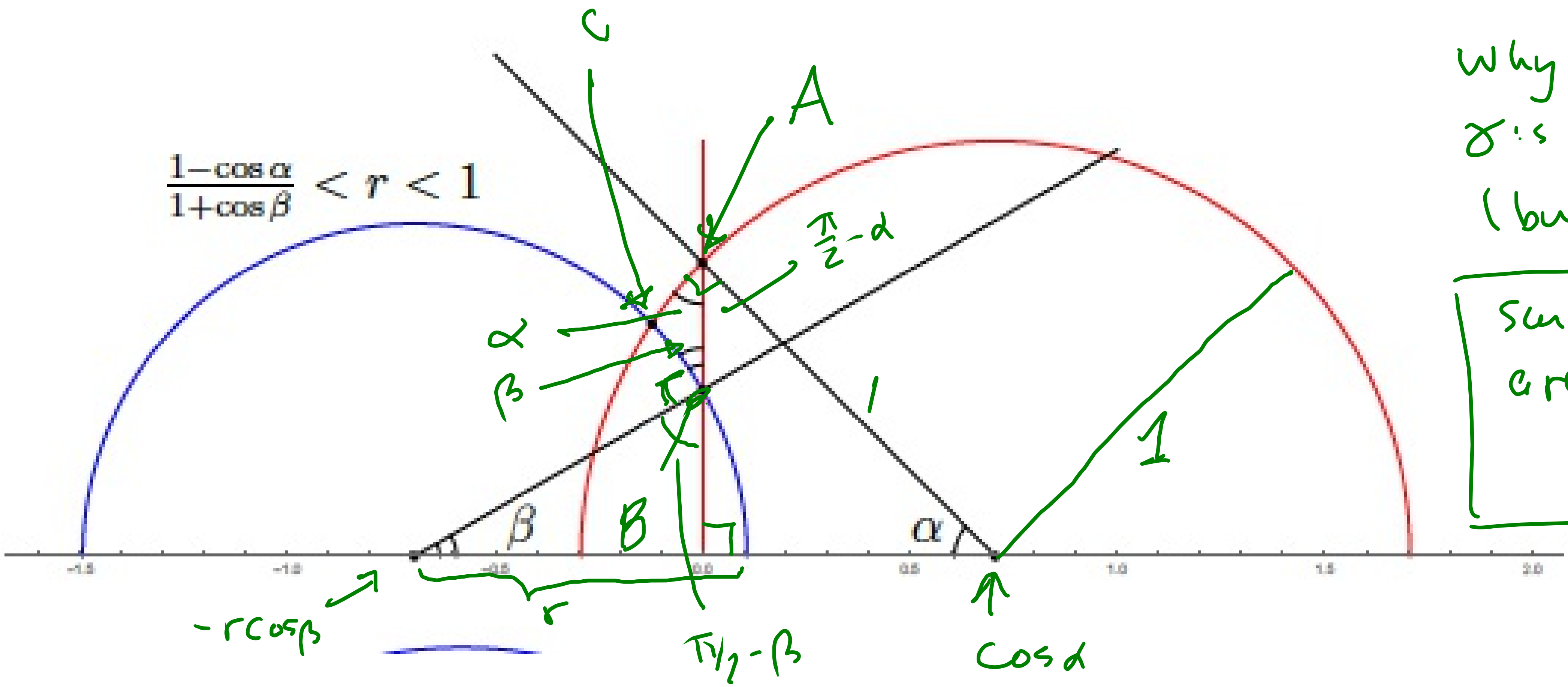
Neg. curved spaces 1.10.2020

Prop 5.10(2) $0<\alpha,\beta,\gamma<\pi$ s.t. $\alpha+\beta+\gamma<\pi$.
Then $\exists$ a triangle in $\mathbb{H}^2$ with angles α,β,γ.

In the Poincare and UHS models angles are the same as in Euclidean space (Prop 5.4, 5.9)

$\frac{(u|v)}{x_n^2}$ at $x\in\mathbb{R}^n_+$

Proof in UHS model of $\mathbb{H}^2$



Why do we get all γ's up to $\pi-(\alpha+\beta)$ (but not $\pi-(\alpha+\beta)$)?

small triangles are almost Euclidean

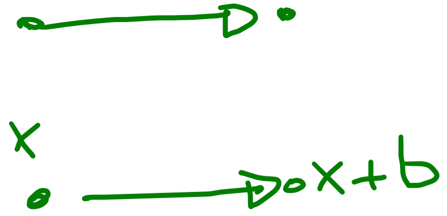
1

Isometries Know from Ch. 4 that Isom $\mathbb{H}^n$ acts **transitively**

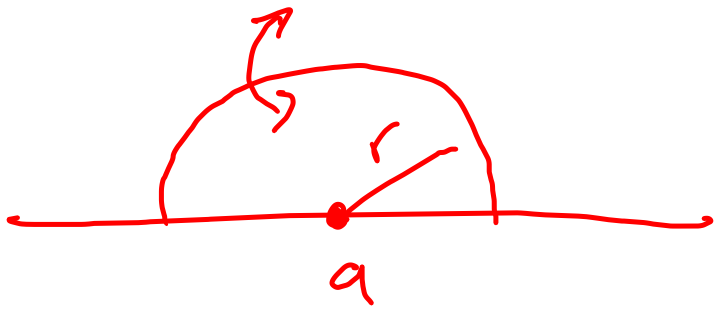
($\forall\, x, y \in \mathbb{H}^n \ \exists\, \gamma \in \mathrm{Isom}\,\mathbb{H}^n : \ \gamma(x) = y$)

Prop. 5.12 The following are isometries of the UHS model:

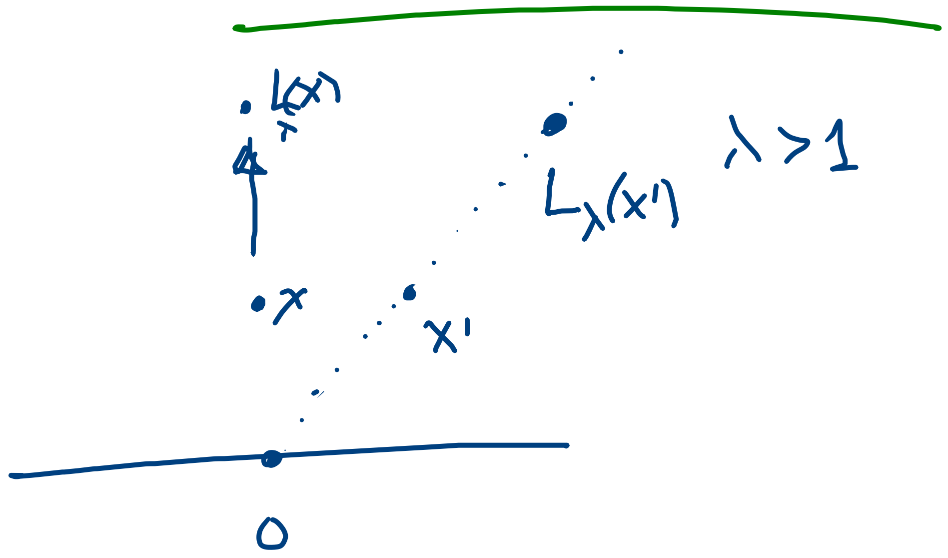
1) horizontal translations: If $b \in \mathbb{R}^{n-1} \times \{0\}$, $T_b(x) = x + b$



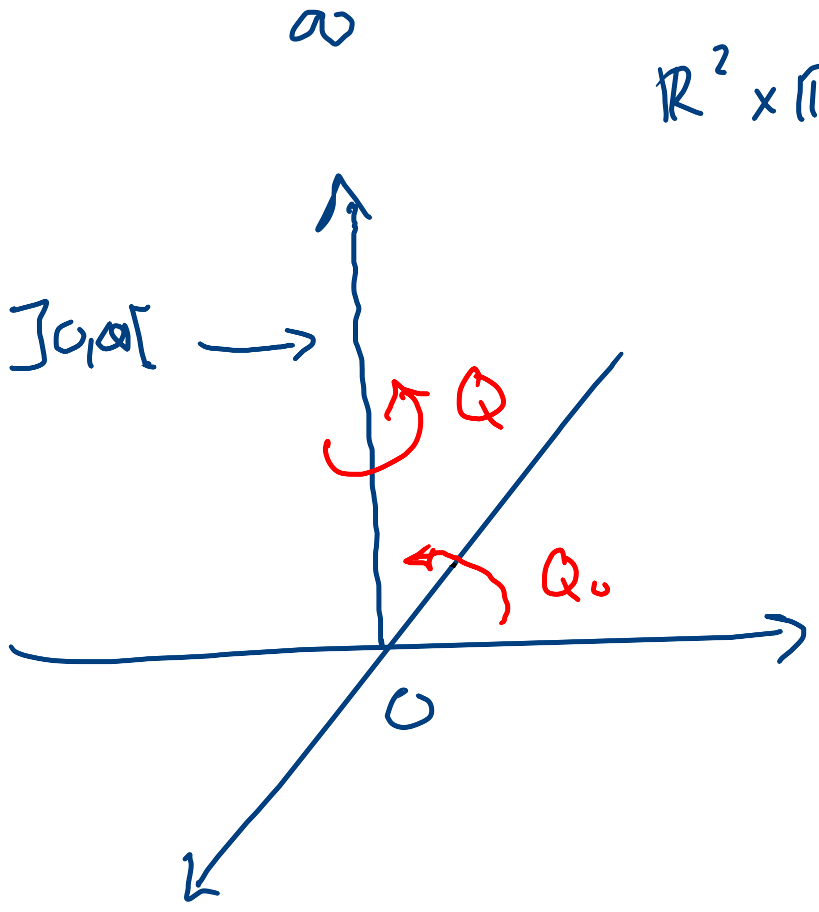
2) inversions: If $a \in \mathbb{R}^{n-1} \times \{0\}$, $r > 0$ ι_{a, r^2}



3) dilations: if $\lambda > 0$ $L_\lambda(x) = \lambda x$



4) $Q_0 \in O(n-1)$ $Q = \begin{pmatrix} Q_0 & 0 \\ 0 & 1 \end{pmatrix}$



$\mathbb{R}^2 \times \mathbb{R}_+$

$Q_0 \in O(2)$ rotation

$\Rightarrow$ Q rotation around the geod. line $]0,\infty[$

Proof. See arg. for $\iota_{0,r}$.

Use Lemma 5.11!

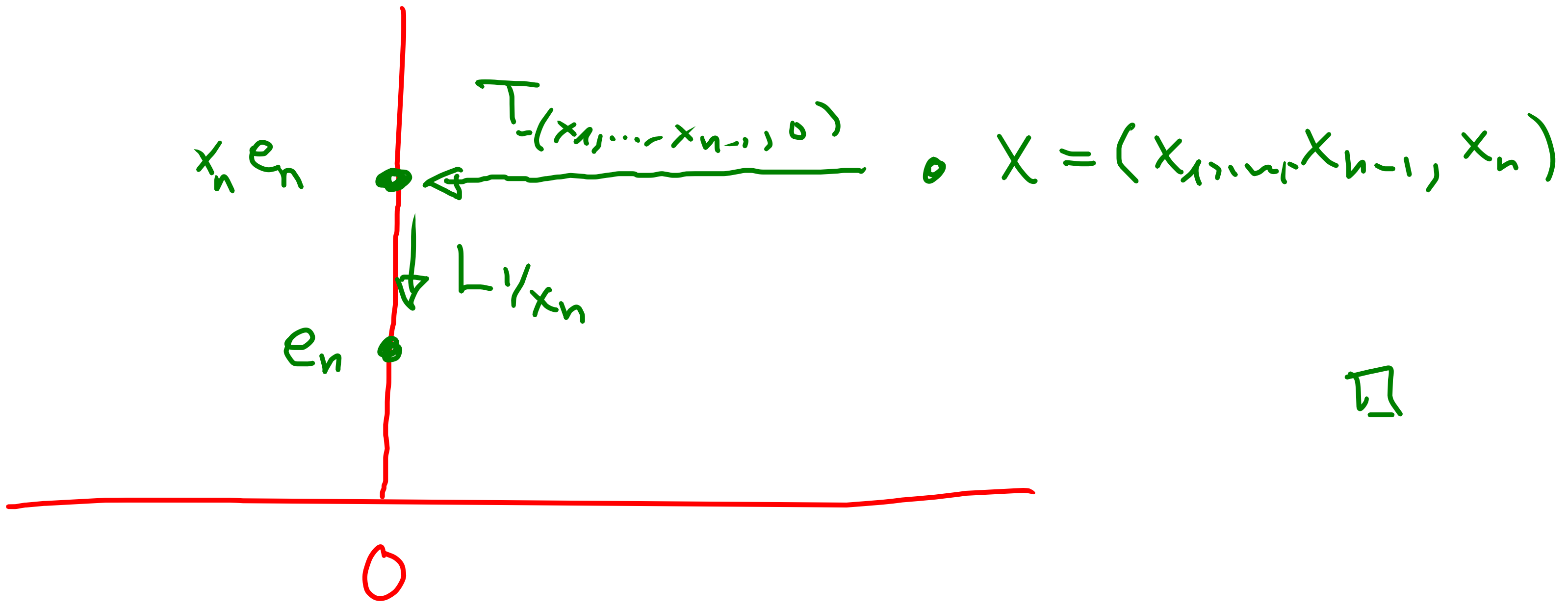
$\to$ $\underline{T_b \circ \iota_{a,r^2} \circ T_{-b}} = \iota_{a+b,r^2}$

$\Rightarrow$ $L_\lambda \circ \iota_{0,r^2} \circ L_{\frac{1}{\lambda}} = \iota_{0,(\lambda r)}$

$\therefore$ show that T_b, L_λ are isometries.

Cor. 5.13 The subgp of Isom $\mathbb{H}^n$ generated by horiz. translations and dilations acts transitively in UHS.

Proof.



Let $X \in$ UHS.

$$L_{1/x_n}\Big(\underbrace{T_{-(x_1,\dots,x_{n-1},0)}X}_{=(0,\dots,0,x_n)}\Big) = e_n$$

□

Prop. 5.14 Balls in the Poincaré model and UHS model are Euclidean balls.

Proof. L. 5.2: Balls in Poincaré model centered at 0 are Euclidean balls.

$F:\mathbb{B}^n \to$ UHS maps $\underset{\text{Hyp.}}{\overset{\text{Eucl}}{\text{balls}}}$ to $\underset{\text{Hyp}}{\overset{\text{Eucl.}}{\text{balls}}}$. (inversion) isometry

④ $F(0) = e_n \in$ UHS $\Rightarrow$ balls in UHS centered at e_n are Eucl. balls.

We can map any ball $\boxed{B(e_n, \underline{r})}$ to any ball $\boxed{B(x, \underline{r})}$ by translations and dilations. (these are isometries)

preserve Eucl. balls.

$\Rightarrow$ hyp. balls in UHS model are Eucl. balls.

$F: \mathbb{B}^n \to \text{UHS}$ isometry $\Rightarrow$ $F^{-1}(B(x,r))$ is a ball $B(F^{-1}(x), r)$

inversion $\Rightarrow$ F^{-1} maps Eucl. balls to Eucl. balls. □

<u>Cor.</u> $\mathbb{H}^n$ is homeo $\begin{pmatrix} \mathbb{E}^n \\ \mathbb{B}^n \subset \mathbb{E}^n \end{pmatrix}$

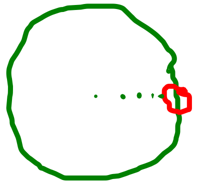
<u>Pf.</u> $id : (\mathbb{B}^n, d_p) \to (\mathbb{B}^n, \text{Eucl.})$ is homeo. □

<u>Cor.</u> $\mathbb{H}^n$ is a proper metric space. (Closed balls are compact)

$\longrightarrow$ complete

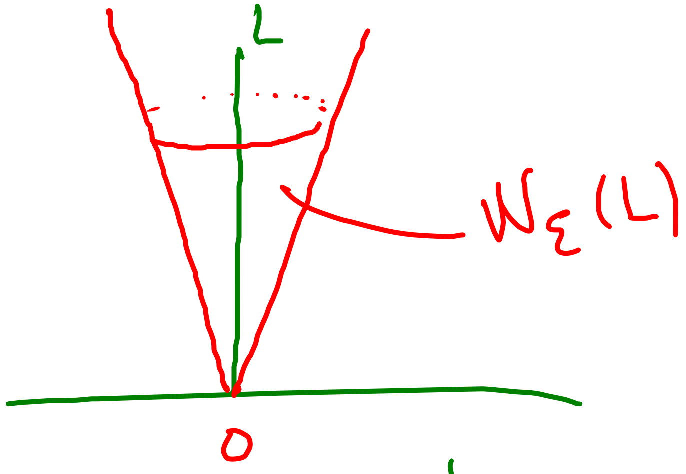
~~4~~ 5 ($\mathbb{E}^2 - \{(1,0)\}$ $\quad \bar{B}(0,1) \subset \mathbb{E}^2 - \{(1,0)\}$ is not compact

$(\mathbb{R}^2, d_{SNCF})$ not proper

)

Lemma 5.20 $L =]0,\infty[\subset \mathbb{H}^n$

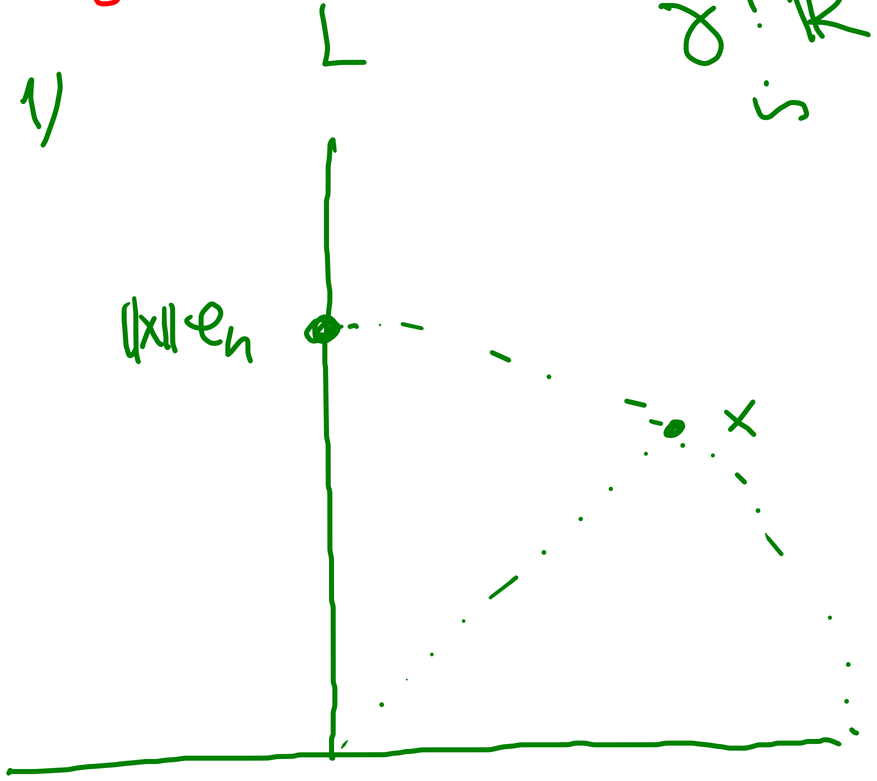
1) $\forall x \in \mathbb{H}^n \ \exists$ unique closest point to x in L.

2) 

$\varepsilon > 0$ $\quad W_\varepsilon(L) = \{x \in \mathbb{H}^n : d(x,L) < \varepsilon\}$

$$= \{x \in \mathbb{R}^n_+ : \cos \underbrace{\measuredangle_0(L,x)}_{\text{angle between } L \text{ and } x \text{ seen from } 0} > \frac{1}{\cosh \varepsilon}\}$$

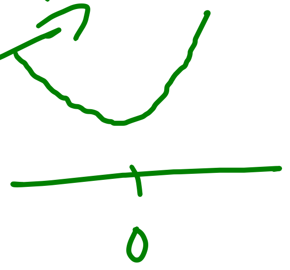
Proof 1)



$\gamma : \mathbb{R} \to \mathbb{R}^n_+$, $\boxed{\gamma(t) = \|x\| e^t e_n}$ is a geod. line parametrizing L

$$\cosh d(x, \gamma(t)) = 1 + \frac{\|x - \gamma(t)\|^2}{2 x_n \gamma(t)_n}$$

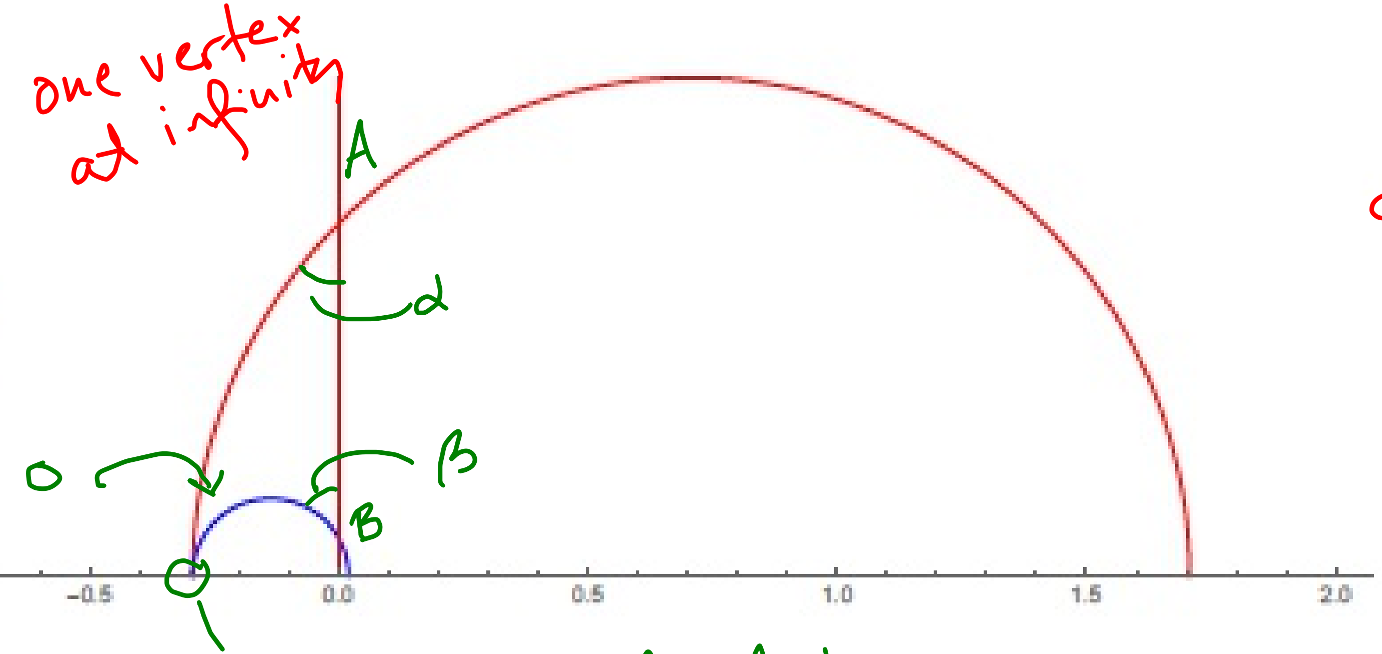
$$= 1 + \frac{x_1^2 + \dots (x_n - \|x\| e^t)^2}{2 x_n \|x\| e^t} = \dots = \frac{\|x\|}{x_n} \cosh t$$



unique min at $t = 0$.

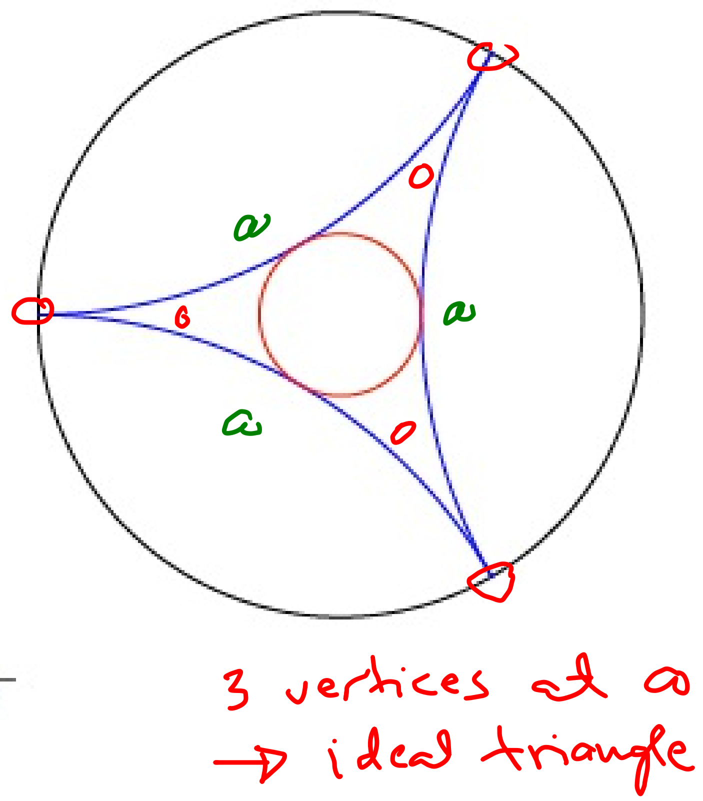
Generalized triangles in $\mathbb{H}^n$

$$r = \frac{1-\cos\alpha}{1+\cos\beta}$$



tangency of red and blue arcs.

geod. rays from A and B to a common pt at ∞.



Defn. $A, B, C \in \mathbb{H}^n \cup \partial_\infty \mathbb{H}^n$ distinct points.
A generalized triangle with vertices A, B, C consists of the geod. arcs, rays or lines with endpts A and B, B and C, C and A.

~~6~~ 7