Number theory 2 2024

Exercises 4

1. Find a solution to the inequality

$$\left|\sqrt{3} - \frac{p}{q}\right| < \frac{1}{q^2}$$

that satisfies $|q\sqrt{3} - p| < \frac{1}{9}$.[1]

2. Let $\frac{a}{b} \in \mathbb{Q}$. Prove that the inequality

$$\left|\frac{a}{b} - \frac{p}{q}\right| < \frac{1}{q^2} \tag{1}$$

has only finitely many solutions.

The *discriminant* of a polynomial $P(X) = aX^2 + bX + c$ of degree 2 is

$$\mathrm{Disc}(P(X)) = b^2 - 4ac\,.$$

3. Let $a, b, c \in \mathbb{R}$ with $a \neq 0$, and let α and α' be the roots of the polynomial $P(X) = aX^2 + bX + c$. Prove that

$$\mathrm{Disc}(P(X)) = a^2(\alpha - \alpha')^2\,.$$

Let $F_0 = 0$ and $F_1 = 1$ and set for all $n \geq 2$

$$F_n = F_{n-1} + F_{n-2}\,.$$

The sequence $(F_n)_{n\in\mathbb{N}}$ is the *Fibonacci sequence*.

The roots of the polynomial $P(X) = X^2 - X - 1$ are the *golden ratio* $\varphi = \frac{1+\sqrt{5}}{2}$ and $\widehat{\varphi} = \frac{1-\sqrt{5}}{2}$.

4. Prove[2] that

$$F_n = \frac{\varphi^n - \widehat{\varphi}^n}{\sqrt{5}}$$

for all $n \in \mathbb{N}$.

5. Prove that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi\,.$$

6. Prove that

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$$

for all $n \in \mathbb{N}$.

[1] Example 10.3.
[2] Induction.

7. What do the previous exercises tell about how well the golden ratio φ is approximated by the sequence of rational numbers $(\frac{F_{n+1}}{F_n})_{n\in\mathbb{N}}$?