LATTICE PROPERTY OF p-ADMISSIBLE WEIGHTS

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Abstract. We show that, for large p's, the maximum of two p-admissible weights remains p-admissible in the terminology of nonlinear potential theory. We also give examples showing that in general, the minimum may fail to remain p-admissible.

1. Introduction

Let $1 < p < \infty$ be fixed. Following [8, Ch. 20], we say that a locally integrable nonnegative function w on $\mathbf{R}^n$, $n \geq 1$, is *p-admissible* if it is the density of a doubling measure measure μ that supports a p-Poincaré inequality. More precisely we require that there exist positive constants C_d and C_P so that for each ball $B(x,r)$ and every Lipschitz function u on $\mathbf{R}^n$ we have that

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)) \tag{1.1}$$

and

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \Big(\fint_{B(x,r)} |\nabla u|^p \, d\mu \Big)^{1/p} . \tag{1.2}$$

Here, and in what follows, we use the notation

$$\mu(A) = \int_A w(x) \, dx$$

and, for any integrable function v,

$$v_A = \frac{1}{\mu(A)} \int_A v \, d\mu = \fint_A v \, d\mu$$

For the significance of the class of p-admissible weights we refer e.g. to [4, 8, 6, 1].

A core class of p-admissible weights is formed by the class of Muckenhoupt A_p-weights [4, 8]. Since the A_p-weights form a lattice:

$$w_1 \wedge w_2 \in A_p$$

and

$$w_1 \vee w_2 \in A_p$$

whenever $w_1, w_2 \in A_p$ (see Appendix below), it is natural to inquire if the same feature is shared by the entire class of p-admissible weights. It is rather surprising to us that this issue does not seem to have been addressed in the literature, not even for A_p-weights. In this note, we discuss this question by establishing the following result:

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1.3. **Theorem.** *The class of p-admissible weights on $\mathbf{R}$ is a lattice. In $\mathbf{R}^n$, $n \geq 2$, the minimum $w_1 \wedge w_2$ of two p-admissible weights w_1 and w_2 may fail to be p-admissible. Further, there is q_0 (depending on w_1) so that the maximum $w_1 \vee w_3$ is q-admissible whenever $q \geq q_0$ and w_3 is q-admissible.*

Let us briefly comment on the proof of Theorem 1.3. First of all, in dimension one, w is p-admissible if and only if $w \in A_p$ ([2] also see [3]), and hence the lattice property is that of the A_p class.

Secondly, in higher dimensions, we have been able to solve the problem only partially. The example in negative direction necessarily deals with non-A_p weights that are p-admissible. For $1 < p < n$, prime examples of such weights are of the form

$$w = J_f^{1-p/n} ,$$

where f is a quasiconformal self-homeomorphism of $\mathbf{R}^n$, $n \geq 2$. In our construction, we employ a planar quasiconformal mapping that generates a singular measure on the real line and simply use $w \equiv 1$ as our second weight. Higher dimensional cases are handled via a lifting procedure. This approach only applies for sufficiently small p; see Example 3.9. It would be interesting to see similar examples for all values of p.

Our proof for the positive direction in the case of the maximum uses a Hölder estimate for Sobolev functions in terms of the gradient. It would be interesting to dispense with it.

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2. Toolbox

In this section we collect auxiliary results that will be used in our proof of Theorem 1.3.

Our first lemma reduces the p-Poincaré inequality into a more checkable condition. The result relies on Mazya's truncation argument [10] and a chaining argument; see e.g. [6, p. 10 and Corollary 9.8].

2.1. **Lemma.** *Let w be a nonnegative, locally integrable function on $\mathbf{R}^n$ such that the associate measure μ with $d\mu = w\,dx$ is doubling (i.e. satisfies (1.1)). Suppose, further, that there is a constant c so that the estimate*

$$\begin{aligned} &\min\left(\mu(\{y \in B(x,r) : u(y) = 0\}), \mu(\{y \in B(x,r) : u(y) = 1\})\right) \\ &\qquad \leq c r^p \int_{B(x,4\sqrt{n}r)} |\nabla u|^p \, d\mu \end{aligned}$$

holds for all Lipschitz functions u and every ball $B(x,r)$. Then w is p-admissible.

Adding dummy variables allows us to lift weights to higher dimensions:

2.2. **Lemma.** *Let w be a p-admissible weight on $\mathbf{R}^n$. Then the weight $\hat{w}$,*

$$\hat{w}(x_1, \dots x_n, x_{n+1}) = w(x_1, \dots, x_n)$$

is p-admissible on $\mathbf{R}^{n+1}$.

Proof. Write μ and $\hat{\mu}$ for the associated measures with $d\mu = w dx$ on $\mathbf{R}^n$ and $d\hat{\mu} = \hat{w} dx$ on $\mathbf{R}^{n+1}$, respectively. Regarding the doubling condition (1.1), simply notice that

$$\hat{\mu}(B^{n+1}(x, 2r)) \leq 4r\mu(\pi(B^{n+1}(x, 2r))) \leq c(C_d, \mu, n)\hat{\mu}(B^{n+1}(x, r)) ,$$

where π is the projection from $\mathbf{R}^{n+1}$ onto $\mathbf{R}^n$ and C_d is the doubling constant of μ; notice that the cylinder

$$B^n(\pi(x), r/2) \times]x_{n+1} - r/2, x_{n+1} + r/2[$$

is contained in the ball $B^{n+1}(x, r)$.

Towards the p-Poincaré inequality (1.2), fix a Lipschitz function u and a ball $B^{n+1}(x, r)$. Write

$$E = \{y \in B^{n+1}(x, r) : u(y) = 0\} , \quad F = \{y \in B^{n+1}(x, r) : u(y) = 1\} .$$

Set

$$E_G = \{z \in \pi(E) : \text{ there is } s \in]x_{n+1} - r, x_{n+1} + r[\text{ with } u(z, s) > \frac{1}{3}\} .$$

If $z \in E_G$, then

$$\begin{aligned} \frac{1}{3} &\leq \int_{]x_{n+1}-r, x_{n+1}+r[} |\nabla u(z, t)| \, dt \\ &\leq (\int_{]x_{n+1}-r, x_{n+1}+r[} |\nabla u(z, t)|^p \, dt)^{1/p} (2r)^{1-1/p} , \end{aligned}$$

and hence

$$r^p \int_{B(x, \sqrt{n} r)} |\nabla u|^p \, d\hat{\mu} \geq 2^{1-p} 3^{-p} \mu(E_G) r .$$

Suppose that

$$\mu(E_G) \geq \frac{1}{2}\mu(\pi(E)).$$

Since

$$\hat{\mu}(E) \leq 2r\mu(\pi(E)) ,$$

it would follow that

$$r^p \int_{B(x, \sqrt{n} r)} |\nabla u|^p \, d\hat{\mu} \geq 2^{1-p} 3^{-p} \mu(E_G) r \geq 2^{-(1+p)} 3^{-p} \hat{\mu}(E) .$$

Hence the estimate assumed in Lemma 2.1 and hence also our claim would follow. Thus we may assume that

$$\mu(E_G) \leq \frac{1}{2}\mu(\pi(E)) .$$

Analogously, defining

$$F_G = \{z \in \pi(F) : \text{ there is } s \in]x_{n+1} - r, x_{n+1} + r[\text{ with } u(z, s) < \frac{2}{3}\} ,$$

we may assume that

$$\mu(F_G) \leq \frac{1}{2}\mu(\pi(F)) .$$

Thus, we are reduced to the case

$$\mu(\pi(E) \setminus E_G) \geq \frac{1}{2}\mu(\pi(E)) \text{ and } \mu(\pi(F) \setminus F_G) \geq \frac{1}{2}\mu(\pi(F))\,.$$

Now by truncating u appropriately, the definition of $\hat{w}$, the p-Poincaré inequality for μ on (copies of) $\mathbf{R}^n$, and the Fubini theorem yield

$$\begin{aligned} r^p \int_{B(x,\sqrt{n}r)} |\nabla u|^p \, d\hat{\mu} &\geq c(C_P, \mu, p) r \min\left(\mu(\pi(E)), \mu(\pi(F))\right) \\ &\geq c(C_P, \mu, p) \min\left(\hat{\mu}(E), \hat{\mu}(F)\right)\,, \end{aligned}$$

and the claim follows from Lemma 2.1. □

The following result due to Tukia [11] gives us the building block for our construction for the negative part in Theorem 1.3.

2.3. **Lemma.** *Let $0 < s < 1$. There is a quasiconformal mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ and a set $E_s \subset \mathbf{R}$ with*

$$f(\mathbf{R}) = \mathbf{R}\,, \quad \dim_H(E_s) \leq s \quad \text{and} \quad \dim_H(f(\mathbf{R} \setminus E_s)) \leq s\,.$$

Here and in what follows $\dim_H(E)$ refers to the Hausdorff dimension of the set E.

2.1. **Sets of (p, μ)-capacity zero.** We need to recall some facts of sets of (p, μ)-capacity zero. For a more thorough discussion the reader is referred to [8].

Suppose that $\Omega \subset \mathbf{R}^n$ is open. The *(p, μ)-capacity* $\mathrm{cap}_{p,\mu}(E, \Omega)$ of any set $E \subset \Omega$ is defined as follows: the (p, μ)-capacity of a compact set $K \subset \Omega$ is

$$\mathrm{cap}_{p,\mu}(K, \Omega) = \inf\left\{ \int_\Omega |\nabla \varphi|^p \, d\mu : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \text{ on } K \right\}.$$

The (p, μ)-capacity of an open set $U \subset \Omega$ is then

$$\mathrm{cap}_{p,\mu}(U, \Omega) = \sup\left\{\mathrm{cap}_{p,\mu}(K, \Omega) : K \text{ compact}, K \subset U\right\};$$

and for an arbitrary set $E \subset \Omega$

$$\mathrm{cap}_{p,\mu}(E, \Omega) = \inf\left\{\mathrm{cap}_{p,\mu}(U, \Omega) : U \text{ open}, E \subset U\right\}.$$

A set E is said to be of *(p, μ)-capacity zero* if

$$\mathrm{cap}_{p,\mu}(E \cap \Omega, \Omega) = 0 \quad \text{for all open } \Omega.$$

The definition seems a bit complicated, but for bounded sets E, one needs only one bounded open set $\Omega \supset E$ to find out if E is of (p, μ)-capacity zero [8, Lemma 2.9]. Moreover, the capacity is subadditive in E, so that E is of (p, μ)-capacity zero if and only if it is a countable union of sets of of (p, μ)-capacity zero.

We shall employ the fact that a bounded set E is of (p, μ)-capacity zero as soon as we find Lipschitz functions η_j (or more generally, quasi continuous functions from the corresponding weighted Sobolev space $W^{1,p}(\mathbf{R}^n; \mu)$), vanishing outside a fixed ball, such that $\max_j \eta_j \geq 1$ on E and

$$\sum_{j=1}^\infty \int_{\mathbf{R}^n} |\nabla \eta_j|^p \, d\mu < \varepsilon\,,$$

whenever $\epsilon > 0$ is a given number; see [8].

2.4. **Lemma.** *Suppose that $1 < p < n$ and that $f : \mathbf{R}^n \to \mathbf{R}^n$ is quasiconformal. Let $w(x) = J_f(x)^{1-p/n}$ and $E \subset \mathbf{R}^n$. If $\dim_H(f(E)) < n-p$, then E is of (p,μ)-capacity zero; recall $d\mu = wdx$.*

Proof. Recall that w is p-admissible. Let $\varepsilon > 0$. Since $\dim_H(f(E)) < n-p$, we may cover $f(E)$ with balls $B(x_j, r_j)$ such that

$$\sum_{j=1}^{\infty} r_j^{n-p} < \varepsilon\,.$$

Next choose Lipschitz functions η_j with compact supports in $B(x_j, 2r_j)$ such that $|\nabla\eta_j| < C/r_j$, $\eta_j = 1$ on $B(x_j, r_j)$. Then

$$\begin{aligned}\int_{\mathbf{R}^n} |\nabla(\eta_j \circ f)|^p \, d\mu &\le \int_{\mathbf{R}^n} |Df|^p |\nabla\eta_j \circ f|^p J_f(x)^{1-p/n} dx \\ &\le c\int_{\mathbf{R}^n} |\nabla\eta_j \circ f|^p J_f(x) dx \\ &= c\int_{B(x_j,2r_j)} |\nabla\eta_j|^p \, dy \\ &\le c r_j^{n-p}\,.\end{aligned}$$

Since $\max(\eta_j \circ f) \ge 1$ on E and

$$\sum_{j=1}^{\infty} \int_{\mathbf{R}^n} |\nabla(\eta_j \circ f)|^p \, d\mu \le c\sum_{j=1}^{\infty} r_j^{n-p} < c\varepsilon\,,$$

we have by referring to discussion above that E is of (p,μ)-capacity zero. □

3. New admissible weights from the old ones

In what follows we use the notation that μ_j stands for the measure with density w_j. Also if $B(x,r)$ is a ball, then $\lambda B = B(x, \lambda r)$ for $\lambda > 0$.

We start with a lemma for sums.

3.1. **Lemma.** *Let w_1 and w_2 be p-admissible and let $w = w_1 + w_2$. Suppose further that*

$$\frac{1}{\mu(B)^2}\int_B\int_B |u(x) - u(y)| d\mu_1(x) d\mu_2(y) \le Cr\Big(⨍_{2B} |\nabla u|^p \, d\mu\Big)^{1/p} \tag{3.2}$$

for all Lipschitz functions u and all balls $B = B(z,r)$; here $\mu = \mu_1 + \mu_2$. Then w is p-admissible.

Proof. The doubling property (1.1) for the sum measure μ immediately follows from the corresponding doubling property with weights w_1 and w_2; indeed,

$$\mu(2B) = \mu_1(2B) + \mu_2(2B) \le C_{D1}\mu_1(B) + C_{D2}\mu_2(B) \le C\mu(B).$$

Towards the Poincaré inequality (1.2), let u_B, u_{B1}, and u_{B2} stand for the averages of u over B with respect to measures μ, μ_1, and μ_2, respectively. In light of [7, Theorem 9.5] it suffices to find the estimate

$$⨍_B |u - u_B| d\mu \le Cr\Big(⨍_{2B} |\nabla u|^p \, d\mu\Big)^{1/p}\,,$$

where the constant C is independent of u and B. To reach this, we first observe that

$$
\begin{aligned}
\fint_B |u-u_B|\,d\mu &\le \fint_B \fint_B |u(x)-u(y)|\,d\mu(x)d\mu(y) \\
&\le \Big(\frac{\mu_1(B)}{\mu(B)}\Big)^2 \fint_B \fint_B |u(x)-u(y)|\,d\mu_1(x)d\mu_1(y) \\
&\quad + \frac{2}{\mu(B)^2}\int_B\int_B |u(x)-u(y)|\,d\mu_1(x)d\mu_2(y) \\
&\quad + \Big(\frac{\mu_2(B)}{\mu(B)}\Big)^2 \fint_B \fint_B |u(x)-u(y)|\,d\mu_2(x)d\mu_2(y)\,,
\end{aligned}
\tag{3.3}
$$

Now we use (3.2) to estimate the second term on the right-hand side:

$$
\frac{2}{\mu(B)^2}\int_B\int_B |u(x)-u(y)|\,d\mu_1(x)d\mu_2(y) \le Cr\Big(\fint_{2B}|\nabla u|^p\,d\mu\Big)^{1/p}.
$$

Hence by (3.3) we need only to find an estimate for the terms

$$
\Big(\frac{\mu_j(B)}{\mu(B)}\Big)^2 \fint_B \fint_B |u(x)-u(y)|\,d\mu_j(x)d\mu_j(y)\,, \quad j=1,2.
$$

To this end, we obtain by using the Poincaré inequality that

$$
\begin{aligned}
&\fint_B \fint_B |u(x)-u(y)|\,d\mu_j(x)d\mu_j(y) \\
&\le \fint_B \fint_B |u(x)-u_{Bj}|\,d\mu_j(x)d\mu_j(y) + \fint_B \fint_B |u_{Bj}-u(y)|\,d\mu_j(x)d\mu_j(y) \\
&\le \fint_B |u(x)-u_{Bj}|\,d\mu_j(x) + \fint_B |u_{Bj}-u(y)|\,d\mu_j(y) \\
&\le 2C_{pj}r\Big(\fint_B |\nabla u|^p\,d\mu_j\Big)^{1/p} \le \Big(\frac{\mu(B)}{\mu_j(B)}\Big)^{1/p} 2C_{pj}r\Big(\fint_B |\nabla u|^p\,d\mu\Big)^{1/p} \\
&\le \Big(\frac{\mu(B)}{\mu_j(B)}\Big)^2 2C_{pj}r\Big(\fint_B |\nabla u|^p\,d\mu\Big)^{1/p}\,,
\end{aligned}
$$

where we also used the simple fact that $\mu(B)\ge\mu_j(B)$. This completes the proof. □

3.4. **Lemma.** *Let w_1 be p-admissible. If w_2 is a function with*

$$
\frac{1}{c_0}w_1 \le w_2 \le c_0 w_1
$$

for a constant $c_0>0$, then w_2 is also p-admissible.

Proof. The doubling property (1.1) follows immediately. For the Poincaré one needs to observe that

$$
\begin{aligned}
\fint_B |u-u_{B_2}|\,d\mu_2 &\le c\fint_B |u-u_{B_1}|\,d\mu_2 \le cc_0^2\fint_B |u-u_{B_1}|\,d\mu_1 \\
&\le cc_0^2 C_{P_1} r\Big(\fint_B |\nabla u|^p\,d\mu_1\Big)^{1/p} \le Cr\Big(\fint_B |\nabla u|^p\,d\mu_2\Big)^{1/p}\,,
\end{aligned}
$$

as desired. □

3.5. **Lemma.** *Let w_1 and w_2 be p-admissible. Suppose further that for all balls $B = B(z,r)$*

$$|u(x) - u(y)| \le Cr\Big(⨍_{2B} |\nabla u|^p \, d\mu_1\Big)^{1/p} \quad x, y \in B, \tag{3.6}$$

for all Lipschitz functions u. Then $w = w_1 + w_2$ is p-admissible.

Proof. The claim follows from Lemma 3.1 once we notice that the condition (3.2) follows from the oscillation estimate (3.6). Indeed,

$$\begin{aligned} &\frac{1}{\mu(B)^2}\int_B\int_B |u(x)-u(y)| d\mu_1(x) d\mu_2(y) \\ &\le Cr\frac{\mu_1(B)\mu_2(B)}{\mu(B)^2}\Big(⨍_{2B}|\nabla u|^p\,d\mu_1\Big)^{1/p} \\ &\le Cr\Big(⨍_{2B}|\nabla u|^p\,d\mu\Big)^{1/p}, \end{aligned}$$

since by the doubling property

$$\begin{aligned} &\frac{\mu_1(B)\mu_2(B)}{\mu(B)^2}\Big(\frac{1}{\mu_1(2B)}\Big)^{1/p} \\ &= \Big(\frac{\mu_1(B)}{\mu(B)}\Big)^{1-1/p}\Big(\frac{\mu_1(B)\mu(2B)}{\mu_1(2B)\mu(B)}\Big)^{1/p}\frac{\mu_2(B)}{\mu(B)}\Big(\frac{1}{\mu(2B)}\Big)^{1/p} \\ &\le C\Big(\frac{1}{\mu(2B)}\Big)^{1/p}. \end{aligned}$$

□

3.7. Remark. Condition (3.6) is the Hölder estimate given by the Sobolev embedding theorem if $w_1 = 1$ and $p > n$. Thus $1 + w_2$ and $1 \vee w_2$ are both p-admissible whenever w_2 is p-admissible and $p > n$.

3.8. **Lemma.** *Let w_1 be p_0-admissible. There is $q_0 > 1$ such that for all $p \ge q_0$ the sum $w_1 + w_2$ and the maximum $w_1 \vee w_2$ are p-admissible whenever w_2 is p-admissible.*

Proof. Since any q-admissible weight is p-admissible for all $p \ge q$ [8, Thm. 1.8], it suffices, by Lemmas 3.4 and 3.5, to observe that the Hölder estimate (3.6) holds for some exponent q_0 depending on the doubling constant of w_1; see [6, Thm. 5.1]. □

If $n = 1$, then the class of p-admissible weights coincides with that of A_p-weights [2] and the claim follows because the class of A_p-weights forms a lattice, see Appendix below.

We conclude the proof of Theorem 1.3 by giving counterexamples.

3.9. **Example.** Fix $1 < p < 2$. First let $n = 2$. For a fixed $0 < s < 2 - p$ Lemma 2.3 provides us with a quasiconformal mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ and a set $E_s \subset \mathbf{R}$ so that

$$\dim_H(E_s) \le s \quad \text{and} \quad \dim_H(f(\mathbf{R} \setminus E_s)) \le s\,.$$

Then the weight $w_1 = J_f^{1-p/2}$ is p-admissible [8, Ch. 15] and $\mathbf{R} \setminus E_s$ is of (p, μ_1)-capacity zero (Lemma 2.4); here $\mu_1 = w_1 dx$. Since

$$\dim_H(E_s) \le s < 2 - p,$$

E_s is of (p, dx)-capacity zero (see the argument at the end of the proof of Lemma 2.4).

Now let $w = w_1 \wedge 1$ and $\mu = w dx$. Then w is not p-admissible. If it were, then both E_s and $\mathbf{R} \setminus E_s$ would be of (p, μ)-capacity zero, and consequently, the whole line $\mathbf{R}$ would be of (p, μ)-capacity zero in $\mathbf{R}^2$ by subadditivity. However, at the presence of the Poincaré inequality, the sets of (p, μ)-capacity zero cannot separate the space [8, Lemma 2.46].

A counterexample for $n \geq 3$ follows by lifting the weights above by using Lemma 2.2 and reasoning similarly as above. The details are left to the reader.

4. Appendix

Recall that the Muckenhoupt class A_p, $p > 1$, consists of all locally integrable functions w with $0 < w < \infty$ a.e., for which there is a constant $c_{p,w}$ so that

$$\fint_B w\, dx \leq c_{p,w} (\fint_B w^{1/(1-p)}\, dx)^{1-p},$$

for each ball B. Set

$$A_\infty = \bigcup_{p>1} A_p.$$

Recall that we set

$$\mu(E) = \mu_w(E) = \int_E w\, dx\,,$$

where w is a weight function. Now we have the following characterization [5, Theorem IV.2.11 and Corollary IV.2.13].

4.1. **Proposition.** *The following two conditions are equivalent:*

(1) $w \in A_\infty$.
(2) *The measure $\mu = \mu_w$ is doubling and there exist positive constants C and δ such that, for each ball B and every measurable set $E \subset B$ it holds that*

$$\frac{\mu(E)}{\mu(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta .$$

We also employ the following characterization from [5, Theorem IV.2.17].

4.2. **Proposition.** *The following two conditions are equivalent:*

(1) $w \in A_p$.
(2) $w \in A_\infty$ *and* $w^{-1/(p-1)} \in A_\infty$.

The above two propositions allow us to verify the lattice property for A_p.

4.3. **Proposition.** *The class of A_p-weights is a lattice.*

Proof. Suppose that $w_j \in A_p$, $j = 1, 2$. We first prove that $w_1 \vee w_2 \in A_p$. By the definition of A_p there exist positive constants $c_{p,j}$, $j = 1, 2$, with

$$\begin{aligned}
\fint_B w_1 \vee w_2 \, dx &\le \fint_B w_1 \, dx + \fint_B w_2 \, dx \\
&\le c_{p,1}\Big(\fint_B w_1^{1/(1-p)} \, dx\Big)^{1-p} + c_{p,2}\Big(\fint_B w_2^{1/(1-p)} \, dx\Big)^{1-p} \\
&\le c_{p,12}\Big(\fint_B (w_1 \vee w_2)^{1/(1-p)} \, dx\Big)^{1-p},
\end{aligned}$$

whenever B is a ball; here we wrote $c_{p,12} = 2 \max\{c_{p,1}, c_{p,2}\}$. Hence

$$w_1 \vee w_2 \in A_p.$$

Next we prove that $w_1 \wedge w_2 \in A_p$. To this end we intend to use Proposition 4.2 and thus show that $w_1 \wedge w_2 \in A_\infty$ and $(w_1 \wedge w_2)^{-1/(p-1)} \in A_\infty$.

To establish the first, we verify condition (2) of Proposition 4.1 for $w_1 \wedge w_2$. To this end, we first observe that $w_1 \wedge w_2$ defines a doubling measure $\mu = w_1 \wedge w_2 \, dx$. Indeed, write

$$\mu_j(E) = \int_E w_j \, dx$$

and, for a fixed ball B choose a set $A \subset B$ with $|A| \ge \frac{1}{2}|B|$ and that $w_1 \wedge w_2 = w_j$ on A, say $w_1 \wedge w_2 = w_1$ on A. Then

$$\mu_1(A) \ge c\Big(\frac{|A|}{|B|}\Big)^p \mu_1(B) \ge c2^{-p}\mu_1(B)$$

by the strong doubling property [8, 15.5] of A_p-weights. Consequently, μ is doubling, since

$$\mu(2B) \le \mu_1(2B) \le c\mu_1(B) \le c\mu_1(A) = c\mu(A) \le c\mu(B).$$

Next, let C_j and δ_j be the positive constants associated to weights w_j, given us by Proposition 4.1. Set $\delta = \min\{\delta_1, \delta_2\}$, $C = C_1 + C_2$ and observe that by the previous estimation

$$\mu(B) \ge c \min(\mu_1(B), \mu_2(B)).$$

Therefore, for all measurable $E \subset B$

$$c\frac{\mu(E)}{\mu(B)} \le \frac{\mu_1(E)}{\mu_1(B)} + \frac{\mu_2(E)}{\mu_2(B)} \le C_1 \left(\frac{|E|}{|B|}\right)^{\delta_1} + C_2 \left(\frac{|E|}{|B|}\right)^{\delta_2} \le C \left(\frac{|E|}{|B|}\right)^{\delta}.$$

Hence by Proposition 4.1, we have that $w_1 \wedge w_2 \in A_\infty$.

To complete the proof, we verify the second condition part (2) of Proposition 4.2. To this end, write

$$w_j' = w_j^{-1/(p-1)}, \quad j = 1, 2.$$

Since $w_j' \in A_\infty$, there exist $p_j > 1$ with $w_j' \in A_{p_j}$. Set $q = \max\{p_1, p_2\}$. Then $w_j' \in A_q$ since $A_{p_j} \subset A_q$ (cf. [8, p. 298]).

By the first part of this proof

$$w_1' \vee w_2' \in A_q \subset A_\infty.$$

Since

$$(w_1 \wedge w_2)^{-1/(p-1)} = w_1' \vee w_2',$$

we infer that

$$(w_1 \wedge w_2)^{-1/(p-1)} \in A_\infty.$$

Hence Proposition 4.2 ensures us that $w_1 \wedge w_2 \in A_p$, as desired. $\square$

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