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Exercises 4, 5.2.2018

Let $T\colon \ell^\infty(\mathbb{K}) \to \ell^\infty(\mathbb{K})$ be the linear mapping defined by setting

$$(T\omega)(k) = \frac{\omega(k)}{k+1}$$

for all $\omega \in \ell^\infty(\mathbb{K})$ and all $k \in \mathbb{N}$.

1. Prove that T is bounded and that $T^{-1}\colon T(\ell^\infty(\mathbb{K})) \to \ell^\infty(\mathbb{K})$ is not bounded.

2. Prove that $T(\ell^\infty(\mathbb{K}))$ is not a closed subspace of $\ell^\infty(\mathbb{K})$.

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3. Let $\|\cdot\|$and $\|\cdot\|'$ be equivalent norms in a vector space V. Prove that $(V, \|\cdot\|)$ is a Banach space if and only if $(V, \|\cdot\|')$ is a Banach space.

4. Let X and Y be Banach spaces and let us use the norm

$$\|(x, y)\| = \max(\|x\|_X, \|y\|_Y)$$

in $X \times Y$. Prove that $X \times Y$ is a Banach space.

5. Prove that $c(\mathbb{K})$ is a closed subspace of $(\ell^\infty(\mathbb{K}), \|\cdot\|_\infty)$. [1]

6. Let $1 \le p < q < \infty$. Prove that $\ell^p(\mathbb{K})$ is a dense subspace of $\ell^q(\mathbb{K})$. [2]

7. Let $p \in [1, \infty]$ and let $\tau \in \ell^{p'}$. Let $L_\tau\colon \ell^p(\mathbb{K}) \to \mathbb{K}$,

$$L_\tau(\omega) = \sum_{i=0}^{\infty} \omega(i)\tau(i)\,.$$

Prove that the mapping $T\colon \ell^{p'}(\mathbb{K}) \to (\ell^p(\mathbb{K}))'$

$$T(\tau) = L_\tau$$

is linear.

8. Prove that $\ell^\infty(\mathbb{K})$ is isometrically isomorphic with $(\ell^1(\mathbb{K}))'$.

[1] Let $(f_k)_{k=1}^\infty$ be a Cauchy sequence in c. As ℓ^∞ is a Banach space, the sequence $(f_k)_{k=1}^\infty$ converges. Prove that the limit is a convergent sequence.

[2] Example 2.11.