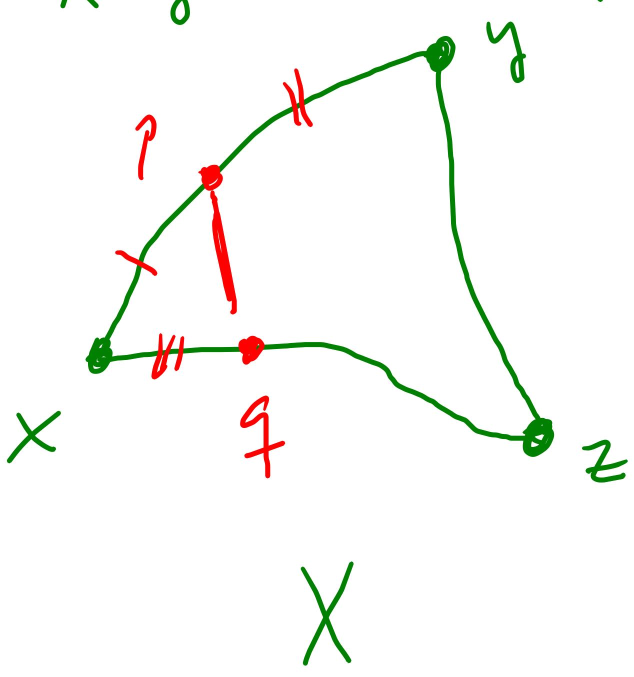
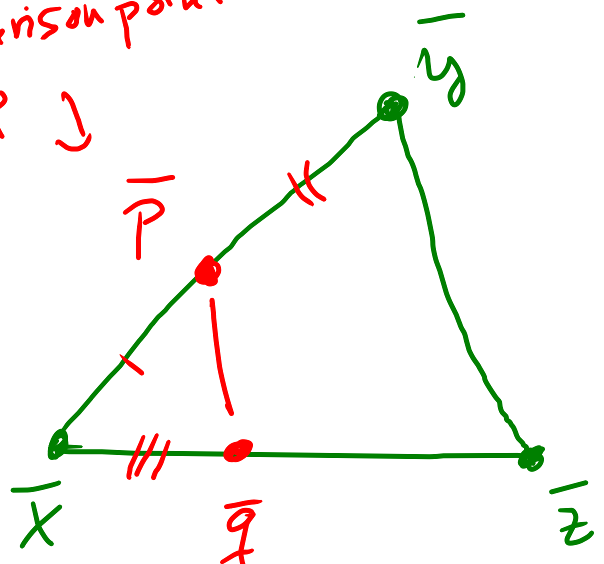
Neg. curved spaces

X geod. metric space, Δ a triangle in X, $\overline{X}_\varkappa = \begin{cases} \mathbb{E}^2 & \text{if } \varkappa = 0 \\ \mathbb{H}^2_\varkappa & \text{if } \varkappa < 0 \end{cases}$

uniquely geod.



comparison point of P



Comparison triangle of Δ.

$\exists\, \bar{x}, \bar{y}, \bar{z} \in \overline{X}_\varkappa$:

$d(\bar{x}, \bar{y}) = d(x, y)$

$d(\bar{y}, \bar{z}) = d(y, z)$

$d(\bar{z}, \bar{x}) = d(z, x)$.

X is a CAT($\varkappa$)-space if $d(p,q) \le d(\bar{p}, \bar{q})$ for all p, q in Δ for all triangles Δ in X.

<u>Yesterday</u>: If $\varkappa < 0$ and X is CAT($\varkappa$), then X is Gromov-hyperbolic.

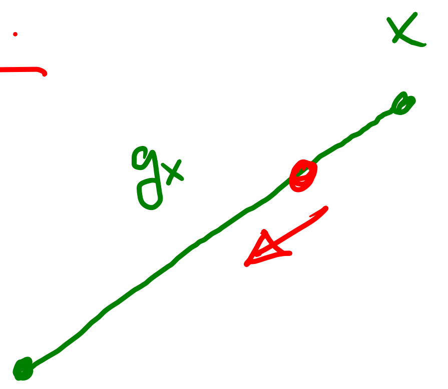
Prop. 10.12. CAT(0)-spaces are contractible.

$\Rightarrow$ CAT(−1) — " —

If $\varkappa_1 \le \varkappa_2$ and X in CAT($\varkappa_1$), then X in CAT($\varkappa_2$). Prop. 10.8

$\rightarrow$ CAT(−1)-spaces are CAT(0)-spaces

Proof.



$x_0 \in X$

Prop. 10.9: CAT(0)-spaces are uniquely geodesic.

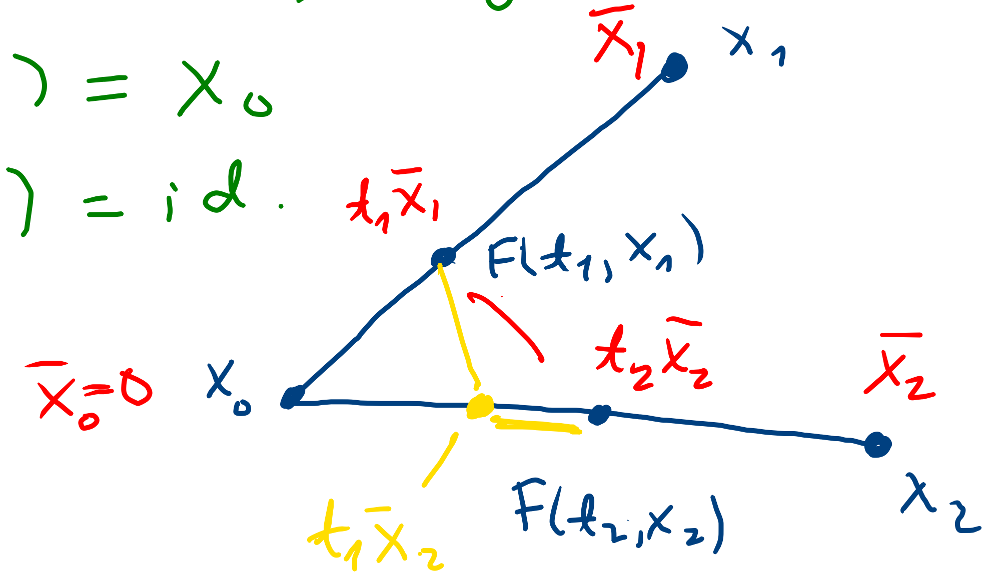
$\forall\ x \in X - \{x_0\}\ \exists_1$ geod. segment $g_x : [0, d(x, x_0)] \to X$.

s.t. $g_x(0) = x_0$, $g_x(d(x, x_0)) = x$

$F : [0,1] \times X \to X$, $F(t, x) = g_x(t\, d(x, x_0))$

$F(0, x) = x_0$ $\quad F(0, \cdot) = x_0$

$F(1, x) = x$ $\quad F(1, \cdot) = id.$



F is continuous:

comparison

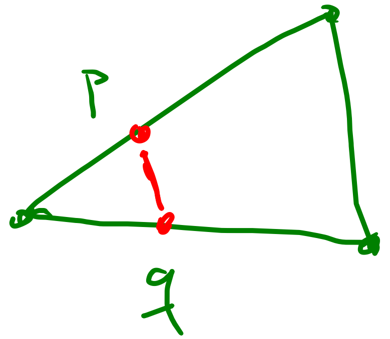
$$d(F(t_1, x_1), F(t_2, x_2)) \le \| t_1\bar{x}_1 - t_2\bar{x}_2 \|$$

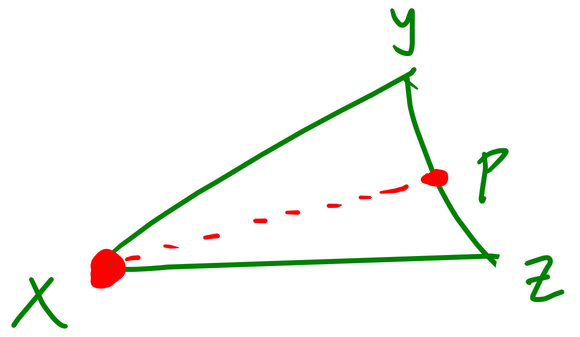
$$\le \| t_1\bar{x}_1 - t_1\bar{x}_2 \| + \| t_1\bar{x}_2 - t_2\bar{x}_2 \|$$

$$= t_1 \| \bar{x}_1 - \bar{x}_2 \| + |t_1 - t_2| \|\bar{x}_2\| = t_1\, d(x_1, x_2) + |t_1 - t_2|\, d(x_0, x_2)$$

continuity ok. $\square$

Prop. 10.3. $\kappa \leq 0$. X is CAT(κ) $\iff$ $\forall\, x, y, z \in X$ $\quad d(x,p) \leq d(\bar{x}, \bar{p})$ $\forall\, p \in [y,z]$

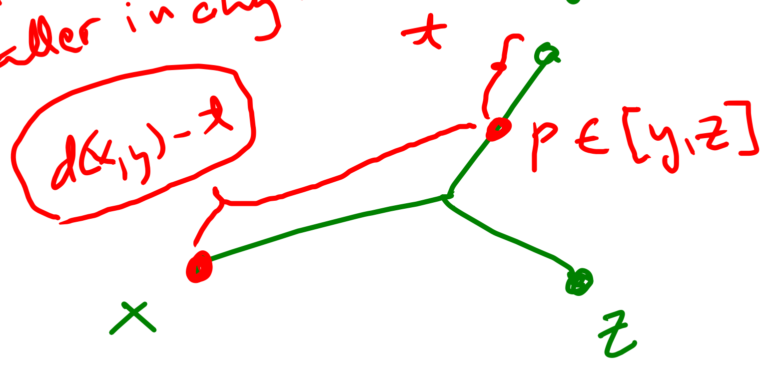




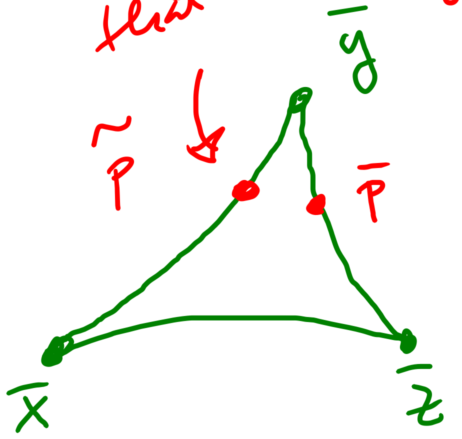
Proof Exercise.

Ex. Trees are CAT(κ) for all κ.

this can't be smaller in any metric space.



that's the only pt with



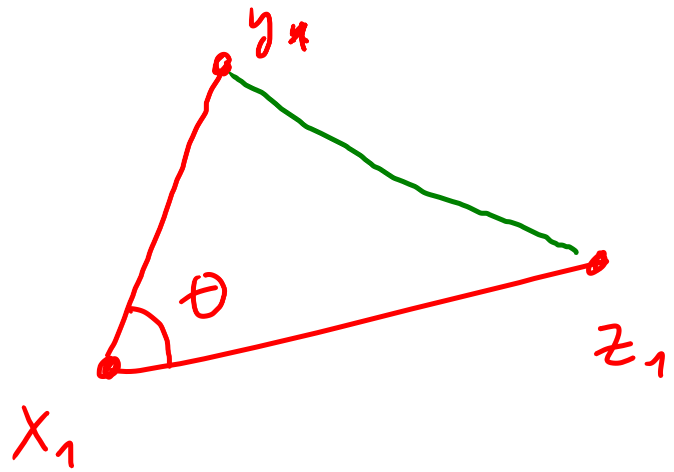
$d(\bar{x}, \tilde{p}) = d(x,p)$

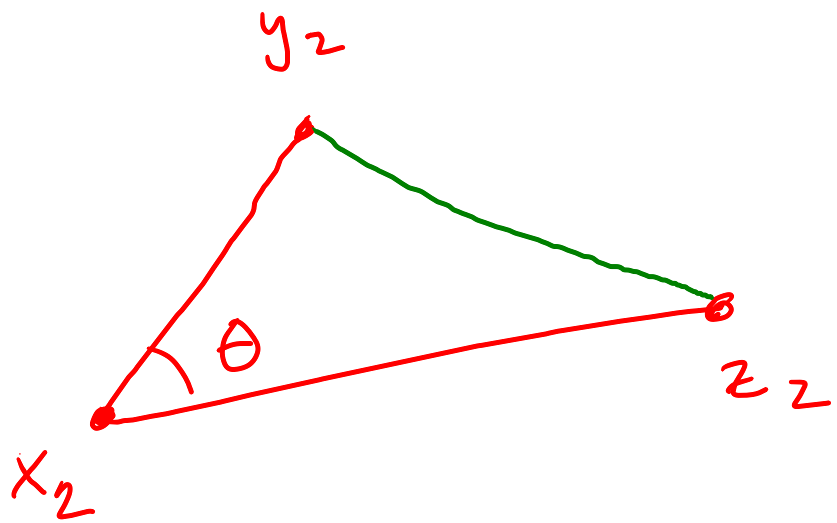
and $d(\bar{y}, \tilde{p}) = d(y,p)$

$\Rightarrow d(\bar{x}, \bar{p}) > d(x,p)$.

X tree $\qquad \overline{X}_\kappa$

Lemma 10.5 $\kappa_1 \le \kappa_2 \le 0$.





$x_1, y_1, z_1 \in \overline{X}_{\kappa_1}$ $\quad$ $x_2, y_2, z_2 \in \overline{X}_{\kappa_2}$

$\measuredangle_{x_1}(y_1, z_1) = \measuredangle_{x_2}(y_2, z_2)$, $\quad d(x_1, y_1) = d(x_2, y_2)$

$d(x_1, z_1) = d(x_2, z_2)$

$\Rightarrow \boxed{d(y_1, z_1) \ge d(y_2, z_2)}$.

Proof. Use polar coordinates centered at x_1 and x_2

Riem metric $ds^2 = dr^2 + \frac{1}{\sqrt{-\kappa}} \sinh^2(\sqrt{-\kappa}\, r)\, d\theta^2$ in $\mathbb{H}_\kappa$

$= dr^2 + r^2\, d\theta^2$ in $\mathbb{E}^2$

$f(\kappa, r)$

$\kappa \mapsto f(\kappa, r)$ strictly decreasing on $]-\infty, 0]$ and continuous.

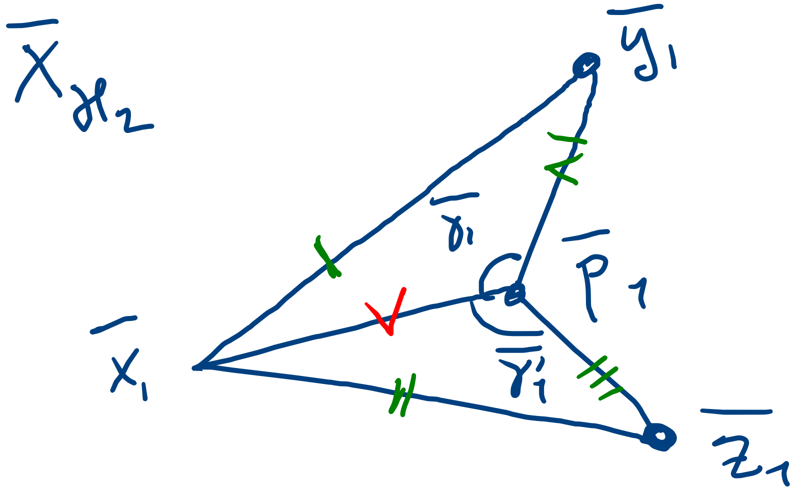
The ~~mapping~~ differential of $(r, \theta) \longmapsto (r, \theta)$

$\overline{X}_{\kappa_1}$ $\quad$ $\overline{X}_{\kappa_2}$

decreases the lengths of tang. vectors that are not radial

$\to$ lengths of non-radial paths shrink. ∎

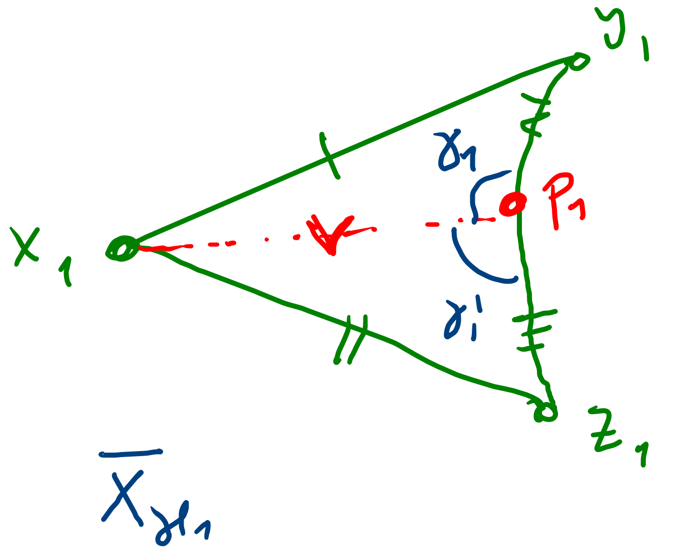
⇒ **Lemma 10.6.** If $d(x_1, y_1) = d(x_2, y_2)$, $d(y_1, z_1) = d(y_2, z_2)$, $d(z_1, x_1) = d(z_2, x_2)$ $\Rightarrow \measuredangle_{x_1}(y_1, z_1) \leq \measuredangle_{x_2}(y_2, z_2)$. $\square$



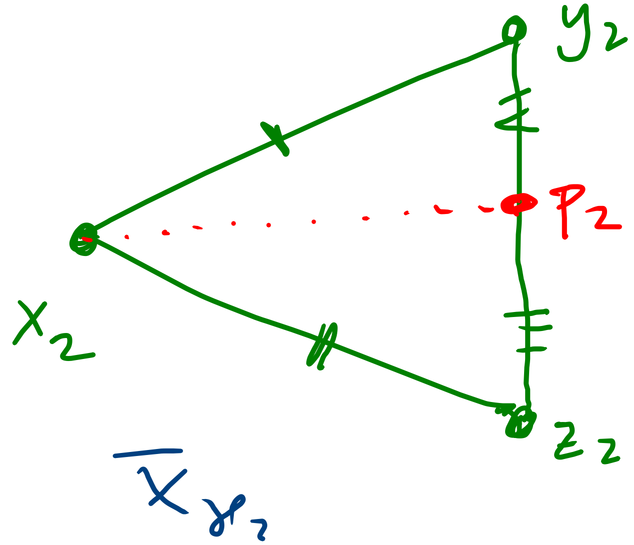
L.10.6 ⇒ $\bar\gamma_1 \geq \gamma_1$, $\bar\gamma_1' \geq \gamma_1'$

Lemma 10.7. $\kappa_1 \leq \kappa_2 \leq 0$. $\bar X_{\kappa_1}$ is a CAT(κ_2)-space.

Proof. Let $x_1, y_1, z_1 \in \bar X_{\kappa_1}$, $x_2, y_2, z_2 \in \bar X_{\kappa_2}$ be comp. pts. $p_1 \in [y_1, z_1]$, p_2 comp. pt.



$\gamma_1 + \gamma_1' = \pi$



Aleksandrov's lemma

$\Rightarrow d(\bar x_1, \bar p_1) \leq d(x_2, p_2)$

$d(\bar x_1, \bar p_1) = d(x_1, p_1)$

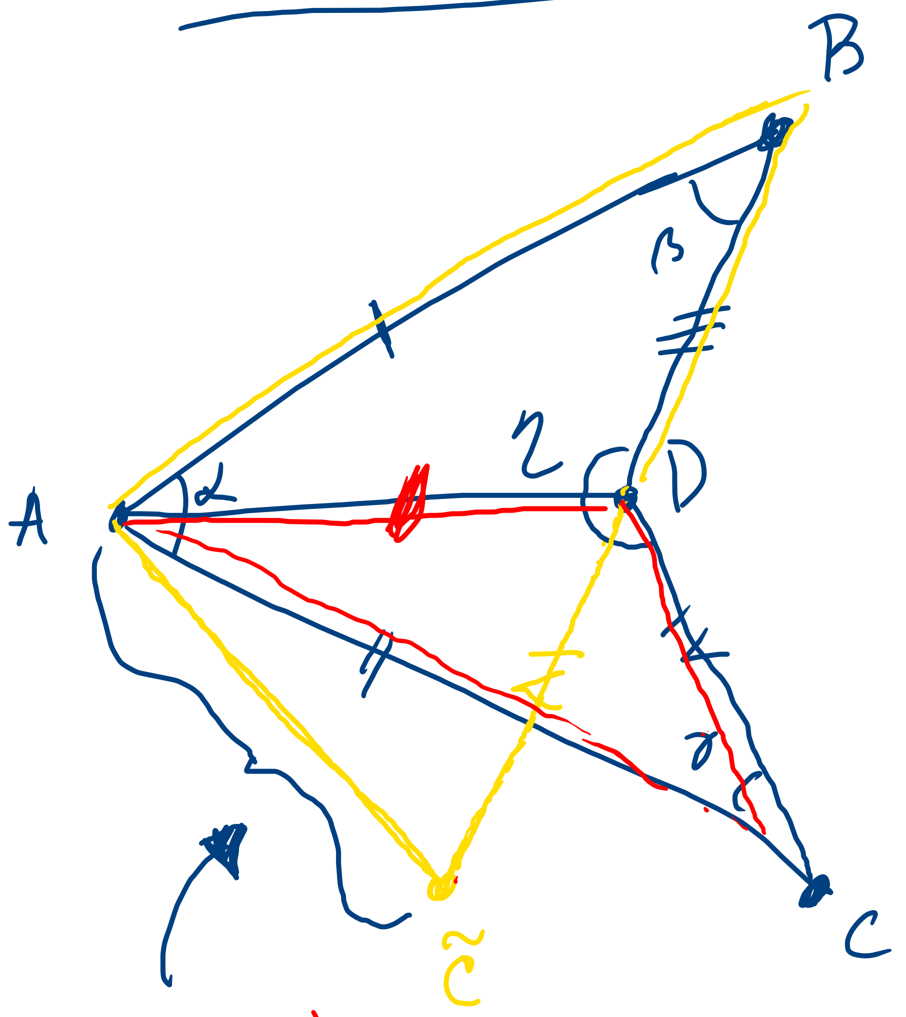
5

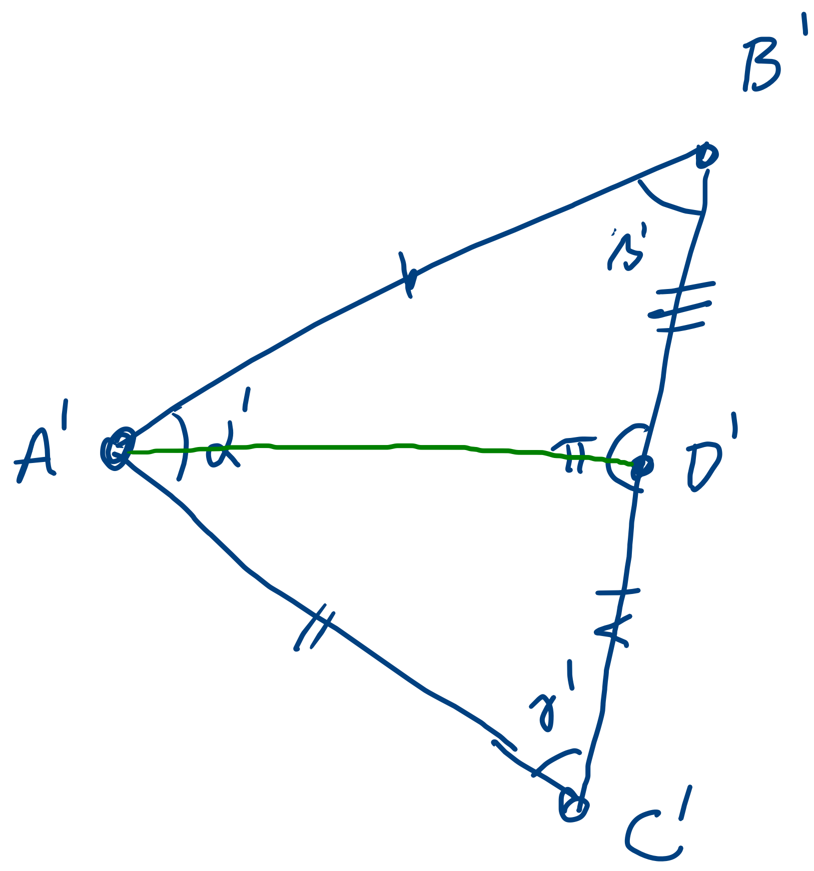
Aleksandrov's lemma

In $\overline{X}_{\varkappa}$, $\varkappa \le 0$.

$d(A,B) = d(A',B')$
$d(A,C) = d(A',C')$
$d(B,D) + d(D,C) = d(B',C')$
$d(B,D) = d(B',D')$. $\eta \geq \pi$

Then $d(A',D') \geq d(A,D)$.





$d(A,\tilde{C}) \le d(A,C) = d(A',C')$

by cos law

triangles with vertices $A, B, \tilde{C}$ and A', B', C' $\longrightarrow$ $\beta' \geq \beta$.

triangles with vertices A, B, D and with vertices A', B', D' $\Rightarrow$ $d(A,D) \le d(A',D')$.

$\square$