Riemannian geometry
Exercises 2, 17.11.2015

1. Check the properties ($\nabla 3$) and ($\nabla 4$) of an affine connection and the compatibility of the connection with the metric in the proof of Theorem 4.2.

2. Determine the Levi-Civita connection in of the Euclidean plane in polar coordinates in terms of Christoffel symbols.

3. Let $\bar{\nabla}\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ be a mapping that has the properties ($\nabla 1$) – –($\nabla 3$) of an affine connection. Let $T\colon \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ be a mapping defined by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]\,.$$

Show that T is a tensor (in the generalized sense). T is called the *torsion tensor* of $\bar{\nabla}$.

4. Determine the parallel transport in the hyperbolic plane $\mathbb{H}^2$along the horizontal path $\gamma\colon \mathbb{R} \to \mathbb{H}^2$, $\gamma(t) = (t,1)$. How does the parallel transport map the tangent vector $\dot{\gamma}(0)$ of γ?

5. Compare parallel transport along the paths $\gamma_0\colon\,]-\pi,\pi[\to \mathbb{R}^2$,

$$\gamma_0(t) = (\tan(t/2), 0)$$

and $\gamma_1\colon \mathbb{R} \to \mathbb{R}^2$,

$$\gamma_1(t) = (\cos t, \sin t)$$

in the plane $\mathbb{R}^2$, with the Euclidean Riemannian metric g_{E} and with the spherical metric $g_S = \frac{4}{(1+x_1^2+x_2^2)^2}\, g_{\mathrm{E}}$ that was discussed in Example 3.2.

6. Determine the parallel transport along the following triangles in $\mathbb{S}^2$: Start at the South Pole. Move to the Equator along the Greenwich meridian. Move along the Equator to the east a distance ϕ. Then return to the South Pole along the meridian.