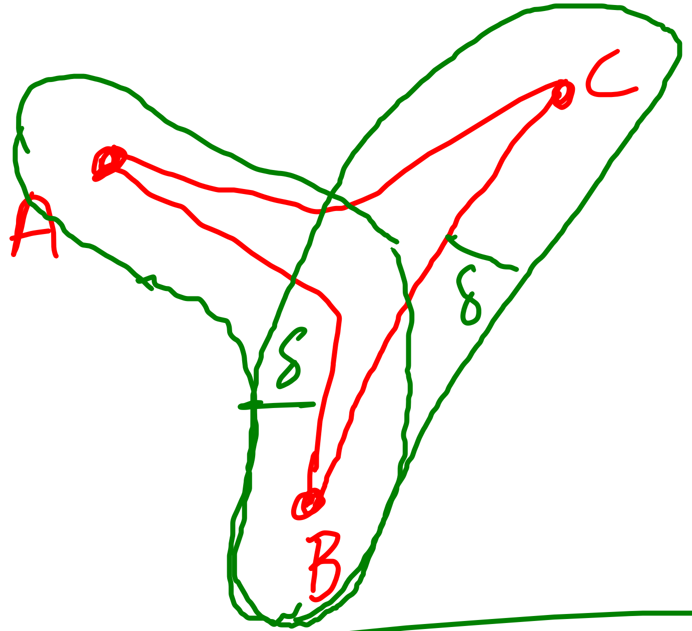
Neg. curved spaces 8.10.2020

Gromov-hyperbolic spaces

Defn X geodesic metric space. A triangle $\Delta \subset X$ satisfies the **Rips condition** w/ const. $\delta > 0$ if any side of Δ is contained in the closed δ-nbhd of the other two sides.

(Δ is δ-slim)

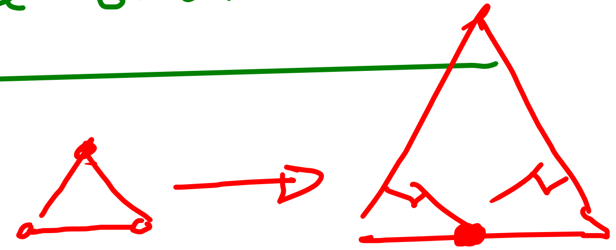


X is **δ-hyperbolic** if all triangles in X satisfy the Rips condition w/ constant δ.

X is Gromov-hyperbolic if it is δ-hyp for some $\delta > 0$.

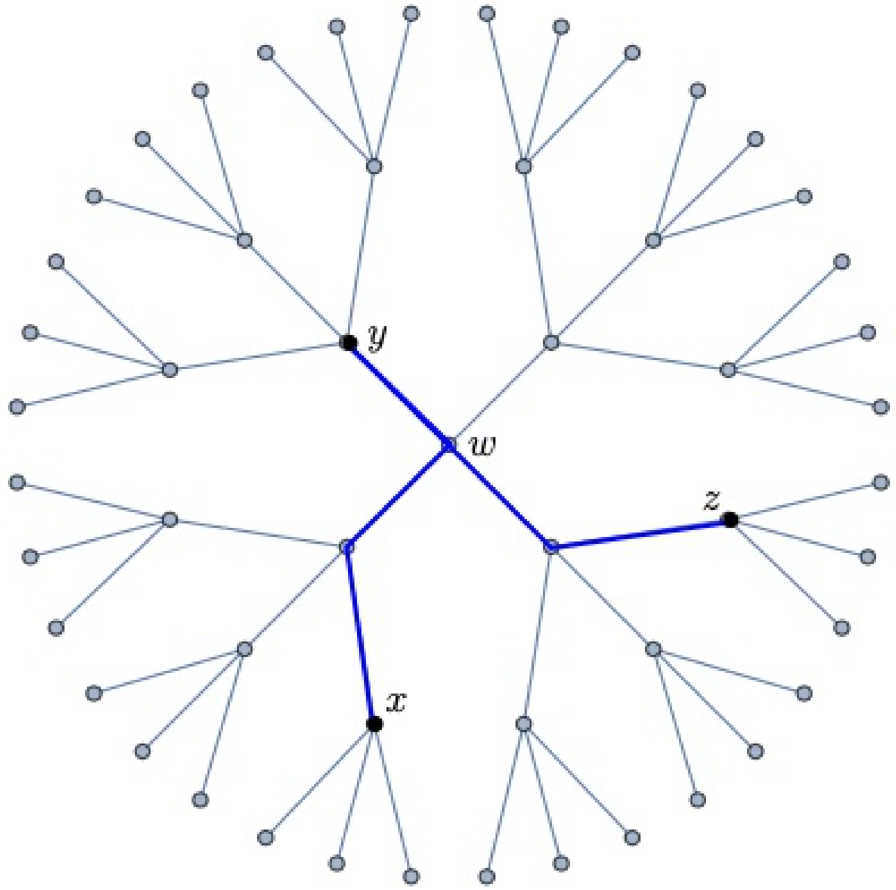
Ex. 1) $\boxed{\mathbb{H}^2 \quad (\mathbb{H}^n) \quad \text{today}}$

① 2) $\mathbb{E}^2$ ($\mathbb{E}^n$, $n \geq 2$) is not δ-hyp for any δ.



3) bounded geod. metric spaces are Gromov-hyp. Not INTERESTING.

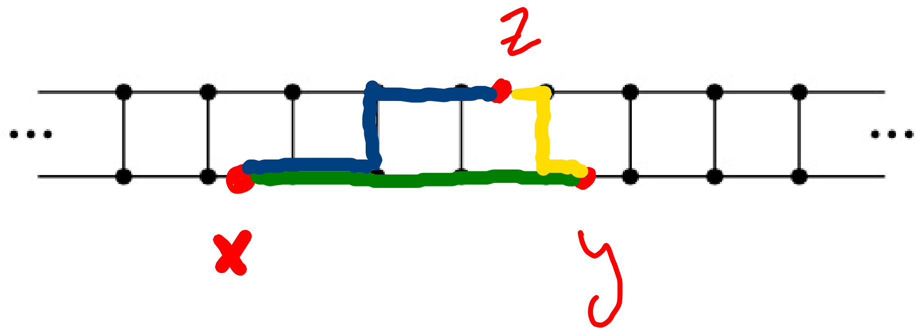
4)



trees are δ-hyp $\forall\ \delta > 0$

$\rightsquigarrow$ 0-hyp.

5)



Gromov-hyperbolic.

Prop. 6.1. $\mathbb{H}^n$ is $\log(1+\sqrt{2})$-hyperbolic.

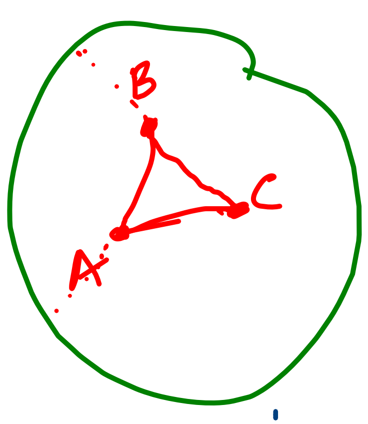
Warning: many different def$^{\underline{n}}$s of δ-hyperbolic $\rightsquigarrow$ equivalent def$^{\underline{n}}$s of Gromov-hyp

Proof. Let $\Delta \subset \mathbb{H}^n$. Prop. 4.29 $\rightsquigarrow$ can take $n=2$.

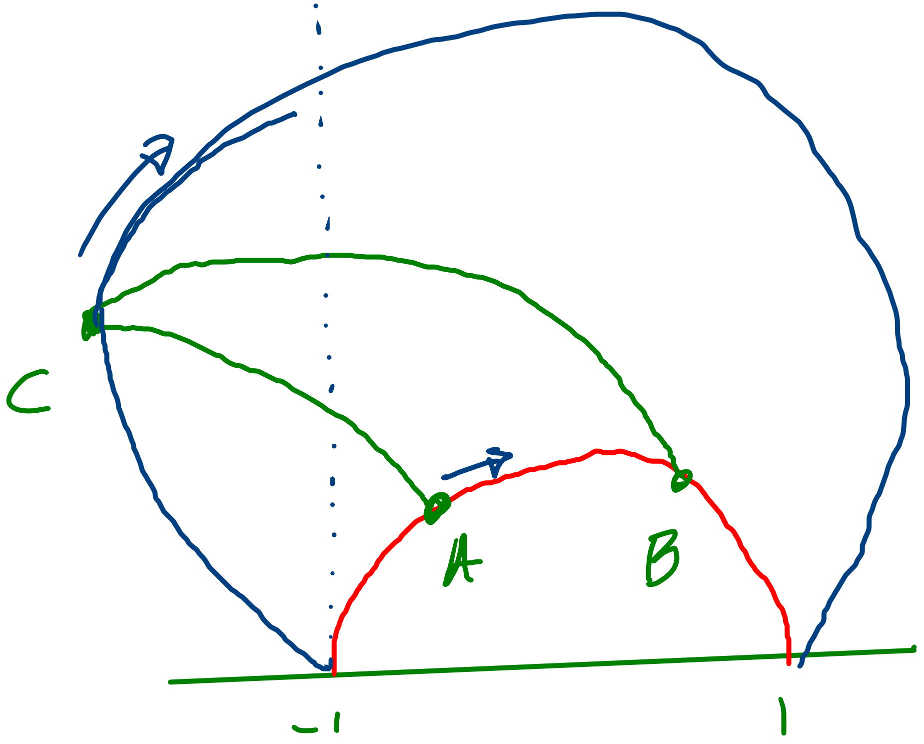
$$\Delta \subset H \subset \mathbb{H}^n$$

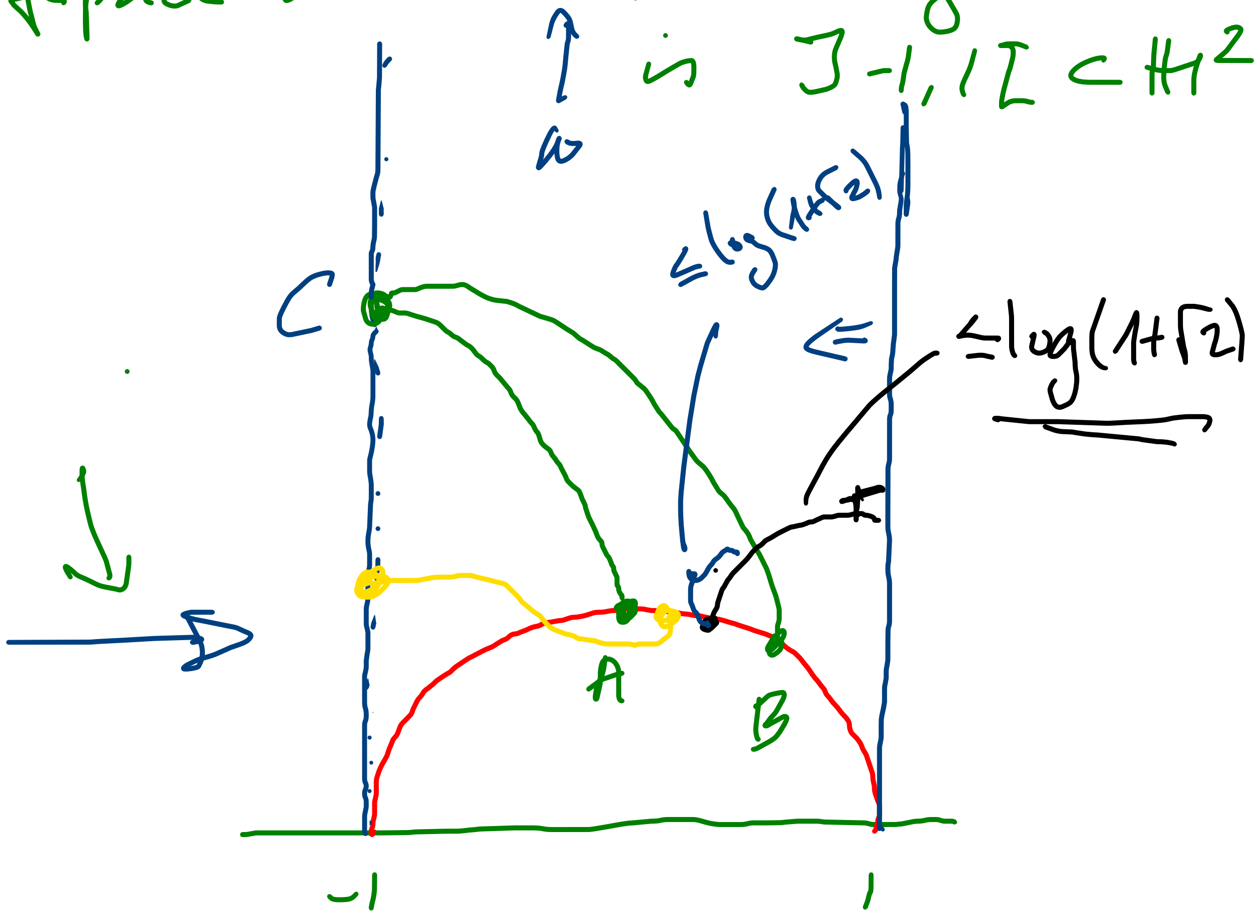
H: isom. embedded copy of $\mathbb{H}^2$.

$\mathbb{H}^2$



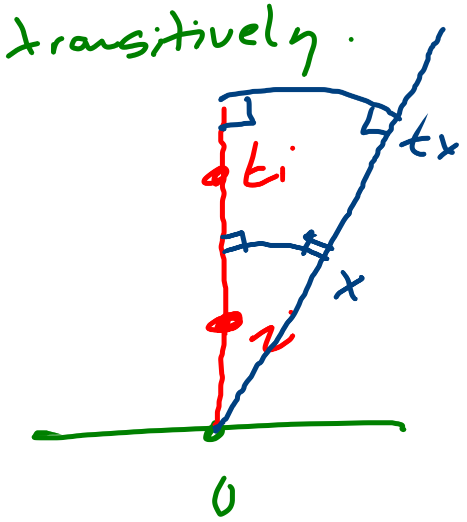
use Upper halfspace model. Assume geod. line containing $[A,B]$ is $]-1,1[\subset \mathbb{H}^2$



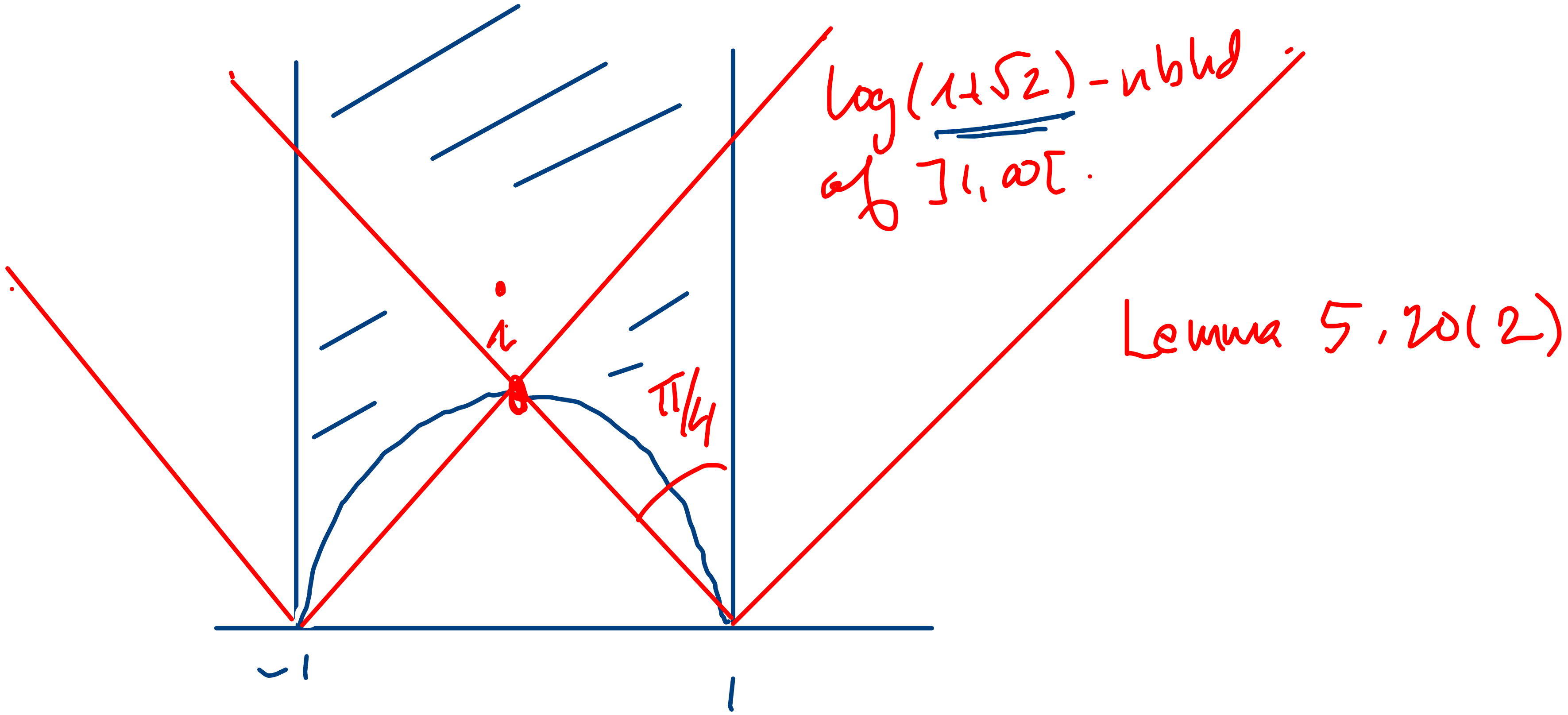


$t>0$

$z \mapsto tz$ isometry fixing $0,\infty$ moving pts on $]0,\infty[\subset \mathbb{H}$ transitively.



(3)



The geodesic lines $]-1,1[$, $]-1,\infty[$, $]1,\infty[$ bound a convex polygon in $\mathbb{H}^2$.

Recall from ch. 4: A hyperplane $\underline{\underline{\mathbb{H}^n}}$ is the bisector of two pts $p,q \in \mathbb{H}^n$.

Prop. 5.23. Hyperplanes in the UHS model are intersections of the UHS with Eucl. hyperplanes orthogonal to $\partial_\infty \mathbb{H}^n - \{\infty\} = \mathbb{E}^{n-1} \times \{0\}$ or with spheres centered in $\mathbb{E}^{n-1} \times \{0\}$.

4 Proof. $p,q \in$ UHS $\quad d(x,p) = d(x,q) \Leftrightarrow \cancel{1+}\dfrac{\|x-p\|^2}{\cancel{x_n}\, p_n} = \cancel{1+}\dfrac{\|x-q\|^2}{\cancel{x_n}\, q_n}$ solve this! $\square$

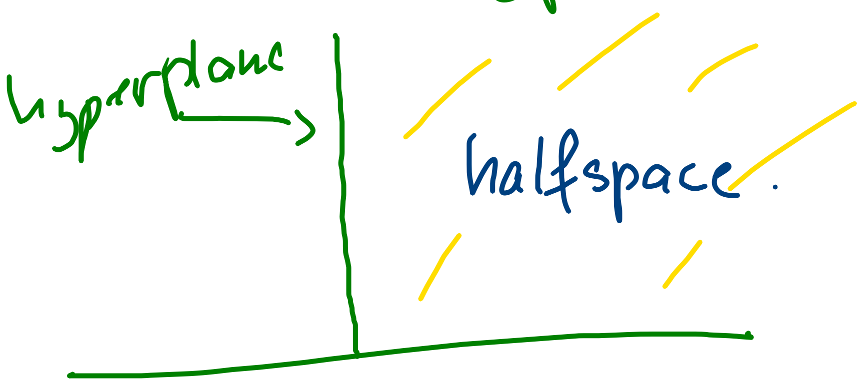
→ the complement of a hyperplane has 2 $\underbrace{\text{open components}}_{\text{open half-spaces}}$.

half-space ∪ hyperplane = closed halfspace

Def$^{\underline{n}}$ Let X be uniquely geodesic. $A \subset X$ ($\neq \emptyset$) is **convex** if $\forall\, a, b \in A$ $\quad [a,b] \subset A$.

Lemma 5.21 Closed and open halfspaces are convex.

Pf. Prove this first for the case where the hyperplane is a Eucl. hyperplane. → Exercise.



Def$^{\underline{n}}$ $(H_i)_{i \in I}$ $\left\{ \begin{array}{l} \text{loc. finite} \\ \text{finite} \end{array} \right\}$ collection of halfspaces in $\mathbb{H}^n$.

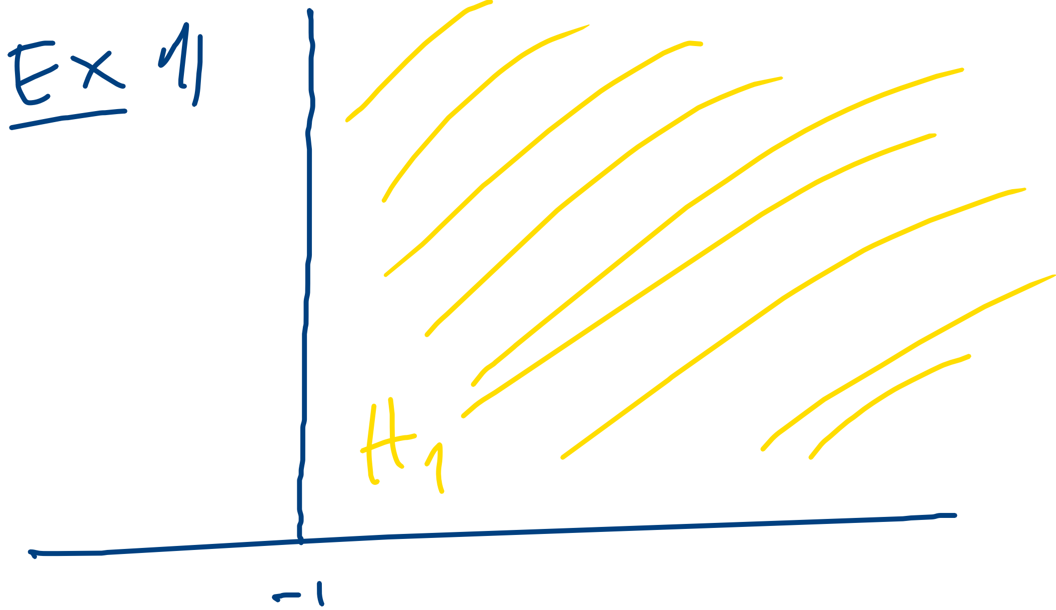
if $K \subset \mathbb{H}^n$ compact $\#\{i \in I : \partial H_i \cap K\} < \infty$

$\bigcap_{i \in I} H_i$ is a polytope (or empty)

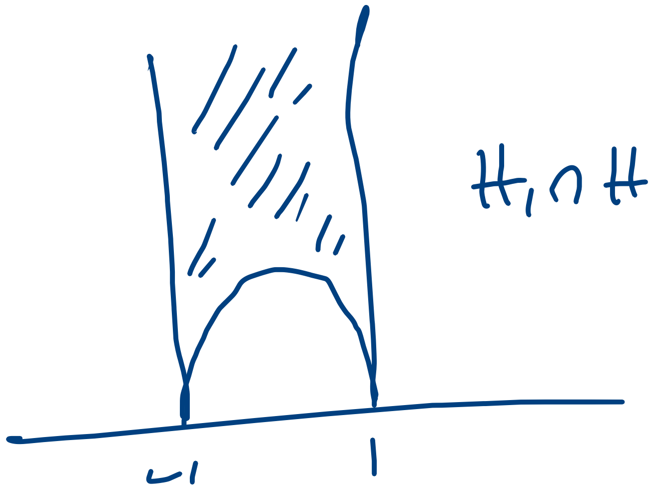
polyhedron if $n=3$

polygon if $n=2$

Ex 1)



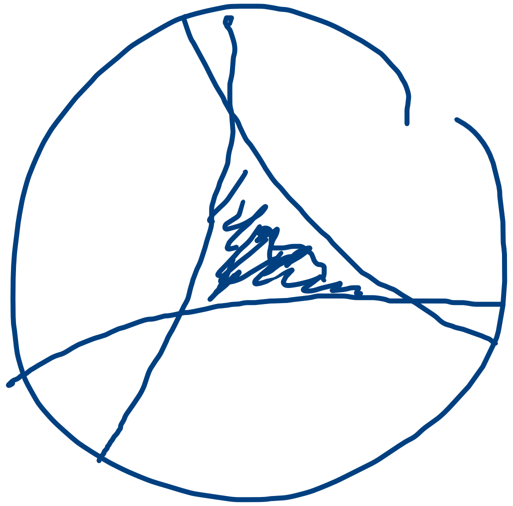
halfspace



$H_1 \cap H_2 \cap H_3$

2)

3)



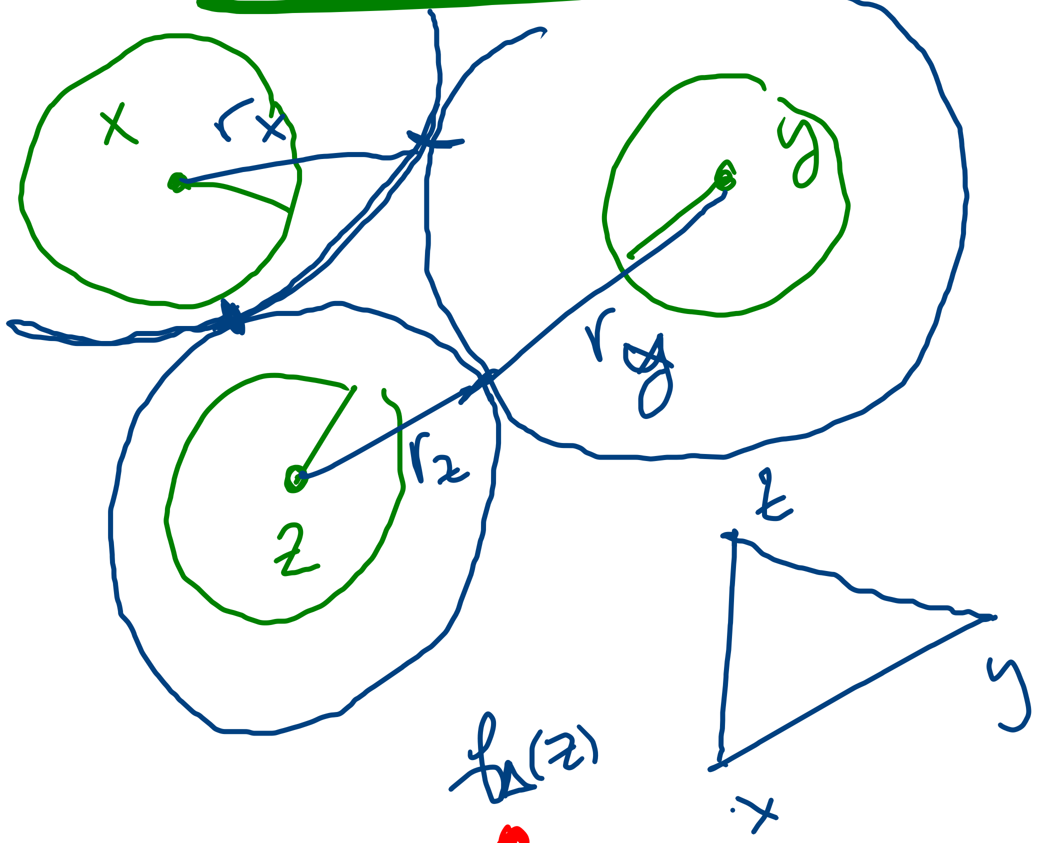
Lemma 5.25 If $K_i \subset \mathbb{R}^n$, $i \in I$ are convex, then $\bigcap_{i \in I} K_i$ is convex (or $\emptyset$)

hedra

6 $\Rightarrow$ Polytopes are convex.

gons

Gromov product



$x, y, z \in X$ X metric space.

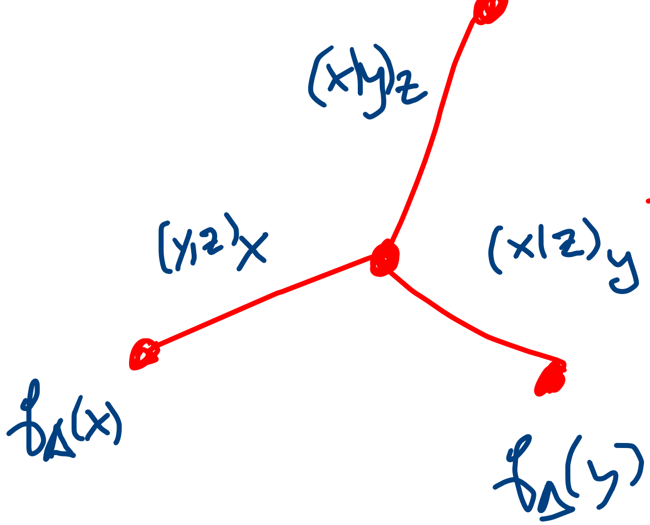
$$\begin{cases} r_x + r_y = d(x,y) \\ r_x + r_z = d(x,z) \\ r_y + r_z = d(y,z) \end{cases}$$

Gromov product of y and z wrt x

$$r_x = \boxed{\tfrac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right) = (y|z)_x}$$

$$r_y = \tfrac{1}{2}\left(d(y,x) + d(y,z) - d(x,z)\right) = (x|z)_y$$

$$r_z = \tfrac{1}{2}\left(d(z,x) + d(z,y) - d(x,y)\right) = (x|y)_z$$



<u>Lemma</u> Δ a triangle with vertices $x, y, z \in X$, (X geod. metric space). Let T_Δ be a tripod w/ edge lengths $(x|y)_z$, $(x|z)_y$, $(y|z)_x$. $\exists\, f_\Delta : \Delta \to T_\Delta$ s.t. $f_\Delta|_{[w_1,w_2]}$ is an isometry $\forall\, w_1, w_2 \in \{x, y, z\}$.

Lemma 6.4 X geod. metric space, $\Delta \subset X$ triangle w/ vertices x, y, z.

$$\boxed{(y|z)_x \leq d(x, [y,z])}$$

Def^n a triangle is **δ-thin** if $d(a,b) \leq \delta$ $\forall\, b \in f_\Delta^{-1}(a)$ $\forall\, a \in \Delta$.

Lemma 6.5. If Δ is δ-thin, then

$$(y|z)_x \leq d(x, [y,z]) \leq (y|z) + \delta$$

Next week: δ-thin $\Rightarrow$ Rips condition w/ δ

Rips condition w/ δ $\Rightarrow$ 4δ-thin