\nu \leq \frac{a}{2}.$$

Recall that $d\nu = \rho_\infty d\mu + \sum_n d\nu_n$ and that $g(x) = f(x)$ on E_∞ and $g(x) = \alpha_n$ on C_n. Then (4.61) is equivalent to

$$(4.62) \qquad \mathrm{Re} \int_{E_\infty \setminus F} f\rho_\infty d\mu + \sum_n \|\nu_n\| \, \mathrm{Re}\,\alpha_n \leq \frac{a}{2}.$$

Note that $\rho_\infty(x) \leq \tau$ on $E_\infty \setminus F$, by the definition of F. Hence

$$(4.63) \qquad \mathrm{Re} \int_{E_\infty \setminus F} f\rho_\infty d\mu \leq \|f\|_\infty \tau\mu(E) \leq \frac{a}{4}.$$

When $Q_n \in \mathrm{PLI}$, $\mathrm{Re}\,\alpha_n = a_0$ by (4.34) and $\|\nu_n\| \leq \mu(H_n)$ by (4.37). Then

$$(4.64) \qquad \|\nu_n\| \, \mathrm{Re}\,\alpha_n \leq a_0\mu(H_n) \quad \text{when } Q_n \in \mathrm{PLI}.$$

When $Q_n \in I_1$, $\|\nu_n\| = \rho_n^*\mu(Q_n)$ by (4.21), where ρ_n^* is the constant value of ρ_{n-1} on Q_n, and $\alpha_n = \mu(Q_n)^{-1} \int_{Q_n} f\,d\mu$ by (4.19). So

$$(4.65) \qquad \|\nu_n\| \, \mathrm{Re}\,\alpha_n = \rho_n^* \, \mathrm{Re} \int_{Q_n} f\,d\mu \quad \text{when } Q_n \in I_1.$$

If furthermore $Q_n \in \mathrm{MI}$, then it satisfies (4.7) and so $\|\nu_n\| \operatorname{Re} \alpha_n \leq \rho_n^* a_1 \mu(Q_n) \leq a_1 \mu(Q_n)$. Otherwise we can only say that $\|\nu_n\| \operatorname{Re} \alpha_n \leq \mu(Q_n)$ (because $|f| \leq 1$), but we have the advantage of knowing that $Q \in \mathrm{HD}$. Altogether,

$$(4.66) \qquad \sum_n \|\nu_n\| \operatorname{Re} \alpha_n \leq a_0 \sum_{Q_n \in \mathrm{PLI}} \mu(H_n) + a_1 \sum_{Q_n \in I_1 \cap \mathrm{MI}} \mu(Q_n) + \sum_{Q_n \in I_1 \cap \mathrm{HD}} \mu(Q_n).$$

Recall from (4.11) that the cubes of $I_1 \cup \mathrm{LI}$ are disjoint. Also, $\sum_{Q_n \in \mathrm{PLI}} \mu(H_n) = \sum_{Q \in \mathrm{LI}} \mu(Q)$, because the sets H_n are obtained from the set of cubes of LI by regrouping brothers. Hence

$$(4.67) \qquad a_0 \sum_{Q_n \in \mathrm{PLI}} \mu(H_n) + a_1 \sum_{Q_n \in I_1} \mu(Q_n) \leq a_1 \mu(E),$$

and we can make this $\leq \frac{a}{8}$ by taking a_1 small enough. We can choose $a_1 = \frac{a}{8\mu(E)}$, for instance.

The last sum is taken care of by noticing that the cubes of $I_1 \cap \mathrm{HD}$ are disjoint and contained in $\mathcal{O}(M)$. Then

$$(4.68) \qquad \sum_{Q_n \in I_1 \cap \mathrm{HD}} \mu(Q_n) \leq \mu\big(\mathcal{O}(M)\big) < \frac{a}{8}$$

by (4.6).

The desired estimate (4.62) follows from (4.67), (4.68), our choice of a_1, (4.66) and (4.63). This completes our proof of (2.8).

We did not choose a_0 yet. In fact, any value of a_0 in $(0, a_1)$ will work, but for the sake of definiteness let us choose $a_0 = \frac{1}{2} a_1$.

We are now finished with the verification of the "easy properties" of g and ν. Note that the constant C in (2.6) and (2.8) can be taken to be $\leq C_1 \, a^{-1} \, \mu(E)$ for some absolute constant C_1, as we announced after the statement of Theorem 2.4. The remaining property, i.e., the L^2-estimate for the maximal Cauchy integral of $g d\nu$, will keep us busy for the rest of the paper.

4.3. L^2-estimates: reduction to positive functions

Our proof that $T^*(g d\nu) \in L^2(d\nu)$ will be in the same spirit as the estimates in [C2], even if we pay a rather high price for the fact that we are not working with a space of homogeneous type.

In this section we reduce the proof of (2.9) to the study of certain positive functions. The L^2-estimates for these functions will be derived in the last chapter, using the small boundary properties of our cubes (which we did not use so far).

We start with a few notations. First, it will be convenient to define H_n even when $Q_n \in I_1$. In this case we simply set $H_n = Q_n$. Also set $G_n = \emptyset$ when $Q_n \in I_1$. With this convention,

$$(4.69) \qquad \text{the sets } H_n, \, n \in I, \text{ are disjoint}$$

because the sets H_n, $Q_n \in \mathrm{PLI}$, were obtained from the cubes $Q \in \mathrm{LI}$ by regrouping brothers, and by (4.11). Also

$$(4.70) \qquad \sum_{\substack{Q_n \subset B(x,r) \\ A^{-kn} \leq r}} \mu(H_n) \leq Cr$$

for all $x \in \mathbb{C}$ and $r > 0$, by (4.51) and (4.52). Set

$$
(4.71) \qquad\qquad \varphi_n = F_n - F_{n-1}
$$

for $n \geq 1$. With our unified notations, (4.20) and (4.36) reduce to

$$
(4.72) \qquad\qquad \varphi_n = -\mathbf{1}_{H_n}\, \rho_{n-1}\, f d\mu - \theta_n\, \mathbf{1}_{G_n}\, \rho_{n-1}\, f d\mu + \alpha_n\, d\nu_n;
$$

we do not have to define θ_n when $Q_n \in I_1$, since G_n is then empty. We shall often use the estimates

$$
(4.73) \qquad\qquad |\alpha_n| \leq C \quad \text{and} \quad \|\nu_n\| \leq \mu(H_n),
$$

which follow from (4.21), (4.23), (4.37) and (4.38).

Denote by BLI (brothers of LI) the union of all the sets $\mathcal{A}_n^*$, where $Q_n \in$ PLI. (The definition of $\mathcal{A}_n^*$ is given just before (4.25).) This is the set of the children of the cubes of PLI that are not in LI. We should be a little more careful with the cubes of BLI, because they are not necessarily disjoint. Set

$$
(4.74) \qquad
\begin{cases}
\theta(Q) = 1 & \text{when } Q \in I_1 \cup \text{LI, and} \\
\theta(Q) = \theta_n & \text{when } Q \in \mathcal{A}_n^* \text{ and } Q_n \in \text{PLI.}
\end{cases}
$$

There is a potential conflict here, because a given cube Q could be in BLI and in I_1 at the same time. In this case $\theta(Q)$ will be defined twice. This will not be a problem; it will always be clear from the context which $\theta(Q)$ we need to take, and the worse thing that may happen is that a given cube will appear twice in a same sum, once with $\theta(Q) = 1$ and once with $\theta(Q) = \theta_n$.

Note that for each $Q_n \in$ PLI,

$$
(4.75) \qquad\qquad \sum_{Q \in \mathcal{A}_n^*} \theta(Q)\, \mu(Q) = \theta_n\, \mu(G_n) \leq C\mu(H_n)
$$

by (4.33). Then

$$
(4.76) \qquad\qquad \sum_{\substack{Q \in \text{BLI} \\ Q \subset B(x,r) \\ A^{-k(Q)} \leq r}} \theta(Q)\, \mu(Q) \leq Cr
$$

when $x \in \mathbb{C}$ and $r > 0$, because the left-hand side of (4.76) is less than

$$
\sum_{\substack{Q_n \in \text{PLI} \\ Q_n \subset B(x,Cr)}} \ \sum_{Q \in \mathcal{A}_n^*} \theta(Q)\, \mu(Q),
$$

which is less than Cr by (4.75) and (4.52).

We are now ready to define the positive functions that will control $T^*(g d\nu)$. For each cube $Q \in \Delta$, define a normalized inverse distance function ρ_Q by

$$
(4.77) \qquad\qquad \rho_Q(x) = \left\{ 1 + A^{k(Q)} \operatorname{dist}(x, Q) \right\}^{-1},
$$

and then set

$$(4.78) \qquad e_Q^*(x) = \rho_Q(x) \int_Q \frac{d\mu(y)}{|x-y|}$$

for all $x \in \mathbb{C}$. Also set

$$(4.79) \qquad 2Q = \left\{ y \in E : \mathrm{dist}(y, Q) \leq A^{-k(Q)} \right\}$$

and

$$(4.80) \qquad h_Q^*(x) = \int_{2Q \backslash Q} \frac{d\mu(y)}{|x-y|}.$$

The next proposition gives the control on $T^*(gd\nu)$ that we want on E_∞.

4.81. Proposition. *If $x \in E_\infty$, then*

$$(4.82) \qquad T^*(gd\nu)(x) \leq C + C \sum_{Q \in I_1 \cup \mathrm{LI} \cup \mathrm{BLI}} \theta(Q)\, \mathbf{1}_{E \backslash Q}(x)\, e_Q^*(x)$$

$$+ C \sum_{Q \in \mathrm{BLI}} \theta(Q)\, \mathbf{1}_Q(x)\, h_Q^*(x).$$

We shall also have to estimate $T^*(gd\nu)$ on the circles $\mathcal{C}_n$, and so we need a variant of Proposition 4.81 for $x \in \mathcal{C}_n$. For each $Q \in I_1 \cup \mathrm{LI} \cup \mathrm{BLI}$, let $J(Q)$ denote the set of integers $m \geq 1$ such that Q_m does not meet Q, or else $Q_m \in \mathrm{PLI}$ and $Q \subset G_m$. Then set

$$(4.83) \qquad \widetilde{e}_Q(x) = \sum_{n \in J(Q)} \mathbf{1}_{\mathcal{C}_n}(x)\, e_Q^*(x),$$

and also

$$(4.84) \qquad \widetilde{h}_Q(x) = \sum_{n : Q_n \subset Q} \mathbf{1}_{\mathcal{C}_n}(x)\, h_Q^*(x)$$

when $Q \in \mathrm{BLI}$.

4.85. Proposition. *If $x \in \bigcup_n \mathcal{C}_n$, then*

$$(4.86) \qquad T^*(gd\nu)(x) \leq C + C \sum_{Q \in I_1 \cup \mathrm{LI} \cup \mathrm{BLI}} \theta(Q)\, \widetilde{e}_Q(x) + C \sum_{Q \in \mathrm{BLI}} \theta(Q)\, \widetilde{h}_Q(x).$$

In this section we shall prove the two propositions; the functions e_Q^* and h_Q^* will be estimated in Part 5.

First we want to use the boundedness of $T(fd\mu)$ to get estimates on truncated integrals. Our assumption from Chapter 2 says that

$$(4.87) \qquad M_0 = \sup \left\{ |T(fd\mu)(x)| : x \in \mathbb{C} \backslash E \right\} < +\infty,$$

where $T(fd\mu)$ is as in (2.3).

For each finite complex measure φ on $\mathbb{C}$ and all $\varepsilon > 0$, set

$$(4.88) \qquad T^\varepsilon \varphi(x) = \int_{\mathbb{C} \backslash B(x,\varepsilon)} \frac{d\varphi(y)}{x-y}$$

for all $x \in \mathbb{C}$.

4.89. Lemma. *We have that*

$$(4.90) \qquad |T^\varepsilon(f d\mu)(x)| \le M_0 + C \sup_{r > \varepsilon} \left\{ r^{-1} \mu(B(x,r)) \right\}$$

for all $x \in \mathbb{C}$.

To prove this, let x and ε be given, and set $h(z) = \int_{\mathbb{C} \setminus B(x,\varepsilon)} \frac{f(y)\, d\mu(y)}{z - y}$ for $z \in B(x, \frac{\varepsilon}{2})$. Then

$$(4.91) \qquad |T^\varepsilon(f d\mu)(x) - h(z)| \le \int_{\mathbb{C} \setminus B(x,\varepsilon)} \frac{|z - x|\, d\mu(y)}{|z - y|\, |x - y|} \le 4 \sup_{r > \varepsilon} \left\{ r^{-1} \mu(B(x,r)) \right\}$$

by the usual computation. (Cut the domain of integration into annuli $A_k : \left\{ y : 2^k \varepsilon \le |x - y| < 2^{k+1} \varepsilon \right\}$, $k = 0, 1, \ldots$, note that $\frac{|z - x|}{|z - y|\,|x - y|} \le 2^{-2k} \varepsilon^{-1}$ on A_k and $\mu(A_k) \le 2^{k+1} \varepsilon \sup_{r > \varepsilon} \left\{ r^{-1} \mu(B(x,r)) \right\}$, and sum a geometric series.) Also, if $z \in B(x, \frac{\varepsilon}{2}) \setminus E$,

$$(4.92) \qquad |T(f d\mu)(z) - h(z)| = \left| \int_{B(x,\varepsilon)} \frac{f(y)\, d\mu(y)}{z - y} \right| \le \int_{B(x,\varepsilon)} \frac{d\mu(y)}{|z - y|}.$$

The mean value on $B(x, \frac{\varepsilon}{2}) \setminus E$ of this last integral is (with $|B|$ denoting the Lebesgue measure of B)

$$(4.93) \qquad |B(x, \tfrac{\varepsilon}{2})|^{-1} \int_{B(x,\varepsilon/2) \setminus E} \left\{ \int_{B(x,\varepsilon)} \frac{d\mu(y)}{|z - y|} \right\} dz$$
$$\le 4\varepsilon^{-2} \int_{B(x,\varepsilon)} \left\{ \int_{B(x,\varepsilon/2)} \frac{dz}{|z - y|} \right\} d\mu(y)$$
$$\le C\varepsilon^{-1} \mu(B(x,\varepsilon)).$$

Thus we may choose $z \in B(x, \frac{\varepsilon}{2}) \setminus E$ so that $|T(f d\mu)(z) - h(z)| \le C\varepsilon^{-1} \mu(B(x,\varepsilon))$. Since we already know that $T(f d\mu)(z) \le M_0$, (4.90) follows from this and (4.91).

Recall from (4.18) and (4.27) that $\rho(x) = 0$ on all cubes of $I_1 \cup \mathrm{LI}$. Thus

$$(4.94) \qquad\qquad E_\infty \quad \text{does not meet any cube of } I_1 \cup \mathrm{LI},$$

and Remark 4.12 says that E_∞ does not even meet any cube Q such that $Q \subset \mathcal{O}(M)$. (See the definitions (4.4) and (4.5).) From this we deduce that

$$(4.95) \qquad\qquad \mathcal{M}_\mu(x) = \sup_{r > 0} \left\{ r^{-1} \mu(B(x,r)) \right\} \le C \quad \text{for } x \in E_\infty.$$

Indeed, we can take $C = 2M$ because if $\mu(B(x,r)) > 2Mr$, then $\mu(B(y, 2r)) > M2r$ for $y \in B(x,r)$, whence $B(x,r) \subset \mathcal{O}(M)$ and all sufficiently small cubes containing x would also be contained in $\mathcal{O}(M)$.

When $x \in \mathcal{C}_n$ for some n, we may use the fact that the parent $\widehat{Q}_n$ of Q_n is not contained in $I_1 \cup \mathrm{LI}$, plus Remark 4.12, to find that $\widehat{Q}_n$ is not contained in $\mathcal{O}(M)$. Because of this

$$(4.96) \qquad\qquad \mu(B(x,r)) \le Cr \quad \text{when } x \in \mathcal{C}_n \text{ and } r \ge A^{-k_n}.$$

(See near (4.14) for the definition of k_n.)

¿From (4.95), (4.96) and Lemma 4.89 we deduce that

$$(4.97) \qquad T^*(f d\mu)(x) \leq C \quad \text{for } x \in E_\infty$$

and

$$(4.98) \qquad |T^\varepsilon(f d\mu)(x)| \leq C \quad \text{for } x \in \mathcal{C}_n \text{ and } \varepsilon \geq \frac{A^{-k_n}}{5},$$

say.

Now we want to estimate $T^\varepsilon \varphi_n$, with $\varphi_n = F_n - F_{n-1}$, when $x \in E_\infty$ or $x \in \mathcal{C}_m$ for some $m \neq n$. In both cases Lemma 4.43 says that $\operatorname{dist}(x, \mathcal{C}_n) \geq \frac{1}{4} A^{-k_n}$. Let us first use (4.72) and estimate brutally:

$$(4.99) \qquad |T^\varepsilon \varphi_n(x)| \leq \int_{H_n} \frac{d\mu(y)}{|x - y|} + \theta_n \int_{G_n} \frac{d\mu(y)}{|x - y|} + \frac{C\mu(H_n)}{\operatorname{dist}(x, \mathcal{C}_n)}$$
$$\leq C \int_{H_n} \frac{d\mu(y)}{|x - y|} + \theta_n \int_{G_n} \frac{d\mu(y)}{|x - y|},$$

by (4.73), and then the fact that $\frac{1}{|x-y|} \geq C^{-1} \operatorname{dist}(x, \mathcal{C}_n)^{-1}$ for all $y \in H_n$, because $\operatorname{dist}(x, \mathcal{C}_n) \geq \frac{1}{4} A^{-k_n}$ and $\operatorname{diam}(H_n \cup \mathcal{C}_n) \leq C A^{-k_n}$.

We do not want to use (4.99) in all cases. In particular, when x is far from Q_n, we wish to "integrate by parts" and get an estimate with a better decay. Let us assume that $\operatorname{dist}(x, Q_n) \geq A^{-k_n+2}$, say, and that $\varepsilon \leq \operatorname{dist}(x, Q_n \cup \mathcal{C}_n)$. In this case $T^\varepsilon \varphi_n(x) = \int \frac{d\varphi_n(y)}{x-y}$, i.e., there is no truncation. Pick any point $y_0 \in \mathcal{C}_n$; then $\int \frac{d\varphi_n(y)}{x-y_0} = 0$, because $\int d\varphi_n = 0$, by (4.22) or (4.39). Hence

$$(4.100) \qquad |T^\varepsilon \varphi_n(x)| = \left| \int \frac{(y - y_0)\, d\varphi_n(y)}{(x - y)\,(x - y_0)} \right|$$
$$\leq \|\varphi_n\| \operatorname{diam}(Q_n \cup \mathcal{C}_n) \operatorname{dist}(x, Q_n \cup \mathcal{C}_n)^{-2}$$
$$\leq C\mu(H_n)\, A^{-k_n} \operatorname{dist}(x, Q_n)^{-2}$$

by (4.72), (4.73), and the end of (4.75) when $Q_n \in \mathrm{PLI}$.

When $\operatorname{dist}(x, Q_n) \geq A^{-k_n+2}$ but

$$(4.101) \qquad \operatorname{dist}(x, Q_n \cup \mathcal{C}_n) < \varepsilon < \operatorname{dist}(x, Q_n \cup \mathcal{C}_n) + \operatorname{diam}(Q_n \cup \mathcal{C}_n),$$

we may have a problem of truncation. In this case we shall prefer to use (4.99) which takes the simpler form

$$(4.102) \qquad |T^\varepsilon \varphi_n(x)| \leq C\mu(H_n) \operatorname{dist}(x, Q_n)^{-1}$$

(we still use (4.75) here when $Q_n \in \mathrm{PLI}$). Set

$$(4.103) \qquad T(n, \varepsilon) = \{x \in \mathbb{C} : \operatorname{dist}(x, Q_n) \geq A^{-k_n+2} \text{ and (4.101) holds}\}.$$

Our last case is when $\varepsilon \geq \operatorname{dist}(x, Q_n \cup \mathcal{C}_n) + \operatorname{diam}(Q_n \cup \mathcal{C}_n)$. In this case $T^\varepsilon \varphi_n(x) = 0$ and we need not worry.

Let us summarize this estimate. It will be convenient to set $\mathcal{A}_n = \{Q_n\}$ and $\mathcal{A}_n^* = \emptyset$ when $Q_n \in I_1$. We claim that

$$(4.104) \qquad |T^\varepsilon \varphi_n(x)| \leq C \sum_{Q \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q)\, e_Q^*(x) + C\, \mathbf{1}_{T(n,\varepsilon)}(x)\, \mu(H_n)\, \mathrm{dist}(x, Q_n)^{-1}$$

when $x \in E_\infty$ or $x \in \mathcal{C}_m$ for some $m \neq n$.

When $\mathrm{dist}(x, Q_n) \leq A^{-k_n+2}$, this follows from (4.99) because $\rho_Q(x) \geq C^{-1}$ for all $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$. When $\mathrm{dist}(x, Q_n) \geq A^{-k_n+2}$ and $x \in T(n, \varepsilon)$, (4.104) follows from (4.102). Otherwise we may use (4.100) (or else $T^\varepsilon \varphi_n(x) = 0$). This completes the proof of (4.104).

There are situations where we do not want to use (4.104), because some of the $e_Q^*(x)$ are loo large. This is typically the case when $Q_n \in \mathrm{PLI}$ and $x \in Q$ for one of the cubes Q that compose G_n. We want to prove an alternate estimate for such cases.

For each $m \geq 1$, denote by Q_m^* the cube such that $\mathcal{C}_m = \partial\left(\frac{1}{2} B(Q_m^*)\right)$. Thus $Q_m^* = Q_m$ if $Q_m \in I_1$, and Q_m^* is one of the cubes of $\mathcal{A}_m$ when $Q_m \in \mathrm{PLI}$.

4.105. Lemma. *Let $Q \in \mathrm{BLI}$, and suppose that $x \in E_\infty \cap Q$ or that*

$$(4.106) \qquad x \in \mathcal{C}_m, \ Q_m^* \subset Q, \ and \ \varepsilon \geq \frac{1}{5} A^{-k_m}.$$

Then

$$(4.107) \qquad |T^\varepsilon(\mathbf{1}_Q\, f d\mu)(x)| \leq C + h_Q^*(x).$$

Set $a = T^\varepsilon(\mathbf{1}_Q\, f d\mu)(x)$. If $\varepsilon \geq A^{-k(Q)+1}$, then $a = 0$. So let us assume that $\varepsilon < A^{-k(Q)+1}$. Write $\mathbf{1}_Q = \mathbf{1}_E - \mathbf{1}_{E \setminus Q}$ to get that

$$(4.108) \qquad a = T^\varepsilon(f d\mu)(x) - \int_{(E \setminus Q) \setminus B(x,\varepsilon)} \frac{f(y)\, d\mu(y)}{x - y}.$$

Because of (4.97) and (4.98), we have that

$$(4.109) \qquad |T^\varepsilon(f d\mu)(x)| \leq C$$

and

$$(4.110) \qquad \left| \int_{(E \setminus Q) \setminus B(x, A^{-k(Q)+1})} \frac{f(y)\, d\mu(y)}{x - y} \right| = \left| T^{A^{-k(Q)+1}}(f d\mu)(x) \right| \leq C.$$

For (4.109) we are also using (4.106) when $x \in \mathcal{C}_m$. Since $\varepsilon < A^{-k(Q)+1}$ we deduce from (4.108), (4.109) and (4.110) that

$$(4.111) \qquad |a| \leq C + \int_{(E \setminus Q) \cap B(x, A^{-k(Q)+1})} \frac{d\mu(y)}{|x - y|}$$

$$\leq C + h_Q^*(x) + \int_{(E \setminus 2Q) \cap B(x, A^{-k(Q)+1})} \frac{d\mu(y)}{|x - y|}$$

(by (4.80)). Observe that $\mathrm{dist}(x, E \setminus 2Q) \geq \frac{1}{2} A^{-k(Q)}$. When $x \in Q$, this comes from the definition (4.79) of $2Q$; if $x \in \mathcal{C}_m$ and $Q_m^* = Q$, then we even have that $\mathrm{dist}(x, E \setminus Q) \geq$

$\frac{1}{2}A^{-k(Q)}$, by (3.8); if $x \in \mathcal{C}_m$ and Q_m^* is strictly smaller than Q, then $\operatorname{dist}(x, Q) \leq C_0 A^{-k_m} < \frac{1}{2}A^{-k(Q)}$ and $\operatorname{dist}(x, E \setminus 2Q) \geq \frac{1}{2}A^{-k(Q)}$ as well. Now the last integral in (4.111) is less than $CA^{k(Q)}\mu\big(B(x, A^{-k(Q)+1})\big)$, which is $\leq C$ by (4.95) or (4.96). This proves the lemma.

We are now ready to prove Proposition 4.81. Let $x \in E_\infty$ and $\varepsilon > 0$ be given. Obviously,

$$(4.112) \qquad |T^\varepsilon(g\,d\nu)(x)| \leq |T^\varepsilon(f\,d\mu)(x)| + \sum_{n \geq 1} |T^\varepsilon \varphi_n(x)|.$$

The first term $|T^\varepsilon(f\,d\mu)(x)|$ is $\leq C$, by (4.97). Next let us take care of the set $\mathcal{N}_1$ of integers n such that $x \notin Q_n$. We apply (4.104) to each $n \in \mathcal{N}_1$, sum, and get that

$$(4.113) \qquad \sum_{n \in \mathcal{N}_1} |T^\varepsilon \varphi_n(x)| \leq C \sum_{Q \in I_1 \cup \mathrm{LI} \cup \mathrm{BLI}} \theta(Q)\,\mathbf{1}_{E \setminus Q}(x)\,e_Q^*(x)$$
$$+ C \sum_{n \in \mathcal{N}_1} \mathbf{1}_{T(n,\varepsilon)}(x)\,\mu(H_n)\operatorname{dist}(x, Q_n)^{-1}$$

because the sets $\mathcal{A}_n \cup \mathcal{A}_n^*$ are all disjoint and contained in $I_1 \cup \mathrm{LI} \cup \mathrm{BLI}$. The first sum is less than the right-hand side of (4.82); the second one is controlled as follows.

4.114. Lemma. *For each $x \in \mathbb{C}$,*

$$\sum_{n \geq 1} \mathbf{1}_{T(n,\varepsilon)}(x)\,\mu(H_n)\operatorname{dist}(x, Q_n)^{-1} \leq C.$$

If $x \in T(n, \varepsilon)$, then $\operatorname{dist}(x, Q_n) \geq A^{-k_n+2}$, which is much larger than $\operatorname{diam}(Q_n \cup \mathcal{C}_n)$. (See (4.103) for the definition of $T(n, \varepsilon)$.) Then (4.101) implies that $\operatorname{dist}(x, Q_n) \geq \frac{\varepsilon}{2}$ and $Q_n \subset B(x, 2\varepsilon)$. The lemma follows from (4.70).

So we control the set $\mathcal{N}_1$. Now let $\mathcal{N}_2$ be the set of integers such that $x \in Q_n$. Since $x \in E_\infty$, the only possibility is that $Q_n \in \mathrm{PLI}$ and $x \in G_n$. Let $\widetilde{Q}$ denote the cube of $\mathcal{A}_n^*$ that contains x. Let us use (4.72) to estimate $T^\varepsilon \varphi_n(x)$. First,

$$(4.115) \qquad |T^\varepsilon(\mathbf{1}_{H_n}\,\rho_{n-1}\,f\,d\mu)(x)| \leq C \sum_{Q \in \mathcal{A}_n} \mathbf{1}_{E \setminus Q}(x)\,e_Q^*(x)$$

by definition (4.78) of e_Q^* and the fact that $\rho_Q(x) \geq C^{-1}$ for $Q \in \mathcal{A}_n$. Similarly,

$$(4.116) \qquad |T^\varepsilon(\theta_n\,\mathbf{1}_{G_n \setminus \widetilde{Q}}\,\rho_{n-1}\,f\,d\mu)(x)| \leq C \sum_{\substack{Q \in \mathcal{A}_n \\ Q \neq \widetilde{Q}}} \theta(Q)\,\mathbf{1}_{E \setminus Q}(x)\,e_Q^*(x).$$

Next observe that ρ_{n-1} is constant on $\widetilde{Q}$; thus we may apply Lemma 4.105 to $\widetilde{Q}$ and get that

$$(4.117) \qquad |T^\varepsilon(\theta_n\,\mathbf{1}_{\widetilde{Q}}\,\rho_{n-1}\,f\,d\mu)(x)| \leq C\theta_n\,\rho_{n-1}(x) + C\theta(\widetilde{Q})\,\mathbf{1}_{\widetilde{Q}}(x)\,h_{\widetilde{Q}}^*(x).$$

Finally

$$(4.118) \qquad |T^\varepsilon(\alpha_n\,d\nu_n)(x)| \leq C\mu(H_n)\operatorname{dist}(x, \mathcal{C}_n)^{-1} \leq C \sum_{Q \in \mathcal{A}_n} \mathbf{1}_{E \setminus Q}(x)\,e_Q^*(x)$$

by (4.73) and the fact that $|x - y| \leq C \operatorname{dist}(x, \mathcal{C}_n)$ for all $y \in H_n$, by Lemma 4.43.

We get an estimate for $|T^\varepsilon \varphi_n(x)|$ by summming (4.115), (4.116), (4.117) and (4.118). When we sum this over all $n \in \mathcal{N}_2$, we get terms that are obviously controlled by the right-hand side of (4.82), plus $C \sum_n \theta_n \rho_{n-1}(x)$, which comes from (4.117). Now (4.27) says that $\rho_n(x) = (1 - \theta_n) \rho_{n-1}(x)$ because $x \in G_n$. Thus $\theta_n \rho_{n-1}(x) = \rho_{n-1}(x) - \rho_n(x)$, and $\sum_n \theta_n \rho_{n-1}(x) \leq 1$. This completes our proof of Proposition 4.81.

We may now turn to the proof of Proposition 4.85. Let $x \in \mathcal{C}_m$ and $\varepsilon > 0$ be given. Let us first observe that we can reduce to the case when $\varepsilon \geq \varepsilon_0 = A^{-k_m}/4$. Indeed, if $\varepsilon < \varepsilon_0$,

$$(4.119) \qquad |T^\varepsilon (g d\nu)(x) - T^{\varepsilon_0}(g d\nu)(x)| = \left| \int_{B(x, \varepsilon_0) \setminus B(x, \varepsilon)} \frac{g(y) \, d\nu(y)}{x - y} \right| \leq C,$$

because Lemma 4.43 tells us that on the range $\varepsilon < |x - y| \leq \varepsilon_0$, $g d\nu$ is reduced to $\alpha_m d\nu_m$; the last inequality in (4.119) then follows from the good boundedness properties of the Cauchy integral on a circle $\mathcal{C}_m$, plus the fact that the density $\alpha_m \mu(H_m) \, H^1(\mathcal{C}_m)^{-1}$ of $\alpha_m d\nu_m$ against Hausdorff measure is $\leq C$ by (4.73) and (4.13).

So let us assume that $\varepsilon \geq \varepsilon_0$ from now on. Then $|T^\varepsilon(f d\mu)(x)| \leq C$ by (4.98), and

$$(4.120) \qquad |T^\varepsilon (g d\nu)(x)| \leq C + \sum_{n \geq 1} |T^\varepsilon \varphi_n(x)|.$$

Let $\mathcal{N}_1$ be the set of integers n such that $m \in J(Q)$ for all $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$. (See just after Proposition 4.81 for the definition of $J(Q)$.) The cubes Q_n, $n \in \mathcal{N}_1$, can be treated as before: we use (4.104) for each $n \in \mathcal{N}_1$, and when we sum we get

$$(4.121) \qquad \sum_{n \in \mathcal{N}_1} |T^\varepsilon \varphi_n(x)| \leq C \sum_{n \in \mathcal{N}_1} \sum_{Q \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q) \, e_Q^*(x)$$
$$+ C \sum_{n \in \mathcal{N}_1} \mathbf{1}_{T(n, \varepsilon)}(x) \, \mu(H_n) \operatorname{dist}(x, Q_n)^{-1}.$$

The first term is controlled by the right-hand side of (4.86) because $\widetilde{e}_Q(x) = e_Q^*(x)$ for all the cubes Q that show up (by the definition of $\mathcal{N}_1$). The second sum is less than C, by Lemma 4.114.

Thus we are left with the set $\mathcal{N}_2$ of integers n for which $m \notin J(Q)$ for some $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$.

4.122. Lemma. *If $n \in \mathcal{N}_2$, then either $m = n$ or else $Q_n \in \mathrm{PLI}$ and $Q_m \subset G_n$.*

Proof. Let $n \in \mathcal{N}_2$. Since $m \notin J(Q)$ for some $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$, Q_m and Q_n must meet. Suppose $m \neq n$. Then either Q_n is strictly contained in Q_m or Q_m is strictly contained in Q_n. Since the cubes in $I_1 \cup \mathrm{LI}$ are disjoint by (4.11), the first case implies that $Q_m \in \mathrm{PLI}$ and $Q_n \subset G_m$. But then $m \in J(Q)$ for all $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$, which is not possible since $n \in \mathcal{N}_2$. Similarly, the second case implies that $Q_n \in \mathrm{PLI}$ and $Q_m \subset G_n$, which is what we want.

Now let $n \in \mathcal{N}_2$, $n \neq m$, be given, and denote by $\widetilde{Q}$ the cube of $\mathcal{A}_n^*$ that contains Q_m. We want to use (4.72) and proceed as in (4.115–118) to estimate $T^\varepsilon \varphi_n(x)$. First let $\varphi = -\mathbf{1}_{H_n} \rho_{n-1} f d\mu - \theta_n \mathbf{1}_{G_n \setminus \widetilde{Q}} \rho_{n-1} f d\mu$. Then a brutal estimate gives that

$$(4.123) \qquad |T^\varepsilon \varphi(x)| \leq C \sum_{\substack{Q \in \mathcal{A}_n \cup \mathcal{A}_n^* \\ Q \neq \widetilde{Q}}} \theta(Q) \, e_Q^*(x)$$

because $\rho_Q(x) \geq C^{-1}$ for $Q \in \mathcal{A}_n \cup \mathcal{A}_n^*$. Notice that if $Q \in \mathcal{A}_n \cup \mathcal{A}_n^* \setminus \{\widetilde{Q}\}$, Q does not meet Q_m and hence $m \in J(Q)$ and $e_Q^*(x) = \widetilde{e}_Q(x)$.

Denote by ρ_n^* the constant value of ρ_{n-1} on $\widetilde{Q}$. Lemma 4.105 (applied to $\widetilde{Q}$) gives that

$$(4.124) \qquad |T^\varepsilon(\theta_n \, \mathbf{1}_{\widetilde{Q}} \, \rho_{n-1} \, f d\mu)(x)| \leq C\theta_n \, \rho_n^* + C\theta(\widetilde{Q}) \, h_{\widetilde{Q}}^*(x).$$

Notice also that $h_{\widetilde{Q}}^*(x) = \widetilde{h}_{\widetilde{Q}}(x)$, because $Q_m \subset \widetilde{Q}$. The last piece from (4.72) is treated as before:

$$(4.125) \qquad |T^\varepsilon(\alpha_n \, d\nu_n)(x)| \leq C\mu(H_n) \, \mathrm{dist}(x, \mathcal{C}_n)^{-1} \leq C \sum_{Q \in \mathcal{A}_n} e_Q^*(x)$$

because $\mathrm{dist}(x, \mathcal{C}_n) \geq \mathrm{dist}(\mathcal{C}_m, \mathcal{C}_n) \geq \frac{1}{4} A^{-k_n} \geq C^{-1}|x - y|$ for all $y \in H_n$, by (4.45). When we sum (4.123), (4.124) and (4.125) over all $n \in \mathcal{N}_2 \setminus \{m\}$, we get terms that are less than the right-hand side of (4.86), plus $C \sum_n \theta_n \, \rho_n^*$ that comes from (4.124). Pick any point $x_0 \in Q_m$. Then $\rho_n^* = \rho_{n-1}(x_0)$ for all $n \in \mathcal{N}_2 \setminus \{m\}$, and $\sum_n \theta_n \, \rho_n^* = \sum_n \theta_n \, \rho_{n-1}(x_0) = \sum_n \left(\rho_{n-1}(x_0) - \rho_n(x_0)\right) \leq 1$, by the same argument using (4.27) as before.

The proof of Proposition 4.85 will be complete as soon as we estimate $T^\varepsilon \varphi_m(x)$. The good properties of the Cauchy integral on $\mathcal{C}_m$, plus the fact that the density $|\alpha_m| \, \|\nu_m\| \, H^1(\mathcal{C}_m)^{-1}$ is less than C by (4.73) and (4.13), yield

$$(4.126) \qquad |T^\varepsilon(\alpha_m d\nu_m)(x)| \leq C.$$

Next let Q_m^* denote the cube such that $\mathcal{C}_m = \partial\left(\frac{1}{2}B(Q_m^*)\right)$. (Thus $Q_m^* = Q_m$ if $Q_m \in I_1$ and $Q_m^* \in \mathcal{A}_m$ otherwise.) Let us check that

$$(4.127) \qquad |T^\varepsilon(\mathbf{1}_{Q_m^*} \, f d\mu)(x)| \leq C.$$

Indeed, $T^\varepsilon(\mathbf{1}_{Q_m^*} \, f d\mu)(x) = 0$ if $\varepsilon > \mathrm{diam}(\mathcal{C}_m \cup Q_m^*)$, and so we are reduced to the case when $A^{-k_m}/4 \leq \varepsilon \leq A^{-k_m+1}$, say. (Recall that we supposed that $\varepsilon \geq A^{-k_m}/4$ using (4.119). A simple computation gives that $|T^\varepsilon(\mathbf{1}_{Q_m^*} \, f d\mu)(x)| \leq CA^{k_m} \, \mu(Q_m^*) \leq C$, by (4.13).

If $Q_m \in I_1$, we can sum (4.126) and (4.127) (multiplied by the constant value of ρ_{m-1} on Q_m) to get that $|T^\varepsilon \varphi_m(x)| \leq C$. Otherwise, we also have to add

$$(4.128) \qquad \begin{aligned} |T^\varepsilon(\mathbf{1}_{H_m \setminus Q_m^*} \, \rho_{m-1} \, f d\mu &+ \theta_m \, \mathbf{1}_{G_m} \, \rho_{m-1} \, f d\mu)(x)| \\ &\leq C\mu(Q_m) \, \mathrm{dist}(x, Q_m \setminus Q_m^*)^{-1} \leq C \end{aligned}$$

by (4.13) and the fact that $\mathrm{dist}(x, Q_m \setminus Q_m^*) \geq \mathrm{dist}(\mathcal{C}_m, E \setminus Q_m) \geq A^{-k_m}/4$. This completes our proof of Proposition 4.85.

4.129. Remark. In Propositions 4.81 and 4.85 we stated estimates for $T^*(gd\nu)(x)$, but our proof also gives estimates for the pieces that compose it. For each set $J \subset \mathbb{N}^*$ of indices, set

$$(4.130) \qquad T_J^*(x) = T^*(f d\mu)(x) + \sup_{\varepsilon > 0}\left\{\sum_{n \in J} |T^\varepsilon \varphi_n(x)|\right\} \quad \text{for } x \in E_\infty,$$

where $\varphi_n = F_n - F_{n-1}$ is as above. If we follow the proof of Proposition 4.81, but forget the estimates for $T^\varepsilon \varphi_n(x)$ when $n \notin J$, we get that

$$(4.131) \qquad T_J^*(x) \le C + C \sum_{n \in J} \sum_{Q \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q) \, \mathbf{1}_{E \setminus Q}(x) \, e_Q^*(x)$$

$$+ C \sum_{\substack{n \in J \\ Q_n \in \mathrm{PLI}}} \sum_{Q \in \mathcal{A}_n^*} \theta(Q) \, \mathbf{1}_Q(x) \, h_Q^*(x)$$

for all $x \in E_\infty$.

There is a similar estimate when $x \in \bigcup_m \mathcal{C}_m$, but we have to be slightly careful because we do not have any good estimate for $T^\varepsilon(f d\mu)(x)$ when $x \in \mathcal{C}_m$ and $\varepsilon \ll A^{-k_m}$. So we set

$$(4.132) \qquad T_J^*(x) = \sup \left\{ |T^\varepsilon(f d\mu)(x)| : \varepsilon \ge \frac{1}{5} A^{-k_m} \right\}$$

$$+ \sup \left\{ \sum_{n \in J} |T^\varepsilon \varphi_n(x)| : \varepsilon \ge \frac{1}{5} A^{-k_m} \right\}$$

when $x \in \mathcal{C}_m$. Then our proof of Proposition 4.85 also gives that

$$(4.133) \qquad T_J^*(x) \le C + C \sum_{n \in J} \sum_{Q \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q) \, \widetilde{e}_Q(x)$$

$$+ C \sum_{\substack{n \in J \\ Q_n \in \mathrm{PLI}}} \sum_{Q \in \mathcal{A}_n^*} \theta(Q) \, \widetilde{h}_Q(x) \quad \text{for } x \in \bigcup_m \mathcal{C}_m.$$

5. The L^2-estimates

Set $\mathcal{R} = I_1 \cup \mathrm{LI} \cup \mathrm{BLI}$, and define four functions by

$$(5.1) \qquad W_1 = \sum_{Q \in \mathrm{BLI}} \theta(Q) \, \mathbf{1}_{Q \cap E_\infty} \, h_Q^*,$$

$$(5.2) \qquad \widetilde{W}_1 = \sum_{Q \in \mathrm{BLI}} \theta(Q) \, \widetilde{h}_Q,$$

$$(5.3) \qquad W_2 = \sum_{Q \in \mathcal{R}} \theta(Q) \, \mathbf{1}_{E_\infty \setminus Q} \, e_Q^*$$

and

$$(5.4) \qquad \widetilde{W}_2 = \sum_{Q \in \mathcal{R}} \theta(Q) \, \widetilde{e}_Q.$$

In this part we show that W_1, $\widetilde{W}_1$, W_2 and $\widetilde{W}_2$ lie in $L^2(d\nu)$; of course Theorem 2.4 will follow from this and Propositions 4.81 and 4.85.

The estimates that follow are a little unpleasant at times. It is probably a good idea to keep in mind that when μ is a nice doubling measure, we know how to prove the desired estimates. Here we shall try to keep the same general organization. A typical thing that we like to do is try to estimate scalar products in terms of the measures $\mu(Q)$ themselves, rather than quantities like $\mu(100B(Q))$, because the $\mu(Q)$'s are easier to sum since the cubes are often disjoint. This is why we often do not want to get "bad cubes", i.e., cubes that do not satisfy (3.14). It will often be useful to decompose bad cubes into smaller good cubes, as in the next section.

5.1. THE FUNCTIONS W_1 AND $\widetilde{W}_1$

First we introduce a few definitions that will make it possible to estimate the functions W_1 and $\widetilde{W}_1$ in a unified way and save some time. Let M_1 be a large constant, to be chosen soon, and define a function r on E by

$$(5.5) \qquad \begin{cases} r(x) = 0 \quad \text{if } \mathcal{M}_\mu(x) \le M_1, \quad \text{and} \\ r(x) = \sup\left\{ r > 0 : \mu\big(B(x,r)\big) > M_1 r \right\} \text{ otherwise.} \end{cases}$$

We choose M_1 so large, compared with the constants from (4.95) and (4.96), that

$$(5.6) \qquad r(x) = 0 \quad \text{when } x \in E_\infty$$

and

$$(5.7) \qquad r(x) \le A^{-k_n} \quad \text{when } x \in Q_n.$$

Next set

$$(5.8) \qquad h_Q(x) = \mathbf{1}_Q(x) \int_{2Q \setminus Q} \frac{d\mu(y)}{r(x) + |x - y|}$$

for each $Q \in \Delta$. A rapid comparison with the definition (4.80) of $h_Q^*(x)$ shows that $h_Q(x) = \mathbf{1}_Q(x)\, h_Q^*(x)$ for $x \in E_\infty$. Set

$$(5.9) \qquad W_3 = \sum_{Q \in \mathrm{BLI}} \theta(Q)\, h_Q.$$

5.10. Lemma. *We have that*

$$(5.11) \qquad \|W_1 + \widetilde{W}_1\|_{L^2(d\nu)} \le C \|W_3\|_{L^2(d\mu)}.$$

It is clear that $\|W_1\|_{L^2(d\nu)}^2 = \int_{E_\infty} W_1^2\, \rho_\infty\, d\mu = \|W_3\|_{L^2(\rho_\infty\, d\mu)}^2 \le \|W_3\|_{L^2(d\mu)}^2$, because $d\nu = \rho_\infty\, d\mu$ on E_∞. So it will be enough to control $\widetilde{W}_1$. We write that

$$(5.12) \qquad \|\widetilde{W}_1\|_{L^2(d\nu)}^2 = \sum_{Q \in \mathrm{BLI}} \sum_{Q' \in \mathrm{BLI}} \theta(Q)\, \theta(Q')\, \langle \widetilde{h}_Q, \widetilde{h}_{Q'} \rangle_\nu$$

with

$$(5.13) \qquad \langle \widetilde{h}_Q, \widetilde{h}_{Q'} \rangle_\nu = \int \widetilde{h}_Q\, \widetilde{h}_{Q'}\, d\nu = \sum_{n: Q_n \subset Q \cap Q'} \int_{\mathcal{C}_n} h_Q^*\, h_{Q'}^*\, d\nu_n.$$

Fix n such that $Q_n \subset Q \cap Q'$. We claim that

$$(5.14) \qquad h_Q^*(x) \le C h_Q(\widetilde{x}) \quad \text{for each } x \in \mathcal{C}_n \text{ and each } \widetilde{x} \in H_n.$$

Indeed, if $x \in \mathcal{C}_n$, $\widetilde{x} \in H_n$ and $y \in 2Q \setminus Q$,

$$(5.15) \qquad r(x) + |\widetilde{x} - y| \le A^{-k_n} + |\widetilde{x} - y| \le C A^{-k_n} + |x - y| \le C|x - y|,$$

by (5.7) and because $|x - y| \geq \operatorname{dist}(\mathcal{C}_n, E \setminus Q) \geq \operatorname{dist}(\mathcal{C}_n, E \setminus Q_n) \geq A^{-k_n}/2$. (We used the fact that $Q_n \subset Q$ and (4.46).) Also, $\widetilde{x} \in Q$ because $H_n \subset Q_n \subset Q$, and (5.14) follows from (5.15) by taking inverses and integrating on $y \in 2Q \setminus Q$. Of course the analogue of (5.14) for Q' holds as well, and so

$$(5.16) \qquad \int_{\mathcal{C}_n} h_Q^*(x)\, h_{Q'}^*(x)\, d\nu_n(x) \leq C \int_{H_n} h_Q(\widetilde{x})\, h_{Q'}(\widetilde{x})\, d\mu(\widetilde{x})$$

because $\|\nu_n\| \leq C\mu(H_n)$, by (4.73). Now we use the fact that the sets H_n are disjoint subsets of Q (by (4.69)) to sum and get that

$$(5.17) \qquad \langle \widetilde{h}_Q, \widetilde{h}_{Q'} \rangle_\nu \leq C \langle h_Q, h_{Q'} \rangle,$$

where

$$(5.18) \qquad \langle h_Q, h_{Q'} \rangle = \int_{Q \cap Q'} h_Q\, h_{Q'}\, d\mu.$$

Lemma 5.10 follows at once from this and (5.12) because

$$(5.19) \qquad \|W_3\|_{L^2(d\mu)}^2 = \sum_{Q \in \mathrm{BLI}} \sum_{Q' \in \mathrm{BLI}} \theta(Q)\, \theta(Q')\, \langle h_Q, h_{Q'} \rangle.$$

Next we want to estimate the functions h_Q. Let us first check that

$$(5.20) \qquad \int_{B(x,r_2) \setminus B(x,r_1)} \frac{d\mu(y)}{r(x) + |x - y|} \leq C\left(1 + \log \frac{r_2}{r_1}\right)$$

for all $x \in E$ and $r_2 > r_1 > 0$. This is easy to prove. First observe that

$$(5.21) \qquad \int_{B(x,r)} \frac{d\mu(y)}{r(x) + |x - y|} \leq \frac{\mu\big(B(x, r(x))\big)}{r(x)} \leq C$$

by the definition of $r(x)$. So we may as well assume that $r_1 \geq r(x)$. Set $A_k = \{y \in E : 2^k r_1 < |x - y| \leq 2^{k+1}r_1\}$ for all $k \geq 0$. Then $\mu(A_k) \leq C 2^k r_1$ because $r(x) \leq r_1$, and hence $\int_{A_k} \frac{d\mu(y)}{r(x) + |x-y|} \leq C$ for all k. Our claim (5.20) follows from this, because the domain of integration in (5.20) can be covered by less than $C(1 + \log \frac{r_2}{r_1})$ domains A_k.

5.22. Lemma. *For each $Q \in \Delta$ and $\lambda > 0$,*

$$(5.23) \qquad \mu\big(\{x \in Q : h_Q(x) \geq \lambda\}\big) \leq Ce^{-c\lambda}\, \mu\big(90B(Q)\big),$$

where c is some small positive constant.

For each $x \in Q$, set $d(x) = \operatorname{dist}(x, E \setminus Q)$. Then $d(x) \leq |x - y| \leq 100C_0 A^{-k(Q)}$, say, for all $y \in 2Q \setminus Q$. Thus (5.20) tells us that

$$(5.24) \qquad h_Q(x) \leq C\big\{1 + \log(1 + d(x)^{-1} A^{-k(Q)})\big\}.$$

The lemma will not be difficult to deduce from this and the small boundary property of our cubes. Set

$$(5.25) \qquad N_l(Q) = \left\{ x \in E \setminus Q : \text{dist}(x, Q) < A^{-k(Q)-l} \right\}$$
$$\cup \left\{ x \in Q : \text{dist}(x, E \setminus Q) < A^{-k(Q)-l} \right\}.$$

This is the same set as in (3.10–12); (3.13) and our choice of $A = CC_0^{100}$ in (4.1) give that

$$(5.26) \qquad \mu\big(N_l(Q)\big) \le C_0^{-93l} \mu\big(90B(Q)\big)$$

for all $Q \in \Delta$ and $l \ge 0$. Of course the power 93 will not be needed precisely here; this is just what we get by being too cautious.

Now let $\lambda > 0$ be reasonably large (otherwise, there is nothing to prove). From (5.24) we deduce that if $h_Q(x) \ge \lambda$, then $d(x) \le C A^{-k(Q)} e^{-\lambda/C}$, and hence $x \in N_l(Q)$ for some $l \ge \frac{\lambda}{C'}$. The lemma follows from this and (5.26).

We shall find Lemma 5.22 more pleasant to use when Q is a good cube, i.e., when

$$(5.27) \qquad \mu\big(100B(Q)\big) \le C_0 \mu\big(B(Q)\big) \le C_0 \mu(Q)$$

(the second inequality is automatic, since Q always contains $E \cap B(Q)$). Otherwise, we shall find it useful to cut Q into a bunch of maximal good cubes, and then apply Lemma 5.22 to each of them separately.

For each $Q \in \Delta$, we denote by $\mathcal{S}(Q)$ the set of maximal good cubes contained in Q. Thus, if Q already satisfies (5.27), then $\mathcal{S}(Q)$ is just composed of Q itself. Obviously the cubes of $\mathcal{S}(Q)$ are disjoint. What is more interesting is that they almost cover Q.

5.28. Lemma. *For each $Q \in \Delta$,*

$$\mu\left(Q \setminus \bigcup_{R \in \mathcal{S}(Q)} R \right) = 0.$$

To prove the lemma we intend to use the property (3.16) of our cubes: if Q is a bad cube (i.e., if (5.27) does not hold), then $\mu\big(100B(Q)\big) \le C_0^{-l} \mu\big(100^{l+1}B(Q)\big)$ for all integers l such that $100^l \le C_0$. We want to apply this with $l = l_0$, where l_0 is the largest integer such that $100^{l+1} \le C_0$. We get that

$$(5.29) \qquad \mu\big(100B(Q)\big) \le C_0^{-l_0} \mu\big(C_0 B(Q)\big) \quad \text{when } Q \text{ is a bad cube.}$$

Recall that we decided to take $A = CC_0^{100}$, where C is an absolute constant, and that we still have the right to choose C_0 as large as we want. Since $C_0^{-l_0}$ decreases faster than any given power of C_0, we can choose C_0 so large that $C_0^{-l_0} < A^{-10}$, say. Then (5.29) implies that

$$(5.30) \qquad \mu\big(100B(Q)\big) \le A^{-10} \mu\big(C_0 B(Q)\big) \le A^{-10} \mu\big(100B(\widehat{Q})\big)$$

whenever Q is a bad cube, and where $\widehat{Q}$ still denotes the parent of Q. Let us derive a consequence of (5.30) and then return to Lemma 5.28.

5.31. Lemma. *Let $Q \in \Delta$, and let $R \subset Q$ be a cube such that all the intermediate cubes S, $R \subsetneqq S \subsetneqq Q$, are bad. Then*

$$(5.32) \qquad \mu\big(100B(R)\big) \le A^{-10(k(R)-k(Q)-1)}\mu\big(100B(Q)\big).$$

When $R = Q$ and when R is a child of Q, (5.32) is trivially true. Otherwise, we even have that

$$(5.33) \qquad \mu\big(100B(R)\big) \le \mu\big(100B(\widehat{R})\big) \le A^{-10(k(R)-k(Q)-1)}\mu\big(100B(Q)\big)$$

by repeated applications of (5.30). This proves Lemma 5.31.

Return to Lemma 5.28. Let $Q \in \Delta$ be given and, for each $k > k(Q)$, denote by Z_k the set of cubes $R \subset Q$ of generation k which are not contained in any cube of $\mathcal{S}(Q)$. Also let U_k be the union of the cubes of Z_k. Thus

$$(5.34) \qquad Q \setminus \bigcup_{R \in \mathcal{S}(Q)} R \subset U_k \quad \text{for all } k > k(Q).$$

Note that Lemma 5.31 applies to each cube of Z_k, and so

$$(5.35) \qquad \mu(R) \le A^{-10(k-k(Q)-1)}\mu\big(100B(Q)\big)$$

for all $R \in Z_k$.

On the other hand, the balls $B(R)$, $R \in Z_k$, are disjoint by (3.9), have radii $\ge A^{-k}$, and are all contained in a ball of radius $100C_0$, say. Thus Z_k has at most CA^{2k} elements. Then (5.35) implies that $\mu(U_k) \le CA^{10(k(Q)+1)}A^{-8k}$ and hence that $\mu(U_k)$ tends to zero when $k \to +\infty$. Lemma 5.28 follows from this and (5.34).

5.36. Lemma. *If $Q \in$ BLI and $R \in \mathcal{S}(Q)$,*

$$(5.37) \qquad h_Q(x) \le h_R(x) + C$$

for all $x \in R$.

Let $Q \in$ BLI, $R \in \mathcal{S}(Q)$ and $x \in R$ be given. For each integer k such that $k(Q) \le k \le k(R)$, let R_k denote the cube of Δ_k that contains R. Also set $D_k = 2R_k \setminus 2R_{k+1}$ for $k(Q) \le k < k(R)$. Then

$$(5.38) \quad h_Q(x) \le h_R(x) + \int_{2Q \setminus 2R} \frac{d\mu(y)}{r(x) + |x-y|} \le h_R(x) + \sum_{k=k(Q)}^{k(R)-1} \int_{D_k} \frac{d\mu(y)}{r(x) + |x-y|}$$

(compare with the definition (5.8)). Now

$$(5.39) \qquad \mu(D_k) \le \mu\big(100B(R_k)\big) \le A^{-10(k-k(Q)-1)}\mu\big(100B(Q)\big)$$

by Lemma 5.31, and $|x-y| \ge A^{-k-1}$ on D_k, because $x \in R_{k+1}$ and by definition (4.79) of $2R_{k+1}$. Then

$$(5.40) \qquad \int_{D_k} \frac{d\mu(y)}{r(x) + |x-y|} \le CA^{-9(k-k(Q))} A^{k(Q)} \mu\big(100B(Q)\big)$$

and (5.37) follows from (5.38) together with our assumption that $Q \in \mathrm{BLI}$ and the density estimate (4.13) (applied to $\widehat{Q}$). This proves the lemma.

We are now ready to estimate scalar products. Fix a cube $Q_0 \in \mathrm{BLI}$, and first consider cubes in the set

$$(5.41) \qquad \mathcal{B}_1(Q_0) = \big\{ Q \in \mathrm{BLI} : Q \subset Q_0 \text{ and } Q \text{ is not strictly}$$
$$\text{contained in any cube } R \in S(Q_0) \big\}.$$

Note that if $Q \in \mathcal{B}_1(Q_0)$ and $R \in \mathcal{S}(Q)$, then $R \in \mathcal{S}(Q_0)$ as well. Then

$$(5.42) \quad \langle h_Q, h_{Q_0} \rangle = \int_Q h_Q \, h_{Q_0} \, d\mu = \sum_{R \in \mathcal{S}(Q)} \int_R h_Q \, h_{Q_0} \, d\mu \leq \sum_{R \in \mathcal{S}(Q)} \int_R (h_R + C)^2 \, d\mu$$

by the definition (5.18) of the scalar product, Lemma 5.28, and Lemma 5.36 applied to Q and to Q_0. Next,

$$(5.43) \qquad \int_R (h_R + C)^2 \, d\mu \leq C \mu\big(90B(R)\big) \leq C\mu(R)$$

for each $R \in \mathcal{S}(Q)$, by Lemma 5.22 and because R is a good cube. Thus (5.42) yields

$$(5.44) \qquad \langle h_Q, h_{Q_0} \rangle \leq C \sum_{R \in \mathcal{S}(Q)} \mu(R) = C\mu(Q) \quad \text{for all } Q \in \mathcal{B}_1(Q_0).$$

Now suppose that $Q \in \mathcal{B}_2(Q_0)$, where

$$(5.45) \qquad \mathcal{B}_2(Q_0) = \big\{ Q \in \mathrm{BLI} : Q \subset Q_0 \quad \text{and there is}$$
$$\text{a cube } R \in \mathcal{S}(Q_0) \text{ such that } Q \subsetneqq R \big\}.$$

Denote by $R = R(Q, Q_0)$ the cube of $\mathcal{S}(Q_0)$ that contains Q. Let us first apply Lemma 5.36 to Q_0 and R to get that

$$(5.46) \qquad \langle h_Q, h_{Q_0} \rangle \leq \int_Q h_Q (h_R + C) \, d\mu.$$

Denote by $l(Q)$ the largest integer $l \geq 0$ such that $Q \subset N_l(R)$, where $N_l(R)$ is as in (5.25). When Q is not even contained in $N_0(R)$, still take $l(Q) = 0$.

5.47. Lemma. *We have that*

$$(5.48) \qquad h_R(x) \leq h_Q(x) + C\big(1 + l(Q)\big) \quad \text{on } Q.$$

By definitions,

$$(5.49) \qquad h_R(x) \leq h_Q(x) + \int_D \frac{d\mu(y)}{r(x) + |x - y|},$$

with $D = 2R \setminus (2Q \cup R)$. We claim that

$$(5.50) \qquad C^{-1} A^{-k(R) - l(Q)} \leq |x - y| \leq C A^{-k(R)} \quad \text{for } x \in Q \text{ and } y \in D.$$

The second inequality is easy, because x and y both lie in $2R$. We also have that $|x - y| \geq A^{-k(Q)}$ because $x \in Q$ and $y \in E \setminus 2Q$. This proves the first inequality when $A^{-k(R)-l(Q)-2} \leq A^{-k(Q)}$, say. So let us assume that $A^{-k(R)-l(Q)-2} > A^{-k(Q)}$. Since Q is not contained in $N_{l(Q)+1}(R)$, we can find $x_0 \in Q$ such that $\text{dist}(x_0, E \setminus R) \geq A^{-k(R)-l(Q)-1}$. Then $|x - y| \geq \text{dist}(x, E \setminus R) \geq A^{-k(R)-l(Q)-1} - \text{diam}\, Q \geq \frac{1}{2}A^{-k(R)-l(Q)-1}$, because we are in the case when $A^{-k(Q)} \leq A^{-k(R)-l(Q)-2}$. This proves our claim (5.50).

The lemma follows at once from (5.50), (5.49), and (5.20).

¿From (5.46) and Lemma 5.47 we deduce that

$$(5.51) \quad \langle h_Q, h_{Q_0} \rangle \leq \int_Q h_Q \left(h_Q + C + Cl(Q) \right) d\mu \leq C\left(1 + l(Q)\right) \int_Q h_Q (h_Q + 1)\, d\mu.$$

Next we use Lemma 5.28 to decompose Q into maximal good cubes S, $S \in \mathcal{S}(Q)$, and then Lemma 5.36 to replace h_Q with $h_S + C$. This gives

$$(5.52) \qquad \int_Q h_Q(h_Q + 1)\, d\mu = \sum_{S \in \mathcal{S}(Q)} \int_S h_Q(h_Q + 1)\, d\mu$$

$$\leq \sum_{S \in \mathcal{S}(Q)} \int_S (h_S + C)^2\, d\mu$$

$$\leq C \sum_{S \in \mathcal{S}(Q)} \mu\left(90B(S)\right)$$

$$\leq C \sum_{S \in \mathcal{S}(Q)} \mu(S) = C\mu(Q),$$

by Lemma 5.22, the goodness of the cubes S, and their disjointness. Because of (5.51), this yields

$$(5.53) \qquad \langle h_Q, h_{Q_0} \rangle \leq C\left(1 + l(Q)\right) \mu(Q) \quad \text{for all } Q \in \mathcal{B}_2(Q_0).$$

Now we want to sum our estimates from (5.44) and (5.53). Let us first check that

$$(5.54) \qquad \sum_{Q \in \mathcal{B}_1(Q_0)} \theta(Q) \langle h_Q, h_{Q_0} \rangle \leq C\mu(Q_0).$$

Of course the contribution of Q_0 itself is $\leq C\mu(Q_0)$, by (5.44). All the other cubes $Q \in \mathcal{B}_1(Q_0)$ lie in $\mathcal{A}_n^*$ for some n such that $Q_n \in \text{PLI}$ and $Q_n \subset Q_0$. For each such Q_n,

$$(5.55) \qquad \sum_{Q \in \mathcal{A}_n^*} \theta(Q) \langle h_Q, h_{Q_0} \rangle \leq C\theta_n\, \mu(G_n) \leq C\mu(H_n)$$

by (5.44) and (4.33). Now we can use the fact that the sets H_n are disjoint (by (4.69)) and contained in Q_0 to sum over n and get (5.54).

We want to prove a similar estimate for $\mathcal{B}_2(Q_0)$. Decompose $\mathcal{B}_2(Q_0)$ into the subsets $\mathcal{B}(R) = \{Q \in \text{BLI} : Q \subsetneq R\}$, $R \in \mathcal{S}(Q_0)$. For each $l \geq 0$, let $\mathcal{B}(R, l)$ denote the set of the cubes in $\mathcal{B}(R)$ such that $l(Q) = l$. If $l \geq 2$, each cube $Q \in \mathcal{B}(R, l)$ is contained in $N_l(R)$ (by the definition of $l(Q)$). In particular the centre of $B(Q)$, which indeed lies in

Q, lies at distance $< A^{-k(R)-l}$ from $E \setminus Q$ (compare with (5.25)). Since it also lies at distance $\geq A^{-k(Q)}$ from $E \setminus Q$ (because $E \cap B(Q) \subset Q$), we get that $A^{-k(Q)} \leq A^{-k(R)-l}$, and then $\widehat{Q} \subset N_{l-2}(R)$. Moreover, by the definition of $\mathcal{B}(R)$, Q is a cube of BLI and its parent is a cube $Q_n \in$ PLI. We may now regroup cubes with a given parent Q_n and get that for $l \geq 2$,

$$(5.56) \qquad \sum_{Q \in \mathcal{B}(R,l)} \theta(Q) \langle h_Q, h_{Q_0} \rangle \leq C(1+l) \sum_{Q \in \mathcal{B}(R,l)} \theta(Q) \, \mu(Q)$$

$$\leq C(1+l) \sum_{\substack{Q_n \in \text{PLI} \\ Q_n \subset N_{l-2}(R)}} \theta_n \, \mu(G_n)$$

$$\leq C(1+l) \sum_{\substack{Q_n \in \text{PLI} \\ Q_n \subset N_{l-2}(R)}} \mu(H_n)$$

$$\leq C(1+l) \, \mu\big(N_{l-2}(R)\big)$$

$$\leq C(1+l) \, C_0^{-93l} \, \mu\big(90B(R)\big)$$

by (5.53), (4.33), the disjointness of the sets H_n, and (5.26). When $l \leq 2$, we simply repeat the argument for (5.54) or replace $N_{l-2}(R)$ by R in (5.56), and get less than $C\mu(R) \leq C\mu\big(90B(R)\big)$.

We may now sum over l and then $R \in \mathcal{S}(Q_0)$ to get that

$$(5.57) \qquad \sum_{Q \in \mathcal{B}_2(Q_0)} \theta(Q) \langle h_Q, h_{Q_0} \rangle \leq \sum_{R \in \mathcal{S}(Q_0)} \sum_{l \geq 0} C(1+l) \, C_0^{-l} \, \mu\big(90B(R)\big)$$

$$\leq C \sum_{R \in \mathcal{S}(Q_0)} \mu\big(90B(R)\big)$$

$$\leq C \sum_{R \in \mathcal{S}(Q_0)} \mu(R) \leq C\mu(Q_0)$$

because the cubes $R \in \mathcal{S}(Q_0)$ are good, disjoint, and contained in Q_0. From this and (5.54) we deduce that

$$(5.58) \qquad \sum_{\substack{Q \in \text{BLI} \\ Q \subset Q_0}} \theta(Q) \langle h_Q, h_{Q_0} \rangle \leq C\mu(Q_0).$$

Let us observe, also for future reference, that

$$(5.59) \qquad \sum_{Q \in \mathcal{R}} \theta(Q) \, \mu(Q) \leq C\mu(E).$$

This follows by the same argument which was used to prove (4.51) and (4.76). Hence

$$(5.60) \qquad \|W_3\|_{L^2(d\mu)}^2 = \sum_{Q_0} \sum_{Q} \theta(Q_0) \, \theta(Q) \langle h_Q, h_{Q_0} \rangle$$

$$\leq 2 \sum_{Q_0} \sum_{Q \subset Q_0} \theta(Q_0) \, \theta(Q) \langle h_Q, h_{Q_0} \rangle$$

$$\leq C \sum_{Q_0} \theta(Q_0) \, \mu(Q_0) \leq C\mu(E)$$

by (5.9), the fact that $\langle h_Q, h_{Q_0} \rangle = 0$ when Q does not meet Q_0, (5.58) and (5.59). Because of Lemma 5.10, this also implies that

$$(5.61) \qquad \|W_1 + \widetilde{W}_1\|^2_{L^2(d\nu)} \le C\mu(E),$$

and completes our study of W_1 and $\widetilde{W}_1$.

5.2. THE FUNCTIONS e_Q, THE FARAWAY PIECE

We would like to estimate the functions W_2 and $\widetilde{W}_2$ in a unified way, and to this effect we introduce a variant of the functions e_Q^* and $\widetilde{e}_Q$ that will be easier to deal with. For each cube $Q \in \Delta$, define a function e_Q on E by

$$(5.62) \qquad e_Q(x) = \mathbf{1}_{E \setminus Q}(x)\, \rho_Q(x) \int_Q \frac{d\mu(y)}{|x - y|}.$$

Set

$$(5.63) \qquad W(x) = \sum_{Q \in \mathcal{R}} \theta(Q)\, e_Q,$$

with $\mathcal{R} = I_1 \cup \mathrm{LI} \cup \mathrm{BLI}$. Let us first check that W dominates W_2 and $\widetilde{W}_2$. First $W_2(x) = W(x)$ for $x \in E_\infty$ (compare (5.62) with (5.3) and (4.78)). Then

$$(5.64) \qquad \|W_2\|^2_{L^2(d\nu)} \le \|W\|^2_{L^2(d\mu)},$$

because $d\nu = \rho_\infty\, d\mu$ on E_∞. Next set

$$(5.65) \qquad \langle \widetilde{e}_Q, \widetilde{e}_{Q'} \rangle_\nu = \int \widetilde{e}_Q\, \widetilde{e}_{Q'}\, d\nu$$

and

$$(5.66) \qquad \langle e_Q, e_{Q'} \rangle = \int_E e_Q\, e_{Q'}\, d\mu$$

for all $Q, Q' \in \mathcal{R}$. We claim that

$$(5.67) \qquad \langle \widetilde{e}_Q, \widetilde{e}_{Q'} \rangle_\nu \le C \langle e_Q, e_{Q'} \rangle \quad \text{for all } Q, Q' \in \mathcal{R}.$$

To prove this claim, first observe that

$$(5.68) \qquad \langle \widetilde{e}_Q, \widetilde{e}_{Q'} \rangle_\nu = \sum_{n \in J(Q) \cap J(Q')} \int_{\mathcal{C}_n} e_Q^*\, e_{Q'}^*\, d\nu_n,$$

where $J(Q)$ and $J(Q')$ are defined just after Proposition 4.81.

Let $n \in J(Q)$ be given. First observe that $H_n \subset E \setminus Q$ by definitions. (Recall that $H_n = Q_n$ when $Q_n \in I_1$, and $H_n = Q_n \setminus G_n$ when $Q_n \in \mathrm{PLI}$.) Then $\mathrm{dist}(\mathcal{C}_n, Q) \ge \mathrm{dist}(\mathcal{C}_n, E \setminus H_n) \ge A^{-k_n}/2$ by (4.47). Thus for all choices of $\widetilde{x} \in H_n$, $x \in \mathcal{C}_n$ and $y \in Q$, we have that

$$(5.69) \qquad |\widetilde{x} - y| \le CA^{-k_n} + |x - y| \le C|x - y|.$$

If we integrate over y and observe that $H_n \subset E \setminus Q$, we get that

$$(5.70) \qquad e_Q^*(x) \leq C e_Q(\tilde{x}) \quad \text{for } x \in \mathcal{C}_n \text{ and } \tilde{x} \in H_n.$$

Of course a similar estimate for Q' holds if $n \in J(Q')$ as well, and

$$(5.71) \qquad \int_{\mathcal{C}_n} e_Q^* \, e_{Q'}^* \, d\nu_n \leq C \int_{H_n} e_Q \, e_{Q'} \, d\mu$$

because $\|\nu_n\| \leq \mu(H_n)$ by (4.73). We may now use the fact that the sets H_n are disjoint (by (4.69)) to sum (5.71) and prove our claim (5.67).

Of course a brutal expansion of $\|\widetilde{W}_2\|_{L^2(d\nu)}^2$ and $\|W\|_{L^2(d\mu)}^2$ from their definitions (5.4) and (5.63) now yields

$$(5.72) \qquad \|\widetilde{W}_2\|_{L^2(d\nu)}^2 \leq C \|W\|_{L^2(d\mu)}^2.$$

Thus it will be enough to control the function W.

We start with the "faraway piece"

$$(5.73) \qquad W^\infty = \sum_{Q \in \mathcal{R}} \theta(Q) \, \mathbf{1}_{E \setminus 2Q} \, e_Q,$$

which we even split further as follows. For each integer $m \geq 0$, set

$$(5.74) \qquad e_Q^m = \mathbf{1}_{\mathcal{D}_m(Q)} \, e_Q,$$

where

$$(5.75) \qquad \mathcal{D}_m(Q) = \left\{ x \in E : A^{-k(Q)+m} < \text{dist}(x, Q) \leq A^{-k(Q)+m+1} \right\}.$$

It is clear that $E \setminus 2Q = \bigcup_{m \geq 0} \mathcal{D}_m(Q)$ (compare with (4.79)), and hence

$$(5.76) \qquad W^\infty = \sum_{m=0}^{\infty} \left\{ \sum_{Q \in \mathcal{R}} \theta(Q) \, e_Q^m \right\}.$$

A brutal estimate using the definitions (5.62) and (4.77) gives that

$$(5.77) \qquad e_Q^m(x) \leq C A^{k(Q)-2m} \, \mu(Q) \, \mathbf{1}_{\mathcal{D}_m(Q)}.$$

We also have that

$$(5.78) \qquad \mu\big(\mathcal{D}_m(Q)\big) \leq C A^{-k(Q)+m},$$

because $Q \in I_1 \cup \text{LI} \cup \text{BLI}$ and by (4.13) (applied to Q or one of its ancestors).

Next let Q and Q' be two cubes of $\mathcal{R}$, and assume for definiteness that $k(Q') \geq k(Q)$. Then

$$(5.79) \quad \langle e_Q^m, e_{Q'}^m \rangle \leq C A^{k(Q)+k(Q')-4m} \mu(Q) \, \mu(Q') \, \mu\big(\mathcal{D}_m(Q')\big) \leq C A^{k(Q)-3m} \mu(Q) \, \mu(Q')$$

by (5.77) and (5.78).

Denote by $\mathcal{R}_m(Q)$ the set of cubes $Q' \in \mathcal{R}$ such that $\langle e_Q^m, e_{Q'}^m \rangle \neq 0$. If $Q' \in \mathcal{R}_m(Q)$, then $\mathcal{D}_m(Q')$ meets $\mathcal{D}_m(Q)$, and hence $Q' \subset B(x_Q, CA^{-k(Q)+m})$, where x_Q is for instance the centre of $B(Q)$. Then (4.51) and (4.76) tell us that

$$(5.80) \qquad \sum_{Q' \in \mathcal{R}_m(Q)} \theta(Q')\, \mu(Q') \leq CA^{-k(Q)+m}.$$

Now

$$(5.81) \qquad \left\| \sum_{Q \in \mathcal{R}} \theta(Q)\, e_Q^m \right\|_{L^2(d\mu)}^2$$

$$\leq 2 \sum_{Q \in \mathcal{R}} \sum_{\substack{Q' \in \mathcal{R} \\ k(Q') \geq k(Q)}} \theta(Q)\, \theta(Q')\, \langle e_Q^m, e_{Q'}^m \rangle$$

$$\leq C \sum_{Q \in \mathcal{R}} \sum_{Q' \in \mathcal{R}_m(Q)} \theta(Q)\, \theta(Q')\, A^{k(Q)-3m}\, \mu(Q)\, \mu(Q')$$

$$\leq C \sum_{Q \in \mathcal{R}} \theta(Q)\, A^{-2m}\, \mu(Q)$$

$$\leq CA^{-2m}\, \mu(E)$$

by (5.79), (5.80), and (5.59). Of course (5.81) and the definition (5.76) yield

$$(5.82) \qquad \|W^\infty\|_{L^2(d\mu)}^2 \leq C\mu(E).$$

5.3. Estimates for the semi-local part of e_Q

Now we wish to study the remaining part of the functions e_Q, i.e., the functions $e_Q - \mathbf{1}_{E \setminus 2Q}\, e_Q = \mathbf{1}_{2Q \setminus Q}\, e_Q$. Here again it will be helpful to use the almost decomposition of the domain Q into maximal good cubes R, $R \in \mathcal{S}(Q)$. Thus we set

$$(5.83) \qquad g_R(x) = \int_R \frac{d\mu(y)}{|x-y|}$$

for $x \in E \setminus R$, and then observe that

$$(5.84) \qquad \mathbf{1}_{2Q \setminus Q}\, e_Q \leq \sum_{R \in \mathcal{S}(Q)} \mathbf{1}_{2Q \setminus Q}\, g_R.$$

We further decompose each $\mathbf{1}_{2Q \setminus Q}\, g_R$ into its very local part

$$(5.85) \qquad e_{Q,R}^0 = \mathbf{1}_{2R \setminus Q}\, g_R$$

and its less local part

$$(5.86) \qquad e_{Q,R} = \mathbf{1}_{2Q \setminus (2R \cup Q)}\, g_R.$$

Set

$$(5.87) \qquad W_4 = \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q)\, e_{Q,R}$$

and

$$(5.88) \qquad W_5 = \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q) \, e^0_{Q,R}.$$

¿From the discussion above it is clear that

$$(5.89) \qquad W - W^\infty = \sum_{Q \in \mathcal{R}} \theta(Q) \, \mathbf{1}_{2Q \setminus Q} \, e_Q \le W_4 + W_5.$$

In this section we want to estimate the semi-local part W_4.

Let $Q, Q' \in \mathcal{R}$ and $R \in \mathcal{S}(Q)$, $R' \in \mathcal{S}(Q')$ be given. We want to estimate the scalar product

$$(5.90) \qquad \langle e_{Q,R}, e_{Q',R'} \rangle = \int_{D(R,R')} g_R \, g_{R'} \, d\mu,$$

where

$$(5.91) \qquad D(R, R') = (2Q \cap 2Q') \setminus (2R \cup 2R' \cup Q \cup Q').$$

Obviously $g_R(x) \le \operatorname{dist}(x, R)^{-1} \mu(R)$, and similarly for R'. Thus

$$(5.92) \qquad \langle e_{Q,R}, e_{Q',R'} \rangle \le \mu(R) \, \mu(R') \, a(R, R'),$$

where

$$(5.93) \qquad a(R, R') = \int_{D(R,R')} \operatorname{dist}(x, R)^{-1} \operatorname{dist}(x, R')^{-1} \, d\mu(x).$$

Next we continue and cut each $a(R, R')$ into pieces $a_{k,l}$, as follows. For each $k \ge 0$ and $l \ge 0$, set

$$(5.94) \qquad U_k = \left\{ x \in 2Q : A^{-k(Q)-k} < \operatorname{dist}(x, R) \le A^{1-k(Q)-k} \right\},$$

$$(5.95) \qquad V_l = \left\{ x \in 2Q' : A^{-k(Q')-l} < \operatorname{dist}(x, R') \le A^{1-k(Q')-l} \right\}$$

and

$$(5.96) \qquad D_{k,l} = D(R, R') \cap U_k \cap V_l.$$

Clearly, $2Q \setminus Q$ is covered by the U_k's and the set $\{x \in E \setminus Q : \operatorname{dist}(x, Q) = 0\}$ which has μ measure zero by (3.13). Using a similar fact for $2Q' \setminus Q'$ and the V_l's we see that $D(R, R')$ is the union of the $D_{k,l}$'s and a set of μ measure zero. Thus

$$(5.97) \qquad a(R, R') = \sum_{k \ge 0} \sum_{l \ge 0} a_{k,l},$$

where

$$(5.98) \qquad a_{k,l} = \int_{D_{k,l}} \operatorname{dist}(x, R)^{-1} \operatorname{dist}(x, R')^{-1} \, d\mu(x) \le A^{k+l} \, A^{k(Q)} \, A^{k(Q')} \, \mu(D_{k,l}).$$

If $D_{k,l}$ is not empty, then U_k meets $E \setminus 2R$. Since $\operatorname{dist}(x, R) \geq A^{-k(R)}$ on $E \setminus 2R$, this forces $A^{-k(R)} \leq A^{1-k(Q)-k}$. Of course there is a similar condition on l, and so

$$(5.99) \quad 0 \leq k \leq k(R) - k(Q) + 1 \quad \text{and} \quad 0 \leq l \leq k(R') - k(Q') + 1 \quad \text{when } D_{k,l} \neq \emptyset.$$

For $0 < k \leq k(R) - k(Q) + 1$, let R_k denote the cube of generation $k(Q) + k - 1$ that contains R, and for $k = 0$ let $R_k = Q$. Since R is a maximal good subcube of Q, we can apply Lemma 5.31 to R_k. Also, $U_k \subset 100B(R_k)$ by definitions and so

$$(5.100) \qquad\qquad \mu(U_k) \leq \mu\big(100B(R_k)\big) \leq CA^{-10k}\mu\big(100B(Q)\big).$$

Because $Q \in \mathcal{R} = I_1 \cup \mathrm{LI} \cup \mathrm{BLI}$, we may apply (4.13) to Q or its parent and get that $\mu\big(100B(Q)\big) \leq CA^{-k(Q)}$. So

$$(5.101) \qquad\qquad \mu(U_k) \leq CA^{-10k} A^{-k(Q)} \quad \text{for } 0 \leq k \leq k(R) - k(Q) + 1.$$

Of course the same argument applies to V_l and gives that

$$(5.102) \qquad\qquad \mu(V_l) \leq CA^{-10l}\mu\big(100B(Q')\big) \leq CA^{-10l} A^{-k(Q')}$$

for $0 \leq l \leq k(R') - k(Q') + 1$.

We want to distinguish two cases, depending on the size of

$$(5.103) \qquad\qquad\qquad\qquad \delta = \operatorname{dist}(R, R').$$

We start with the case when

$$(5.104) \qquad\qquad\qquad k(Q) \leq k(Q') \quad \text{and} \quad \delta \leq A^{-k(Q')+1}.$$

(The first condition is just here for the sake of definiteness, but the second one is serious.) In this first case we want to refine the estimate (5.102) before we apply it. The idea will be that if Q' is somewhat smaller than Q, Lemma 5.31 will tell us that $\mu\big(100B(Q')\big) \ll A^{-k(Q')}$.

Let us assume that we are in Case 1, and that $D(R, R')$ is not empty. If $x \in D(R, R')$,

$$(5.105) \qquad \operatorname{dist}(x, R) \leq \operatorname{dist}(x, Q') + \operatorname{diam} Q' + \operatorname{dist}(R', R)$$
$$\leq A^{-k(Q')} + 100C_0 A^{-k(Q')} + \delta \leq 2A^{-k(Q')+1}$$

because $R' \subset Q'$, $x \in 2Q'$, and by (5.104). Since $x \in D(R, R') \subset E \setminus 2R$, we also have that $\operatorname{dist}(x, R) \geq A^{-k(R)}$, and so (5.105) yields

$$(5.106) \qquad\qquad\qquad\qquad k(Q') - 1 \leq k(R).$$

Now suppose that in addition $k(Q') - 1 \geq k(Q)$, and denote by $\widetilde{R}$ the cube of generation $k(Q') - 1$ that contains R. Then $R \subset \widetilde{R} \subset Q$, and we may apply Lemma 5.31 to $\widetilde{R}$ and get that

$$(5.107) \qquad\qquad \mu\big(100B(\widetilde{R})\big) \leq CA^{-10(k(Q')-k(Q))}\mu\big(100B(Q)\big).$$

Since $D(R, R')$ meets $2Q'$, (5.105) tells us that all points of $100B(Q')$ lie at distance $\leq 3A^{-k(Q')+1} = 3A^{-k(\widetilde{R})}$ from R. Hence $100B(Q') \subset 100B(\widetilde{R})$. We also know (from (4.13) applied to Q or its parent) that $\mu\big(100B(Q)\big) \leq CA^{-k(Q)}$, and so (5.107) implies that

$$(5.108) \qquad \mu\big(100B(Q')\big) \leq CA^{-10(k(Q')-k(Q))} A^{-k(Q)}.$$

This was when $k(Q') - 1 \geq k(Q)$. In this first case where (5.104) holds, the only other possibility is that $k(Q') = k(Q)$, and in this case (5.108) also holds, more trivially, by (4.13). So (5.108) always holds, provided that $D(R, R') \neq \emptyset$ and we are in Case 1. Now we use the first half of (5.102) to get that

$$(5.109) \qquad \mu(V_l) \leq CA^{-10(l+k(Q')-k(Q))} A^{-k(Q)}.$$

Then we say that $\mu(D_{k,l}) \leq \mathrm{Min}\{\mu(U_k), \mu(V_l)\}$, and use (5.98), (5.101) and (5.109) to obtain

$$(5.110) \qquad a_{k,l} \leq CA^{k+l} A^{k(Q')} \mathrm{Min}\left\{A^{-10k}, A^{-10(l+k(Q')-k(Q))}\right\}.$$

Next $\mathrm{Min}(u^{10}, v^{10}) \leq u^2 v^8$, and so

$$(5.111) \qquad a_{k,l} \leq CA^{k+l} A^{k(Q')} A^{-2k} A^{-8(l+k(Q')-k(Q))}$$
$$= CA^{-k-7l} A^{-7k(Q')+8k(Q)} \leq CA^{-k-l} A^{k(Q)}.$$

(The other convergence factors will not help, and so we drop them.) Now use (5.92) and (5.97), and sum over k and l to get that

$$(5.112) \qquad \langle e_{Q,R}, e_{Q',R'} \rangle \leq \mu(R)\,\mu(R')\,a(R, R')$$
$$= \sum_k \sum_l \mu(R)\,\mu(R')\,a_{k,l}$$
$$\leq C\mu(R)\,\mu(R')\,A^{k(Q)}.$$

Now denote by Z_1 the set of all quadruples (Q, R, Q', R') where $Q \in \mathcal{R}$, $R \in \mathcal{S}(Q)$, $Q' \in \mathcal{R}$, $R' \in \mathcal{S}(Q')$, $D(R, R') \neq \emptyset$ and (5.104) holds (i.e., we are in Case 1). Also set

$$(5.113) \qquad \sigma_1 = \sum_{(Q,R,Q',R') \in Z_1} \theta(Q)\,\theta(Q')\,\langle e_{Q,R}, e_{Q',R'} \rangle.$$

Then

$$(5.114) \qquad \sigma_1 \leq C \sum_{Z_1} \theta(Q)\,\theta(Q')\,\mu(R)\,\mu(R')\,A^{k(Q)}$$
$$\leq C \sum_{Q \in \mathcal{R}} \sum_{Q' \in \mathcal{R}(Q)} \theta(Q)\,\theta(Q')\,\mu(Q)\,\mu(Q')\,A^{k(Q)},$$

where

$$(5.115) \qquad \mathcal{R}(Q) = \big\{Q' \in \mathcal{R} : k(Q') \geq k(Q) \text{ and } 2Q \cap 2Q' \neq \emptyset\big\}.$$

(The last condition is needed if we want any of the sets $D(R, R')$ to be nonempty.)
Notice that all the cubes Q', $Q' \in \mathcal{R}(Q)$, are contained in a fixed ball centered on Q
and with radius $CA^{-k(Q)}$. Then (4.51) and (4.76) imply that

$$
(5.116) \qquad \sum_{Q' \in \mathcal{R}(Q)} \theta(Q') \, \mu(Q') \leq C A^{-k(Q)}
$$

and then

$$
(5.117) \qquad \sigma_1 \leq C \sum_{Q \in \mathcal{R}} \theta(Q) \, \mu(Q) \leq C \mu(E)
$$

by (5.59).

Now we come to our second case when $k(Q') \geq k(Q)$, but (5.104) fails, i.e., when

$$
(5.118) \qquad k(Q') \geq k(Q) \quad \text{and} \quad \delta > A^{-k(Q')+1}.
$$

In this case there is an additional constraint that we want to use: if $D_{k,l}$ is not empty,
then U_k meets V_l and

$$
(5.119) \qquad \delta = \text{dist}(R, R') \leq A^{1-k(Q)-k} + A^{1-k(Q')-l}.
$$

(Compare with (5.94) and (5.95).) Since in this case $\delta > A^{-k(Q')+1} \gg A^{-k(Q')-l}$
(because $l \geq 0$), we even have that

$$
(5.120) \qquad \delta \leq 2 A^{1-k(Q)-k}.
$$

Here diam Q' is much smaller than δ (by (5.118)), and hence not much larger than the
"scale" $A^{-k(Q)-k}$ of U_k (by (5.120) and the definition (5.94)). Thus it will be reasonable
to derive our estimate of $\mu(D_{k,l})$ only from the estimate (5.102) on $\mu(V_l)$. Let us do
this. From (5.98) and (5.102) we deduce that

$$
(5.121) \qquad a_{k,l} \leq C A^{k-9l} A^{k(Q)}.
$$

Sum this first over l, with k fixed. We get less than $CA^{k+k(Q)}$. Then sum again over
all $k \geq 0$ such that (5.120) holds. This gives $\leq C\delta^{-1}$. Thus (5.92) and (5.97) yield

$$
(5.122) \qquad \begin{aligned}
\langle e_{Q,R}, e_{Q',R'} \rangle &\leq \mu(R) \, \mu(R') \, a(R, R') \\
&= \mu(R) \, \mu(R') \sum_k \sum_l a_{kl} \\
&\leq C \mu(R) \, \mu(R') \, \delta^{-1}.
\end{aligned}
$$

Let Z_2 denote the set of quadruples (Q, R, Q', R') with $Q \in \mathcal{R}$, $R \in \mathcal{S}(Q)$, $Q' \in \mathcal{R}$,
$R' \in \mathcal{S}(Q')$, and for which (5.118) holds and $D(R, R') \neq \emptyset$. Also set

$$
(5.123) \qquad \sigma_2 = \sum_{(Q,R,Q',R') \in Z_2} \theta(Q) \, \theta(Q') \, \langle e_{Q,R}, e_{Q',R'} \rangle.
$$

Then

$$
(5.124) \qquad \sigma_2 \leq \sum_{Z_2} \theta(Q) \, \theta(Q') \, \mu(R) \, \mu(R') \, \text{dist}(R, R')^{-1}.
$$

Next we fix $Q \in \mathcal{R}$ and $R \in \mathcal{S}(Q)$, and we wish to sum over Q' and R'. We want to
compare the sum with an integral.

5.125. Lemma. *If (5.118) holds and $D(R, R') \neq \emptyset$,*

(5.126)   *the parent $\widehat{Q}'$ of Q' is contained in*

$$\{x \in E : \operatorname{dist}(x, Q) \leq CA^{-k(Q)}\},$$

(5.127) $\operatorname{dist}(R, R') \geq \dfrac{1}{2}A^{-k(R)},$

and

(5.128) $\operatorname{dist}(x, R) \leq C \operatorname{dist}(R, R')$ *for all $x \in \widehat{Q}'$.*

Recall from (5.91) that $D(R, R') \subset 2Q \cap 2Q'$ and that $k(Q') \geq k(Q)$ by (5.118). Our first assertion follows from this.

Next $D(R, R') \subset 2Q' \setminus 2R$, and so $2Q'$ meets $E \setminus 2R$ and in particular there are points in $2Q'$ at distance $\geq A^{-k(R)}$ from R. On the other hand all points of $2Q'$ lie at distance $\leq \operatorname{diam} 2Q' + \operatorname{dist}(R', R)$ from R, and this number is $\leq 2 \operatorname{dist}(R, R')$ by (5.118). This proves (5.127).

Finally, if $x \in \widehat{Q}'$, then $\operatorname{dist}(x, R) \leq \operatorname{dist}(x, R') + \operatorname{dist}(R', R) \leq CA^{-k(Q')+1} + \operatorname{dist}(R', R) \leq C \operatorname{dist}(R, R')$ by (5.118). The lemma follows.

5.129. Lemma. *For each choice of Q and R, set*

(5.130) $\sigma(Q, R) = \displaystyle\sum_{Q'} \sum_{R'} \theta(Q') \, \mu(R') \operatorname{dist}(R, R')^{-1},$

where we sum over pairs (Q', R') such that $(Q, R, Q', R') \in Z_2$. Then

(5.131) $\sigma(Q, R) \leq C \displaystyle\int_{\widetilde{Q}} \varphi_R \, d\mu,$

where $\widetilde{Q} = \{x \in E : \operatorname{dist}(x, Q) \leq CA^{-k(Q)}\}$ is the same set as in (5.126), and

(5.132) $\varphi_R(x) = \left[A^{-k(R)} + \operatorname{dist}(x, R)\right]^{-1}.$

To prove the lemma, first observe that if $(Q, R, Q', R') \in Z_2$, the hypotheses of Lemma 5.125 are satisfied. Then $\operatorname{dist}(R, R')^{-1} \leq Cm_R(\widehat{Q}')$, where $m_R(\widehat{Q}') = \inf \{\varphi_R(x) : x \in \widehat{Q}'\}$. This follows from (5.127) and (5.128). Next

(5.133) $\displaystyle\sum_{R' \in \mathcal{S}(Q')} \mu(R') \operatorname{dist}(R, R')^{-1} \leq C\mu(Q') \, m_R(\widehat{Q}').$

Now all cubes Q' that show up in (5.130) lie in $\mathcal{A}_n \cup \mathcal{A}_n^*$ for some cube Q_n such that $Q_n \subset \widetilde{Q}$. (This comes from (5.126).) Thus

(5.134) $\sigma(Q, R) \leq C \displaystyle\sum_{n:Q_n \subset \widetilde{Q}} \sum_{Q' \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q') \, \mu(Q') \, m_R(\widehat{Q}')$

$$\leq C \sum_{n:Q_n \subset \widetilde{Q}} \mu(H_n) \, m_R(Q_n)$$

$$\leq C \sum_{n:Q_n \subset \widetilde{Q}} \int_{H_n} \varphi_R \, d\mu$$

$$\leq C \int_{\widetilde{Q}} \varphi_R(x) \, d\mu(x)$$

with a definition of $m_R(Q_n)$ similar to the definition of $m_R(\widehat{Q}')$ above, and by (4.75) and the disjointness property (4.69) for the H_n's. This proves the lemma.

Our next task is to estimate the integral in (5.131). As usual, we cut it in slices and use the fact that there is no good cube between R and Q. Let us start with the contribution of the "annuli" U_k, $0 \leq k \leq k(R) - k(Q) + 1$, from (5.94). We can use (5.101) to get that

$$(5.135) \qquad \int_{U_k} \varphi_R \, d\mu \leq A^{k(Q)+k} \, \mu(U_k) \leq C A^{-9k}.$$

The contribution of the region near R can be estimated in the same way. If $D_0 = \{x \in E : \text{dist}(x, R) \leq A^{-k(R)-1}\}$, then $\varphi_R(x) \leq A^{k(R)}$ on D_0 (and even everywhere), and $D_0 \subset 100B(R)$. Then $\mu(D_0) \leq \mu(100B(R)) \leq C A^{-10(k(R)-k(Q))} \mu(100B(Q)) \leq C A^{-10k(R)+9k(Q)}$ by Lemma 5.31 and the density estimate (4.13), applied to Q or its parent. Thus

$$(5.136) \qquad \int_{D_0} \varphi_R \, d\mu \leq A^{k(R)} \, \mu(D_0) \leq C A^{-9(k(R)-k(Q))}.$$

For the exterior "annulus" $D_\infty = \{x \in \widetilde{Q} : \text{dist}(x, R) > A^{-k(Q)}\}$, we use (4.13) again (applied to an ancester of Q, possibly its grandparent) to get that $\mu(\widetilde{Q}) \leq C A^{-k(Q)}$, and then

$$(5.137) \qquad \int_{D_\infty} \varphi_R \, d\mu \leq C.$$

¿From (5.135), (5.136), (5.137) and the fact that $\widetilde{Q} = D_\infty \cup D_0 \cup \left(\bigcup_{k=0}^{k(R)-k(Q)} U_k \right)$, we deduce that

$$(5.138) \qquad \int_{\widetilde{Q}} \varphi_R \, d\mu \leq C,$$

and then Lemma 5.129 tells us that $\sigma(Q, R) \leq C$. If we compare (5.124) with the definition (5.130), we obtain that

$$(5.139) \qquad \sigma_2 \leq C \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q) \, \mu(R) \leq C \sum_{Q \in \mathcal{R}} \theta(Q) \, \mu(Q) \leq C \mu(E)$$

by (5.59). Note that σ_1 and σ_2 control all the terms where $k(Q') \geq k(Q)$, and so

$$(5.140) \qquad \|W_4\|_{L^2(d\mu)}^2 = \| \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q) \, e_{Q,R} \|_2^2 \leq 2\sigma_1 + 2\sigma_2 \leq C \mu(E)$$

by the definition (5.87) of W_4, the definitions (5.113) and (5.123) of σ_1 and σ_2, and the estimates (5.117) and (5.139). This completes our study of the function W_4.

5.4. THE VERY LOCAL PART W_5

We are left with $W_5 = \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q) \, e^0_{Q,R}$, with $e^0_{Q,R}$ as in (5.85). We start with an estimate of $e^0_{Q,R}$ for Q and R fixed. Of course we may assume that $x \in 2R \setminus Q$, since otherwise $e^0_{Q,R}(x) = 0$. Set

$$(5.141) \qquad d_R(x) = A^{k(R)} \operatorname{dist}(x, R).$$

5.142. Lemma. *We have that*

$$(5.143) \qquad e^0_{Q,R}(x) \le C d_R(x)^{-\frac{1}{10}} A^{-9(k(R)-k(Q))}.$$

Before we prove the lemma, observe that we could have proved an estimate with only a logarithmic singularity in $d_R(x)$, but we prefer (5.143) because of the convergence factor $A^{-9(k(R)-k(Q))}$.

To prove the lemma, recall first from (5.26) that $\mu(\overline{Q} \setminus Q) = 0$. Let $x \in 2R \setminus \overline{Q}$ be given and denote by $l(x)$ the largest integer l such that $x \in N_l(R)$, where $N_l(R)$ is as in (5.25) (or Chapter 3). Note that the only case when $x \notin N_0(R)$ is when $\operatorname{dist}(x, R) = A^{-k(Q)}$, because $x \in 2R$. In this case we still set $l(x) = 0$. In all events

$$(5.144) \qquad A^{-k(R)-l(x)-1} \le \operatorname{dist}(x, R) \le A^{-k(R)-l(x)}.$$

We split the domain of integration R into regions

$$(5.145) \qquad D_k = \left\{ y \in R : A^{-k} \le |x - y| < A^{-k+1} \right\}.$$

Let us check first that if D_k is not empty, then

$$(5.146) \qquad k(R) \le k \le k(R) + l(x) + 1.$$

If $y \in R$, then $|x - y| < A^{-k(R)+1}$ because $x \in 2R$, and so $A^{-k} < A^{-k(R)+1}$ if D_k is not empty; this proves the first inequality. Similarly, the first inequality in (5.144) forces $A^{-k(R)-l(x)-1} < A^{-k+1}$ if D_k is not empty; this completes the proof of (5.146).

Next $d(y, E \setminus R) \le |x - y| < A^{-k+1}$ for $y \in D_k$, and so $D_k \subset N_{k-k(R)-1}(R)$ if $k > k(R)$. (If $k = k(R)$, we shall simply remember that $D_k \subset R$.) Then

$$(5.147) \qquad \int_{D_k} \frac{d\mu(y)}{|x - y|} \le A^k \mu(D_k) \le C A^k C_0^{-93(k-k(R))} \mu(90B(R))$$

by (5.26). Recall from (4.1) that $A = CC_0^{100}$ for some absolute constant C, and so $AC_0^{-93} = CC_0^7 \le C^{1/10}C_0^{10} = A^{1/10}$ if C_0 is large enough, which we are happy to assume. Because of this,

$$(5.148)$$
$$\int_{D_k} \frac{d\mu(y)}{|x - y|} \le A^{\frac{1}{10}(k-k(R))} A^{k(R)} \mu(90B(R))$$
$$\le C A^{\frac{1}{10}(k-k(R))} A^{-9(k(R)-k(Q))} A^{k(Q)} \mu(100B(Q))$$
$$\le C A^{\frac{1}{10}(k-k(R))} A^{-9(k(R)-k(Q))}$$

by Lemma 5.31, the fact that R is a maximal good subcube of Q, and the usual density estimate (4.13) applied to Q or its parent.

When we sum (5.148) over k, we get a geometric series whose leading term is when k is largest, i.e., $k = k(R) + l(x) + 1$. Also note that $A^{-l(x)} \sim d_R(x)$ by (5.144). Thus

$$
(5.149) \qquad e^0_{Q,R}(x) = \int_R \frac{d\mu(y)}{|x-y|} = \sum_{k=k(R)}^{k(R)+l(x)+1} \int_{D_k} \frac{d\mu(y)}{|x-y|}
$$
$$
\leq C A^{\frac{l(x)}{10}} A^{-9(k(R)-k(Q))}
$$
$$
\leq C d_R(x)^{-\frac{1}{10}} A^{-9(k(R)-k(Q))}.
$$

This proves Lemma 5.142.

Now we fix the cubes Q, Q', R, R', and we estimate

$$
(5.150) \qquad I(R, R') = \langle e^0_{Q,R}, e^0_{Q',R'} \rangle = \int_{D(R,R')} e^0_{Q,R}\, e^0_{Q',R'}\, d\mu,
$$

where

$$
(5.151) \qquad D(R, R') = (2R \cap 2R') \setminus (Q \cup Q').
$$

Obviously by Lemma 5.142

$$
(5.152) \qquad I(R, R') \leq C A^{-9(k(R)-k(Q)+k(R')-k(Q'))} J(R, R'),
$$

where

$$
(5.153) \qquad J(R, R') = \int_{D(R,R')} \big(d_R(x)\, d_{R'}(x)\big)^{-\frac{1}{10}}\, d\mu(x)
$$

and $d_{R'}$ is defined like d_R (i.e., $d_{R'}(x) = A^{k(R')} \operatorname{dist}(x, R')$.)

5.154. Lemma. *Suppose for definiteness that $k(R') \geq k(R)$. Then*

$$
(5.155) \qquad J(R, R') \leq C \left[\frac{A^{-k(R)}}{A^{-k(R')} + \operatorname{dist}(R, R')} \right]^{\frac{1}{10}} \mu(R').
$$

We start the proof of Lemma 5.154 in the case when

$$
(5.156) \qquad \operatorname{dist}(R, R') \geq \operatorname{diam} R' + A^{-k(R')}.
$$

In this case the function $d_R(x)$ is essentially constant on $D(R, R')$ (because $D(R, R') \subset 2R'$) and

$$
(5.157) \qquad J(R, R') \leq C \big[A^{k(R)} \operatorname{dist}(R, R')\big]^{-\frac{1}{10}} \int_{D(R,R')} d_{R'}(x)^{-\frac{1}{10}}\, d\mu(x).
$$

The last integral is easy to estimate. Since we are going to need a variant later, let us state a slightly more general lemma.

5.158. Lemma. *For each cube $R \in \Delta$, set*

$$\delta_R(x) = A^{k(R)}\big\{ \operatorname{dist}(x, R) + \operatorname{dist}(x, E \setminus R)\big\}$$

for all $x \in E \cap 100B(R)$. Then

(5.159)
$$\int_{E \cap 100B(R)} \delta_R(x)^{-\frac{1}{5}} \, d\mu(x) \leq C\mu\big(100B(R)\big).$$

To prove the lemma, we use the sets $N_l(R)$ of (5.25) and Chapter 3. For each $l \geq 0$,

(5.160)
$$\int_{N_l(R) \setminus N_{l+1}(R)} \delta_R(x)^{-\frac{1}{5}} \, d\mu(x) \leq A^{\frac{1}{5}(l+1)} \, \mu\big(N_l(R)\big)$$

$$\leq CA^{\frac{1}{5}} \, C_0^{-93l} \, \mu\big(90B(R)\big)$$
$$\leq C(C' \, C_0^{20} \, C_0^{-93})^l \, \mu\big(90B(R)\big)$$

by (5.26), and because $A = C'^5 \, C_0^{100}$ for some absolute constant C', as in (4.1). We may sum over $l \geq 0$ and get that

(5.161)
$$\int_{N_0(R)} \delta_R(x)^{-\frac{1}{5}} \, d\mu(x) \leq C\mu\big(90B(R)\big).$$

(The set where $\delta_R(x) = 0$ has measure zero by (5.26), so we can forget about it.)

On the rest of $E \cap 100B(R)$, that is, on $\big(E \cap 100B(R)\big) \setminus N_0(R)$, $\delta_R(x) \geq C^{-1}$ and the corresponding integral is $\leq C\mu\big(100B(R)\big)$. The lemma follows.

Notice that $\delta_R = d_R$ on $2R \setminus R$, and so it follows from (5.157) and Lemma 5.158 (applied to R') that

(5.162)
$$J(R, R') \leq C\big[A^{k(R)} \operatorname{dist}(R, R')\big]^{-\frac{1}{10}} \mu\big(100B(R')\big).$$

This is less than the right-hand side of (5.155) because of (5.156) and the fact that $\mu\big(100B(R')\big) \leq C_0\mu(R')$ since R' is a good cube. This proves (5.155) when (5.156) holds.

Now assume that $\operatorname{dist}(R, R') < \operatorname{diam} R' + A^{-k(R')}$. Cover $2R' \setminus (R \cup R')$ by the cubes S of generation $k(R') + 1$ that meet $2R' \setminus (R \cup R')$. Note that $100B(S) \subset 100B(R')$ for each such cube S, and also that we need less than C cubes S. If $x \in S$, $d_R(x) \, d_{R'}(x) \geq A^{k(R)} A^{k(R')} \operatorname{dist}(x, E \setminus S)^2$, because S does not meet R or R'. (Recall that $k(S) > k(R') \geq k(R)$.) Thus

(5.163)
$$\int_{D(R,R')} (d_R \, d_{R'})^{-\frac{1}{10}} \, d\mu$$

$$\leq A^{-\frac{1}{10}(k(R)+k(R'))} \sum_S \int_S \operatorname{dist}(x, E \setminus S)^{-\frac{1}{5}} \, d\mu(x)$$
$$\leq CA^{-\frac{1}{10}(k(R)+k(R'))} \sum_S A^{\frac{1}{5}k(R')} \, \mu\big(100B(S)\big)$$
$$\leq CA^{-\frac{1}{10}(k(R)-k(R'))} \, \mu\big(100B(R')\big)$$
$$\leq CA^{-\frac{1}{10}(k(R)-k(R'))} \, \mu(R')$$

by Lemma 5.158 (applied to each S), and the fact that R' is a good cube. This time $A^{-k(R')} + \text{dist}(R, R') \sim A^{-k(R')}$ because $\text{dist}(R, R') < \text{diam } R' + A^{-k(R')}$, and (5.155) follows from (5.163). This completes our proof of Lemma 5.154.

Now let us fix Q, Q' and R, and try to sum $I(R, R')$ over all cubes $R' \in S(Q')$. We want to do a comparison with an integral. If $R' \in \mathcal{S}(Q')$ and x is any point of $\widehat{Q}'$ (the parent of Q'), and if $D(R, R')$ is not empty,

(5.164)
$$\text{dist}(x, R) + \text{dist}(x, E \setminus R)$$
$$\leq \text{dist}(x, R') + \text{diam } R' + \text{dist}(R', R) + \text{dist}(x, E \setminus R)$$
$$\leq CA^{-k(Q')} + \text{dist}(R', R) + \text{dist}(x, E \setminus R)$$
$$\leq CA^{-k(Q')} + \text{dist}(R, R') + \text{dist}\big(x, D(R, R')\big)$$
$$\leq CA^{-k(Q')} + \text{dist}(R, R') + CA^{-k(Q')}$$
$$\leq CA^{k(R')-k(Q')}[A^{-k(R')} + \text{dist}(R, R')]$$

because $R' \subset Q'$, and $D(R, R') = (2R \cap 2R') \setminus (Q \cup Q')$ is contained in $E \setminus R$ and meets $2Q'$. With the notation of Lemma 5.158,

(5.165)
$$\delta_R(x) \leq CA^{k(R')-k(Q')}\left[\frac{A^{-k(R')} + \text{dist}(R, R')}{A^{-k(R)}}\right] \quad \text{for all } x \in \widehat{Q}'.$$

Now suppose that

(5.166)
$$\widehat{Q}' \subset 100B(R),$$

and let $\mathcal{S}_R(Q')$ denote the set of cubes $R' \in \mathcal{S}(Q')$ such that $k(R') \geq k(R)$. (We need not prove that (5.166) implies that $k(Q') \geq k(R)$, so let us not bother.) Then

(5.167)
$$\sum_{R' \in \mathcal{S}_R(Q')} I(R, R')$$
$$\leq C \sum_{R'} A^{-9(k(R')-k(Q'))}\left[\frac{A^{-k(R)}}{A^{-k(R')} + \text{dist}(R, R')}\right]^{\frac{1}{10}} \mu(R')$$
$$\leq C \sum_{R'} \inf\left\{\delta_R(x)^{-\frac{1}{10}} : x \in \widehat{Q}'\right\} \mu(R')$$
$$\leq C \inf\left\{\delta_R(x)^{-\frac{1}{10}} : x \in \widehat{Q}'\right\} \mu(Q')$$

by (5.152), (5.155) and (5.165). We have dropped most of the convergence factor in (5.152) because it will no longer be useful here. Now we want to sum over Q' as well. Set

(5.168)
$$I_1(R) = \sum_{Q'} \sum_{R'} \theta(Q')\, I(R, R'),$$

where we sum on pairs Q', R' such that (5.166) holds and $k(R') \geq k(R)$. For each cube Q_n such that $Q_n \subset 100B(R)$,

(5.169)
$$\sum_{Q' \in \mathcal{A}_n \cup \mathcal{A}_n^*} \theta(Q')\, \mu(Q')\, \inf\left\{\delta_R(x)^{-\frac{1}{10}} : x \in Q_n\right\}$$
$$\leq C\mu(H_n) \inf\left\{\delta_R(x)^{-\frac{1}{10}} : x \in H_n\right\}$$
$$\leq C \int_{H_n} \delta_R(x)^{-\frac{1}{10}}\, d\mu(x)$$

by definitions, and (4.33) when $Q_n \in$ PLI. Then

$$(5.170) \qquad I_1(R) \leq C \sum_{n: Q_n \subset 100B(R)} \int_{H_n} \delta_R^{-\frac{1}{10}} \, d\mu$$

$$\leq C \int_{100B(R)} \delta_R^{-\frac{1}{10}} \, d\mu \leq C\mu\big(100B(R)\big) \leq C\mu(R)$$

because all the cubes Q' for which (5.166) holds lie in $\mathcal{A}_n \cup \mathcal{A}_n^*$ for some Q_n such that $Q_n \subset \widehat{Q}' \subset 100B(R)$, by (5.167), (5.169), Lemma 5.158, and the fact that R is a good cube.

Let Z_1 be the set of all quadruples (Q, R, Q', R') for which $Q, Q' \in \mathcal{R}$, $R \in \mathcal{S}(Q)$, $R' \in \mathcal{S}(Q')$, (5.166) holds, and $k(R') \geq k(R)$. Then

$$(5.171) \qquad \sum_{Z_1} \theta(Q)\,\theta(Q')\, \langle e_{Q,R}^0, e_{Q',R'}^0 \rangle = \sum_{Z_1} \theta(Q)\,\theta(Q')\, I(R, R')$$

$$\leq C \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q)\,\mu(R)$$

$$\leq C \sum_{Q \in \mathcal{R}} \theta(Q)\,\mu(Q) \leq C\mu(E)$$

by (5.150), (5.168), (5.170), and (5.59).

Now let Z_2 denote the set of quadruples (Q, R, Q', R') with $Q, Q' \in \mathcal{R}$, $R \in \mathcal{S}(Q)$, $R' \in \mathcal{S}(Q')$, and $k(R') \geq k(R)$ as before, but (5.166) does not hold.

If (5.166) does not hold and $D(R, R') \neq \emptyset$, then $k(Q') \leq k(R) + 1$ because $\widehat{Q}'$ is not contained in $100B(R)$ while $2Q'$ meets $2R$. Then

$$(5.172) \qquad \frac{A^{-k(R)}}{A^{-k(R')} + \mathrm{dist}(R, R')} \leq A^{k(R')-k(R)} \leq A^{k(R')-k(Q')+1},$$

and we deduce from (5.155) that $J(R, R') \leq C A^{k(R')-k(Q')} \mu(R')$ and then from (5.152) that

$$(5.173) \qquad I(R, R') \leq C A^{-k(R')+k(Q')} \mu(R').$$

(Here again we may drop most of the convergence factor.) Because $D(R, R') \subset 2R \cap 2R'$ and $k(R') \geq k(R)$, we only need to consider cubes R' that are contained in $100B(R)$.

For a given cube $R' \subset 100B(R)$, the sum over all cubes Q' of generations $\leq k(R)+1$ of the right-hand side of (5.173) is less than $C A^{-k(R')+k(R)} \mu(R')$, because there is at most one cube Q' per generation. If we sum this bound over all cubes R' of a given generation $k' \geq k(R)$, we get less than $C A^{-k'+k(R)} \mu\big(100B(R)\big)$, and if we sum again over all possible generations $k' \geq k(R)$, we get less than $C\mu\big(100B(R)\big) \leq C\mu(R)$ because R is a good cube. Thus for each $Q \in \mathcal{R}$ and each $R \in \mathcal{S}(Q)$,

$$(5.174) \qquad \sum_{Q'} \sum_{R'} I(R, R') \leq C\mu(R),$$

where we sum over all pairs Q', R' such that $(Q, R, Q', R') \in Z_2$. Hence

$$(5.175) \qquad \sum_{Z_2} \theta(Q)\,\theta(Q')\,\langle e^0_{Q,R}, e^0_{Q',R'} \rangle = \sum_{Z_2} \theta(Q)\,\theta(Q')\,I(R, R')$$

$$\leq C \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q)\,\mu(R)$$

$$\leq C \sum_{Q \in \mathcal{R}} \theta(Q)\,\mu(Q) \leq C\mu(E),$$

by (5.150), (5.174), and (5.59).

When we collect the sum over $Z_1 \cup Z_2$, we get all the terms for which $k(R') \geq k(R)$; the other ones can of course be obtained by symmetry, and so

$$(5.176) \qquad \|W_5\|^2_{L^2(d\mu)} = \left\| \sum_{Q \in \mathcal{R}} \sum_{R \in \mathcal{S}(Q)} \theta(Q)\,e^0_{Q,R} \right\|^2 \leq C\mu(E)$$

by (5.171) and (5.175).

This completes our estimate of the last piece of W. (See (5.82), (5.89), (5.140) and now (5.176) for the control of W.) Since we have seen also that W controls W_2 and $\widetilde{W}_2$ (see (5.64) and (5.72)), we get the desired bounds on these functions as well. Thus the proof of Theorem 2.4 is complete.

5.177. Remark. Suppose we give ourselves a set $J \subset \mathbb{N}^*$ and define functions W^J_1, $\widetilde{W}^J_1$, W^J_2 and $\widetilde{W}^J_2$ as in (5.1)–(5.4), but where we sum only on those cubes $Q \in \mathcal{R}$ that lie in $\mathcal{A}_n \cup \mathcal{A}^*_n$ for some $n \in J$. A close look at the arguments in this chapter shows that we also get that

$$(5.178) \qquad \|W^J_1\|^2_{L^2(d\nu)} + \|\widetilde{W}^J_1\|^2_{L^2(d\nu)} + \|W^J_2\|^2_{L^2(d\nu)} + \|\widetilde{W}^J_2\|^2_{L^2(d\nu)}$$

$$\leq C \sum_{n \in J} \mu(H_n) \leq C\mu\left(\bigcup_{n \in J} Q_n \right),$$

where the last inequality comes from (4.69). The proof is the same; we simply have to replace the usual estimate that $\sum_{\mathcal{R}} \theta(Q)\,\mu(Q) \leq C\mu(E)$ at the end of our various estimates (i.e., in (5.60), (5.81), (5.117), (5.171) and (5.175)) with the corresponding fact that $\sum_n \left\{ \sum_{\mathcal{A}_n \cup \mathcal{A}^*_n} \theta(Q)\,\mu(Q) \right\} \leq C \sum_n \mu(H_n)$.

5.179. Remark. In our L^2-estimates we have used the measure ν, but we could also have used the following slightly larger measure

$$(5.180) \qquad \nu^+ = \mathbf{1}_{E_\infty}\,d\mu + \sum_{n \geq 1} (\rho^*_n)^{-1}\,d\nu_n = \mathbf{1}_{E_\infty}\,d\mu + \sum_{n \geq 1} \frac{\mu(H_n)}{H^1(\mathcal{C}_n)}\,dH^1|_{\mathcal{C}_n},$$

where ρ^*_n denotes the constant value of ρ_{n-1} on Q_n, as in the definitions (4.17) and (4.26) of ν_n. (Compare (5.180) with the definition (4.41) of ν.) Note that

$$(5.181) \qquad (\rho^*_n)^{-1}\,\|\nu_n\| = \mu(H_n)$$

by definitions (or by (4.21) and (4.37)).

We claim that our proof also allows the same control on the norms of W_1, $\widetilde{W}_1$, W_2 and $\widetilde{W}_2$ in $L^2(d\nu^+)$, and similarly that

(5.182)
$$\|W_1^J\|_{L^2(d\nu^+)}^2 + \|\widetilde{W}_1^J\|_{L^2(d\nu^+)}^2 + \|W_2^J\|_{L^2(d\nu^+)}^2 + \|\widetilde{W}_2^J\|_{L^2(d\nu^+)}^2$$
$$\leq C \sum_{n \in J} \mu(H_n) \leq C\mu\left(\bigcup_{n \in J} Q_n\right).$$

This time we only have to recall the way we obtained our control on W_1, $\widetilde{W}_1$, W_2 and $\widetilde{W}_2$ in terms of the L^2-norms (for $d\mu$) of the functions W_3 and W defined in (5.9) and (5.63). For W_1, we only noticed that $h_Q(x) = \mathbf{1}_Q(x) \, h_Q^*(x)$ for $x \in E_\infty$, so that $W_1 = W_3$ on E_∞. Thus $\|W_1\|_{L^2(d\nu^+)}^2 \leq \|W_3\|_{L^2(d\mu)}^2$ directly (i.e., without using the weight ρ_∞). As for $\widetilde{W}_1$, let us note that the proof of Lemma 5.10 only uses the information that $\|\nu_n\| \leq \mu(H_n)$, and not any more precise informations that would have used the fact that $\|\nu_n\| = \rho_n^* \, \mu(H_n)$ and the precise size of ρ_n^*. Thus we may replace ν with ν^+ in Lemma 5.10 and use (5.181). Similar remarks apply to our control on W_2 and $\widetilde{W}_2$ using the function W: we only used the fact that $W_2(x) = W(x)$ on E_∞ (and not the size of ρ_∞ on E_∞) and the fact that $\|\nu_n\| \leq \mu(H_n)$ (and not the precise value of ρ_n^*). Except for these initial reductions (i.e., Lemma 5.10, (5.64) and (5.72)), the argument of Chapter 5 does not need to be modified; we were only working with the measure μ. The same modifications also work with W_1^J and its colleagues, and gives (5.182).

Remarks 5.177 and 5.179 are probably the first step towards a BMO-type estimate on the $T^\varepsilon(g d\nu)$.

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