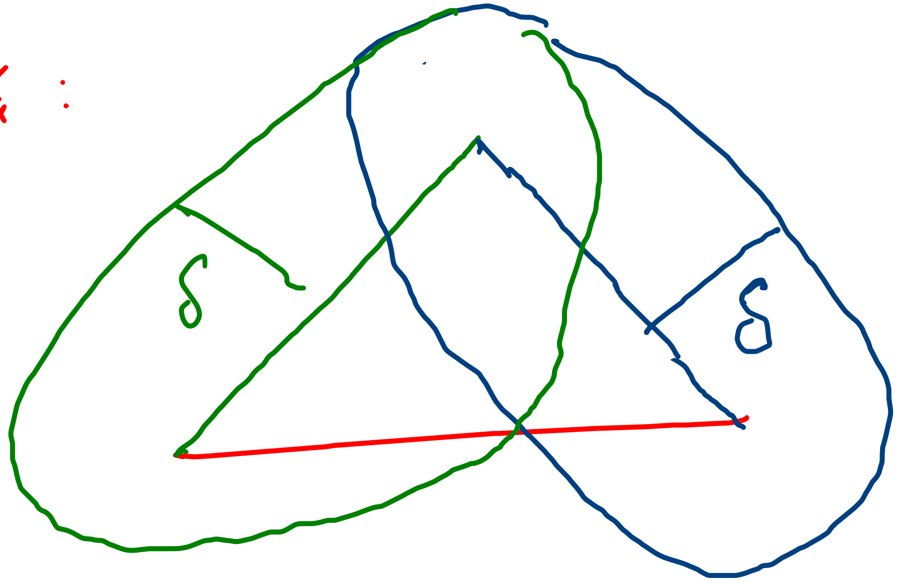
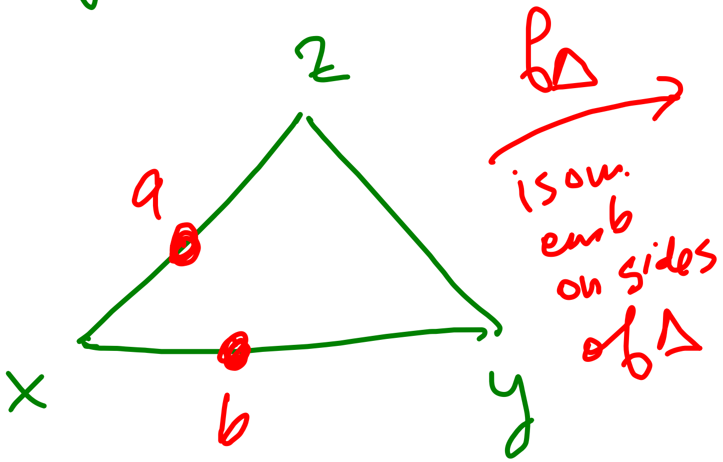
Neg. curved spaces 15.10.2020

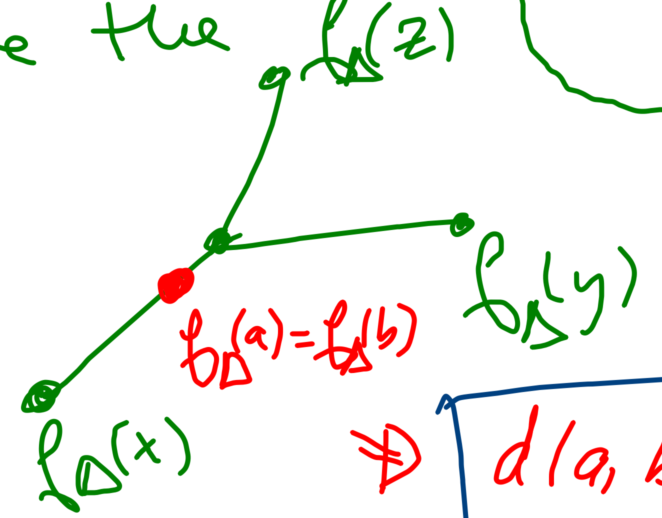


δ-hyp: all triangles in X satisfy Rips cond:

any side is contained in $\overline{\mathcal{N}_\delta}$(other two sides)

δ-thin triangle: If x, y, z are the vertices of Δ / ∃ $f_\Delta : \Delta \to T_\Delta$





⇒ $\boxed{d(a,b) \le \delta.}$

δ-thin means "triangles are similar to triangles in a tree up to δ".

Δ is δ-thin ⇒ Δ satisfies Rips cond with δ. (Ex)

all triangles in a δ-hyp space are 4δ-thin.

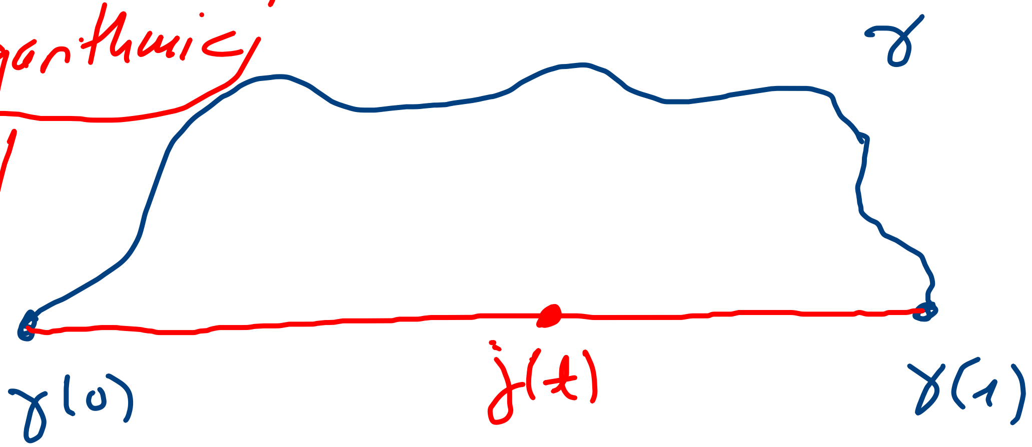
Prop. 6.8 X δ-hyp, $\gamma : [0,1] \longrightarrow X$, $\ell(\gamma) < \infty$ (γ is rectifiable).

Let $j : [0, d(\gamma(0), \gamma(1))] \longrightarrow X$ geod. segment s.t. $j(0) = \gamma(0)$, $j(d(\gamma(0),\gamma(1))) = \gamma(1)$.

$\forall t \in$

$$d\big(j(t), \gamma([0,1])\big) \leq \delta \log_2 \ell(\gamma) + 1 .$$

logarithmic



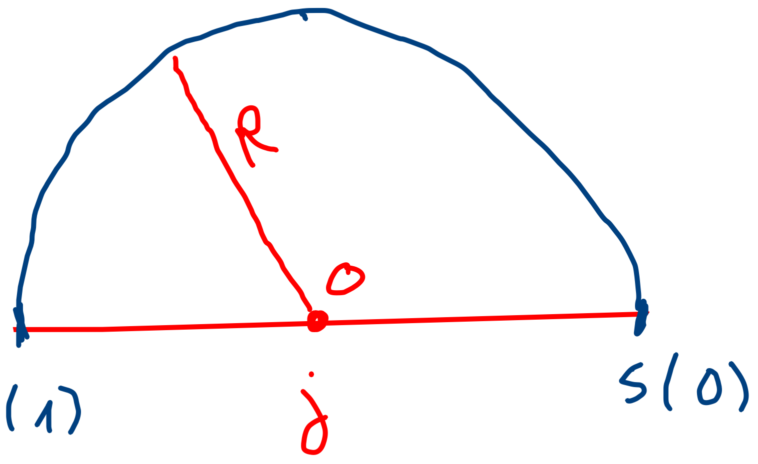
Remarks: 1) in $\mathbb{E}^2$ the path $s : [0,1] \to \mathbb{E}^2 = \mathbb{C}$

$s(t) = R e^{i \pi t}$ parametrizes a half-circle of radius R.

$$\ell(s) = \pi R$$

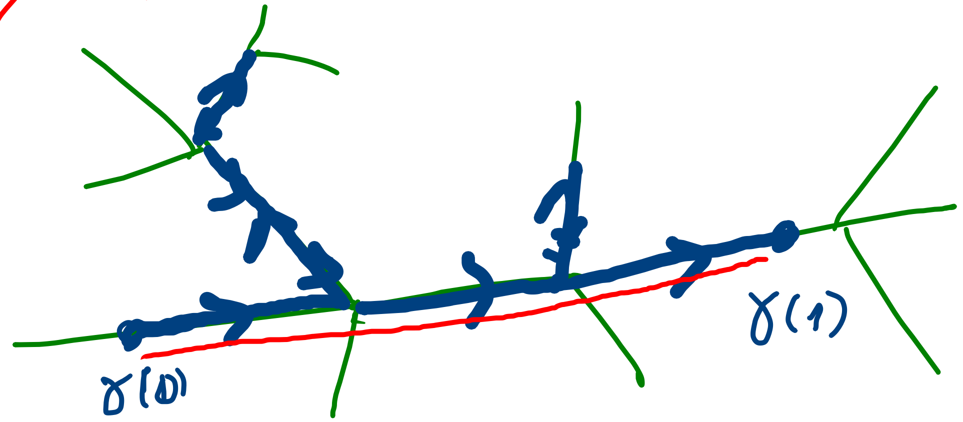
$$d(0, s([0,1])) = R = \frac{\ell(s)}{\pi}$$

linear



$[\gamma(0), \gamma(1)] \subset \gamma([0,1])$

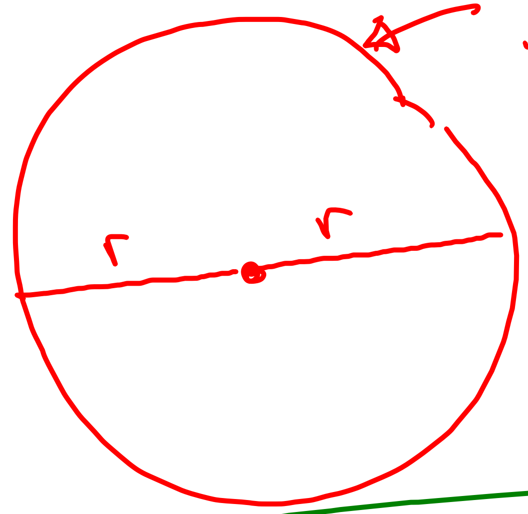
2) If X is a tree then



3) In $\mathbb{H}^2$ the length of a circle of radius r is $\boxed{2\pi \sinh r} \sim \pi e^r$ for big r

$\sinh r = \dfrac{e^r - e^{-r}}{2}$

$l \sim \frac{\pi}{2} e^r$



Exercise

Do it in Poincaré disk

4.2 comp. of length

Proof of Prop 6.8

Let $t \in [0, d(\gamma(0), \gamma(1))]$.

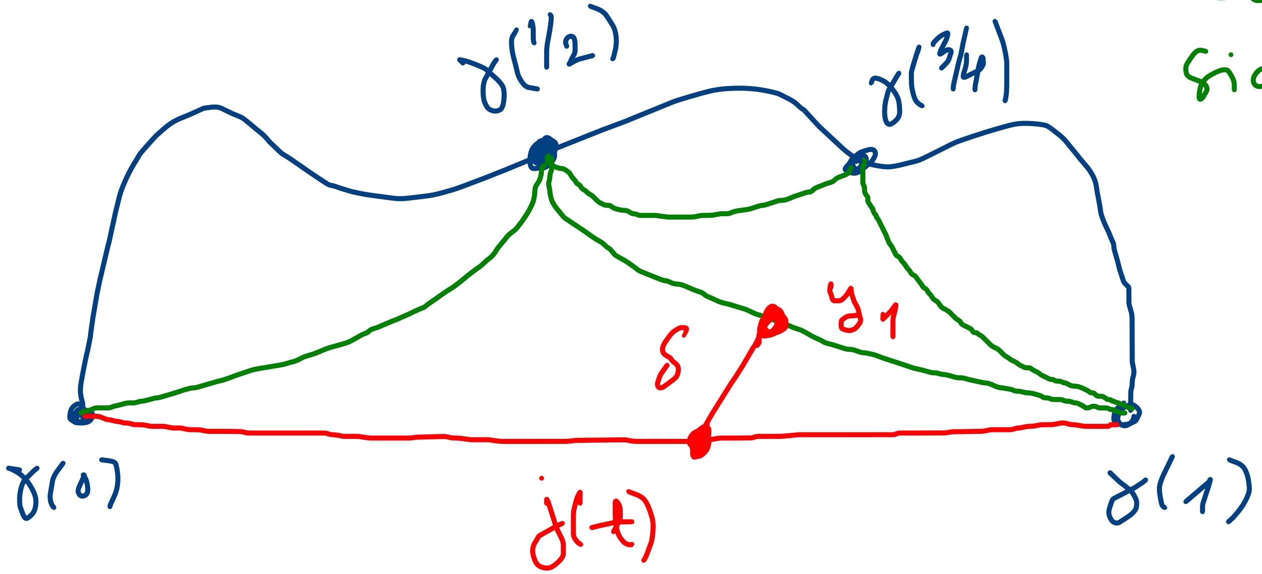
Assume that γ is parametrized proportional to arc length: If $0 \le s \le 1$, then $\ell(\gamma|_{[0,s]}) = s\,\ell(\gamma)$.

(see BH)

Consider a triangle with sides j and two geod. segments $[\gamma(0), \gamma(\frac{1}{2})]$, $[\gamma(\frac{1}{2}), \gamma(1)]$.

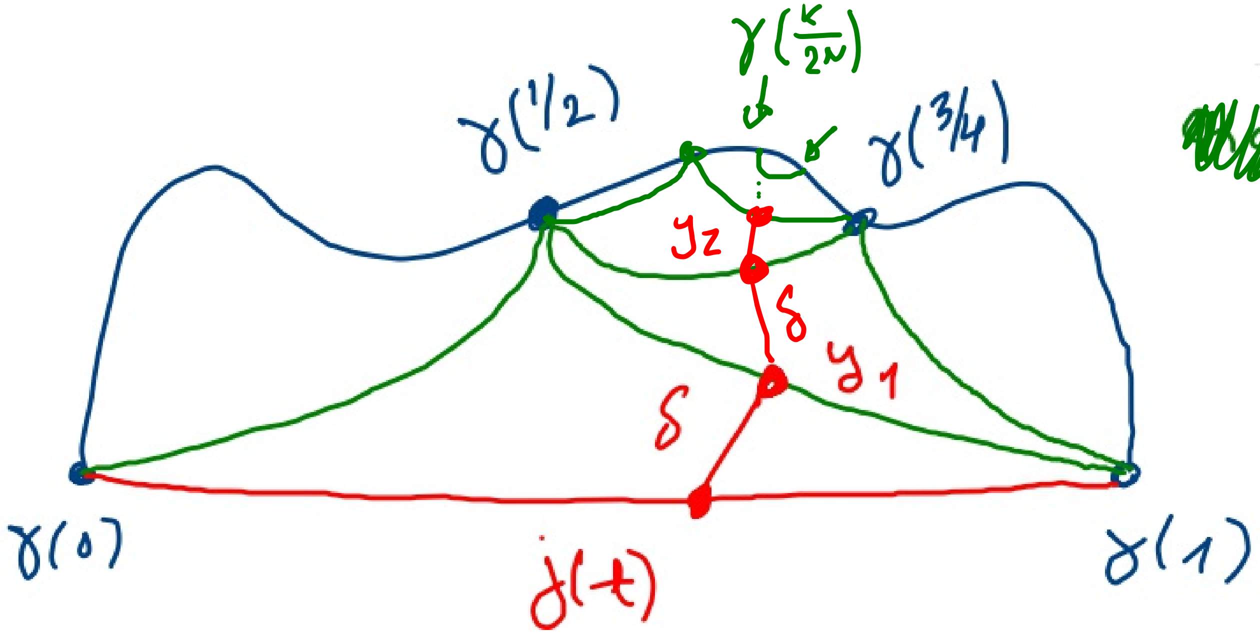
δ-hyp $\Rightarrow \exists\, y_1 \in [\gamma(0), \gamma(\frac{1}{2})] \cup [\gamma(\frac{1}{2}), \gamma(1)]$

s.t. $d(j(t), y_1) \le \delta$. Say $y_1 \in [\gamma(1/2), \gamma(1)]$



Make a triangle with vertices $\gamma(1/2), \gamma(3/4), \gamma(1)$ and $[\gamma(1/2), \gamma(1)]$ one of the sides.

(3)



δ-hyp $\Rightarrow$ $\exists\, y_2 \in [\gamma(1/2), \gamma(3/4)] \cup [\gamma(3/4), \gamma(1)]$

s.t. $d(y_1, y_2) \leq \delta$.

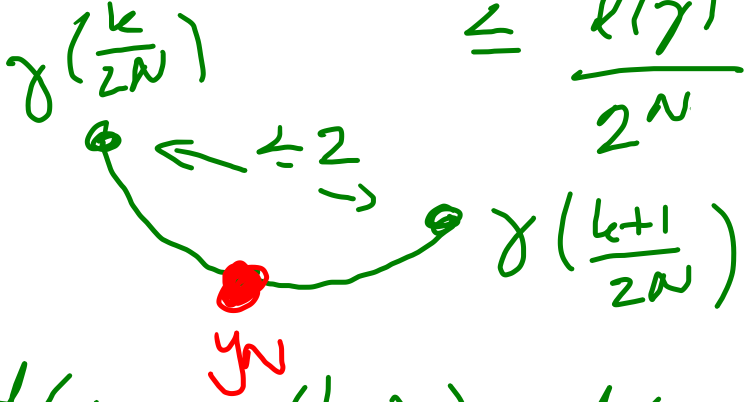
Continue like this.

$\rightsquigarrow$ point $y_N \in [\gamma(\frac{k}{2^N}), \gamma(\frac{k+1}{2^N})]$

Choose N s.t. $\frac{\ell(\gamma)}{2} \leq 2^N \leq \ell(\gamma)$.

$\Rightarrow \frac{\ell(\gamma)}{2^N} \leq 2$

$d(\gamma(\frac{k}{2^N}), \gamma(\frac{k+1}{2^N})) \leq \frac{\ell(\gamma)}{2^N}$



$$\Rightarrow d(j(t), \gamma([0,1])) \leq \underset{\leq \log_2 \ell(\gamma)}{N}\delta + \underbrace{\min\left(d(y_N, \gamma(\tfrac{k}{2^N})), d(y_N, \gamma(\tfrac{k+1}{2^N}))\right)}_{\leq 1}$$

$\square$

7 Quasi-isometries

Def^n (X, d_X), (Y, d_Y), $\lambda \geq 1$, $c \geq 0$, $F : X \to Y$ is a (λ, c)-quasi-isometric embedding if $\forall\ x, x' \in X$

$\underbrace{\text{quasi-isometric embedding}}_{\text{q.i. embedding}}$

important when $d_X(x,x')$ is large.

$$\frac{1}{\lambda} d_X(x,x') - c \leq d_Y(F(x), F(x')) \leq \lambda\, d_X(x, x') + c$$

important when $d_X(x,x')$ is small

Examples. 0) isometric embeddings are $(1,0)$-q.i. embeddings

1) $\mathrm{diam}(X) < \infty \Rightarrow X \mapsto x_0$ is a $(1, \mathrm{diam}\, X)$-q.i. embedding.
$x_0 \in X$

$t \in \mathbb{R}$

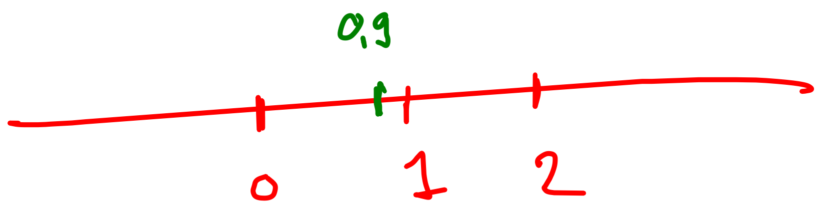
$\lfloor t \rfloor = \max\{m \in \mathbb{Z} : m \leq t\}$ floor

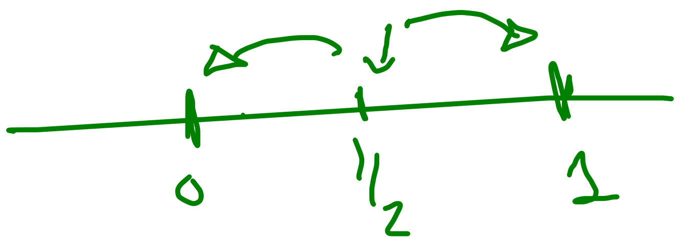
$\lceil t \rceil = \min\{m \in \mathbb{Z} : m \geq t\}$ ceiling

$$[t] = \begin{cases} \lfloor t \rfloor & t \in \bigcup_{n \in \mathbb{Z}} [n, n+\frac{1}{2}] \\ \lceil t \rceil & \text{otherwise} \end{cases}$$

nearest integer

$\underbrace{\lfloor \cdot \rfloor, \lceil \cdot \rceil, [\cdot]}_{(1,1)\text{ q.i. embedding}} : \mathbb{E}^1 \to \mathbb{E}^1$ not continuous, not injective





$d(0,0.9)-1 \leq d(\lfloor 0 \rfloor, \lfloor 0.9 \rfloor) \leq d(0,0.9)+1$

$\lfloor 0 \rfloor = 0$, $\lfloor 0.9 \rfloor = 0$

6

Def$^{\underline{n}}$ $F: X \to Y$ q.i. embeddings and $K \geq 0$ s.t.
$\bar{F}: Y \to X$

$$d_X(x, \bar{F}\circ F(x)) \leq K \quad \forall x \in X$$

$$d_Y(y, F\circ \bar{F}(y)) \leq K \quad \forall y \in Y.$$

$\Rightarrow$ F is a <u>quasi</u>-<u>isometry</u> $\bar{F}$ is its <u>quasi-inverse</u>.
$\bar{F}$ q.i.

<u>Ex.</u> 1) <u>diam $X < \infty$</u>, $x \mapsto x_0$ is a q.i. (identity is its
$x_0 \in X$ quasi-inverse!)

$\Rightarrow$ <u>Thm 7.8</u>. if X, Y geod. metric spaces & $\exists$ <u>$F: X \to Y$ q.i.</u>
then (X Gromov-hyp) $\iff$ (Y Gromov-hyp).