Neg. curved spaces 19.11.2020

Topology on $X \cup \partial_\infty X$ when X is a proper Gromov-hyp. space.

(closed balls are compact ↓ proper)

Comes from the top. of compact convergence in $\check{\mathcal{G}}_+(X, x_0)$ for a basept $x_0 \in X$.

$E : \check{\mathcal{G}}_+(X, x_0) \to X \cup \partial_\infty X$, $E(\rho) = \rho(\infty) = \begin{cases} \text{as usual} & \text{if } \rho \in \mathcal{G}(X, x_0) \\ \lim_{t\to\infty} \rho(t) & (\text{the other endpoint of the geod. segment}) \text{ if } \rho \notin \mathcal{G}(X, x_0). \end{cases}$

surjective mapping.

$\sim$ equiv. relation in $\check{\mathcal{G}}_+(X, x_0)$:

$\rho \sim \rho' \iff E(\rho) = E(\rho')$.

$(\check{\mathcal{G}}_+(X, x_0)$, top. of comp. conv.$)$ — Compact by Thm 9.2 (AA)

$\downarrow \pi$ ← continuous surjection

$(\check{\mathcal{G}}_+(X, x_0)/\sim$, quotient topology$)$ — compact

$\xrightarrow{E}$ $X \cup \partial_\infty X$ (bijection)

This gives the original metric topology of X on X,

⇝ A geometric compactification of X.

1

$$\begin{array}{ccc} \check{G}(X,x_0) & & \\ \downarrow & \searrow^{E} & \\ \check{G}(X,x_0)/\sim & \xrightarrow{\text{bij.}} & X \cup \partial_\infty X \end{array}$$

quotient mappings.

$q: X \to Y$ is a quotient mapping if q is a continuous surjection and $q^{-1}(U)$ [$q^{-1}(F)$] is open [closed] only if $U \subset Y$ is open [$F \subset Y$ is closed].

Proposition. $q: X \to Y$ continuous surj. $x \sim x' \iff q(x) = q(x')$. The spaces $X/\sim$ and Y are homeomorphic $\iff$ q is a quotient mapping.

Proof. See Munkres.

Prop. 9.15. E is a quotient map.

Cor. $\left(\check{G}_+(X,x_0) - G_+(X,x_0)\right)/\sim$ homeo X.

Proof of 9.15. $A \subset X$ closed. Let us show $E^{-1}(A)$ is closed.

Let $g_n \to g$. Show that $g \in E^{-1}(A) \iff g(\infty) \in A$.

$g_n(\infty) \in A$

$\iff g_n \in E^{-1}(A)$

g_n's converge unif. on compact sets $\leadsto g_n(\infty) \to g(\infty)$.

$$\begin{array}{ccc} g_n(\infty) & \to & g(\infty) \\ \cap & & \cap \\ A & & A \\ \text{closed} & & \end{array}$$

$A \subset X$ s.t. $E^{-1}(A)$ is closed. Let $g_n \in \check{G}_+(X, x_0)$: $g_k(\infty) = a_k$.

$a_k \in A$, $a_k \to a \in X$.

(g_k) has a conv. subsequence $g_{k_j} \to g \in E^{-1}(A)$

$$\begin{array}{ccc} g_{k_j}(\infty) & \to & g(\infty) \in A. \\ \| & & \| \\ a_{k_j} & \to & a \end{array}$$

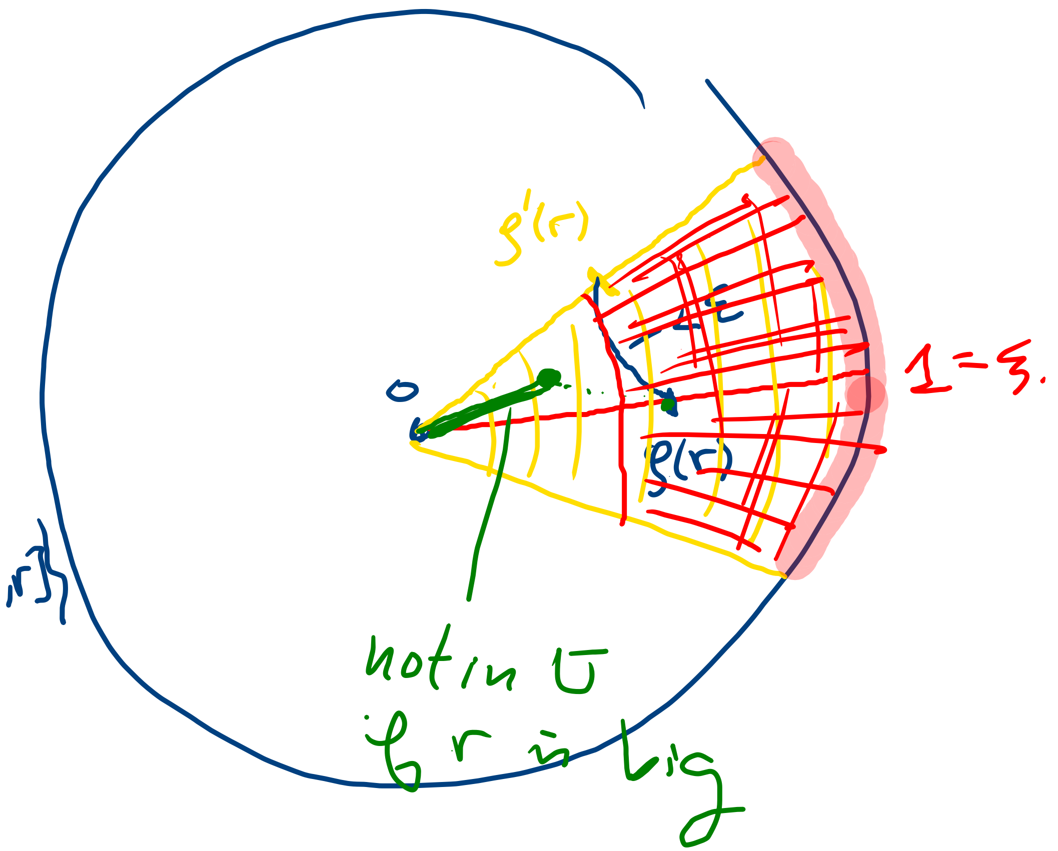
$\square$

Open sets in $\check{\mathcal{G}}_+(X, x_0)$.

Let $\varrho \in \check{\mathcal{G}}_+(X, x_0)$, $r > 0$, $\varepsilon > 0$.

$$B_{[0,r]}(\varrho, \varepsilon) = \{ \varrho' \in \check{\mathcal{G}}_+(X, x_0) : d(\varrho(t), \varrho'(t)) < \varepsilon \ \forall t \in [0,r] \}.$$

$[0,r]$ ← K compact



Ex. $\mathbb{H}^2$: Poincaré model, $x_0 = 0$.

$\xi \in \partial_\infty \mathbb{H}^2 \leadsto \xi = \varrho(\infty)$ for some $\varrho \in \mathcal{G}_+(X, 0)$.

$$U := B_{[0,r]}(\varrho, \varepsilon) = \{ \varrho' \in \tilde{\mathcal{G}}_+(\mathbb{H}^2, 0) : d(\varrho'(t), \varrho(t)) < \varepsilon \ \forall t \in [0,r] \}$$

$\downarrow E$

$X \cup \partial_\infty X$

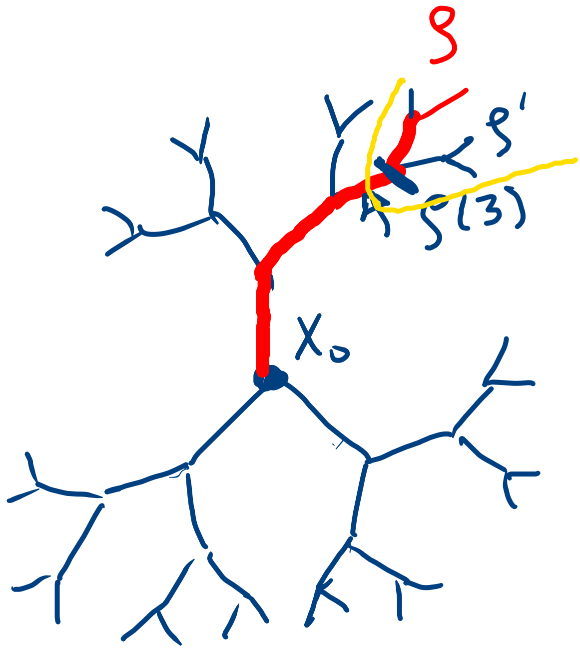
$U \cap \partial_\infty \mathbb{H}^2$ is identified with an open segment in S^1.

$\leadsto$ the basis elts U give a basis of the standard top of S^1

$\Rightarrow \partial_\infty X$ is homeomorphic with standard $S^1 \subset \mathbb{E}^2$.

$\rightsquigarrow \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ homeo $\overline{B}(0,1) \subset \mathbb{E}^2$.

Ex



X simplicial tree with const. degree $= 3$ at the vertices

Let $\rho \in \mathcal{G}_+(X, x_0)$. $0 < \varepsilon < 1$

$$B_{[0,r]}(\rho, \varepsilon) = \{ \rho' \in \check{\mathcal{G}}_+(X, x_0) : d(\rho'(t), \rho(t)) < \varepsilon \ \ \forall\, 0 \le t \le r \}$$

$$U(r, \rho, \varepsilon) = \{ \rho' \in \check{\mathcal{G}}_+(X, x_0) : \rho'|_{[0, r-\varepsilon]} = \rho|_{[0, r-\varepsilon]} \}$$

Topology of $\partial_\infty X$: $U(r, \rho, \varepsilon) \cap \partial_\infty X = \{ \rho \in \mathcal{G}_+(X, x_0) : \rho'|_{[0,r]} = \rho|_{[0,r]} \}$

Ball in the metric based at x_0 $\longrightarrow$ $= \overline{B}(\rho, e^{-r})$ $\qquad \mathcal{G}_+(X,x_0) \longleftrightarrow \partial_\infty X$.