Riemannian geometry
Exercises 5, 1.12.2015

1. Let M be a Riemannian manifold. Let $p \in M$ let q be a point in a normal ball neighbourhood of p. Show that the radial geodesic from p to q is the unique minimizing path up to renormalization.

2. Let M be a Riemannian manifold. Let $p, q \in M$. Show that if $\delta > 0$ is small enough, there is a point $p_0 \in \partial B(p, \delta)$ such that

$$d(p, p_0) + d(p_0, q) = d(p, q) .$$

3. Let γ and η be two geodesics on a complete Riemannian manifold M such that for some $a < b$ we have $\gamma(a) = \eta(a)$, $\gamma(b) = \eta(b)$ and $\ell(\gamma|_{[a,b]}) = \ell(\eta|_{[a,b]})$. Show that $\gamma|_{[a,b+\epsilon]}$ is not minimizing for any $\epsilon > 0$.

Let M be a complete noncompact Riemannian manifold. We say that a mapping $\rho\colon [0, \infty[\to M$ is a *minimizing ray* if $\rho_{[0,T]}$, $T > 0$ is a minimizing geodesic for all $T > 0$.

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4. Let M be a complete noncompact Riemannian manifold. Show that for each $p \in M$ there is a minimizing ray ρ such that $\rho(0) = p$.

5. Show that the complex exponential map $\mathrm{Exp}\colon \mathbb{C} \to \mathbb{C} - \{0\}$,

$$\mathrm{Exp}(z) = \sum_{k=0\infty} \frac{z^k}{k!} = e^{\mathrm{Re}\, z}(\cos(\mathrm{Im}\, z) + i \sin(\mathrm{Im}\, z))$$

is a Riemannian local isometry, when $\mathbb{C}$ has the Euclidean Riemannian metric and in $\mathbb{C}$ we use the Riemannian metric

$$g = \frac{dx^2 + dy^2}{x^2 + y^2} .$$

6. Show that the Riemannian manifold $\left(\mathbb{R}^2 - \{0\}, \frac{dx^2+dy^2}{x^2+y^2}\right)$ is complete.

7. Describe the minimizing rays in $\left(\mathbb{R}^2 - \{0\}, \frac{dx^2+dy^2}{x^2+y^2}\right)$.