Funktionaalianalyysi
Exercises 5, 12.2.2018

1. Prove that $\overline{d^\infty(\mathbb{K})} = c_0(\mathbb{K})$.

2. Prove that any Hamel basis of an infinite-dimensional Banach space is uncountable. [1]

3. Does the space $\mathbb{R}[X]$ of real polynomials have a norm $\|\cdot\|$ such that $(\mathbb{R}[X], \|\cdot\|)$ is a Banach space?

Let V be a normed space. Let

$$p((v_k)_{k\in\mathbb{N}}) = \lim_{k\to\infty} \|v_k\|$$

for any Cauchy sequence $(v_k)_{k\in\mathbb{N}}$ in V. Let $\phi\colon V \to \mathscr{C}(V)/\ker p$ be the mapping

$$\phi(v) = (v)_{k\in\mathbb{N}} + \ker p\,,$$

that maps a vector $v \in V$ to the class of the constant sequence $(v)_{k\in\mathbb{N}} \in \mathscr{C}(V)$ in the quotient space $\mathscr{C}(V)/\ker p$.

4. Prove that ϕ is a linear isometric embedding.

5. Prove that $\phi(V)$ is a dense subspace of $\mathscr{C}(V)/\ker p$.

Let X be a dense subspace of a normed space V. Let W be a Banach space and let $T \in \mathrm{Lin}_b(X, W)$. For every $v \in V$ there is a sequence $(x_k)_{k\in\mathbb{N}}$ in X such that $v = \lim_{k\to\infty} x_k$. Let

$$\widehat{T}(y) = \lim_{k\to\infty} T(x_k)\,.$$

6. Prove that $\widehat{T}\colon V \to W$ is well defined and that the definition is independent of the choice of the sequence $(x_k)_{k\in\mathbb{N}}$. Prove that $\widehat{T}$ is a linear mapping such that $\widehat{T}|_X = T$.

7. Prove that $\widehat{T} \in \mathrm{Lin}_b(V, W)$ and $\|\widehat{T}\| = \|T\|$.

8. Let $\widehat{T}, \widetilde{T} \in \mathrm{Lin}_b(V, W)$ such that $\widehat{T}|_X = \widetilde{T}|_X$. Prove that $\widehat{T} = \widetilde{T}$.

[1] Use Baire's theorem.