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Exercises 3, 29.1.2018

1. Prove that a real normed space V is separable if and only if the subset $S(0,1) = \{x \in V : \|x\| = 1\}$ is separable. [1]

2. Let V and W be vector spaces. Prove that the subset $\mathrm{Lin}(V,W)$ of linear mappings is a vector subspace of $\mathscr{F}(V,W)$.

3. Let V and W be normed spaces. Prove that the subset $\mathrm{Lin}_b(V,W)$ bounded of linear mappings is a vector subspace of $\mathrm{Lin}(V,W)$.

4. Prove that two norms $\|\cdot\|$ and $\|\cdot\|'$ of a vector space V are equivalent if and only if $\mathrm{id}\colon (V,\|\cdot\|) \to (V,\|\cdot\|')$ is a bounded linear mapping whose inverse is bounded.

5. Let

$$V = (\{f \in \mathrm{C}^0([0,1],\mathbb{R}) : f(1) = 0\}, \|\cdot\|_\infty)\,.$$

Prove that the subspace

$$H = \big\{f \in V : \int_{[0,1]} f = 0\big\}$$

is closed.

6. The fundamental theorem of calculus implies that the linear mapping

$$\mathscr{I}\colon \big(\mathrm{C}^0([0,1]),\|\cdot\|_\infty\big) \to \big(\{g \in \mathrm{C}^1([0,1]) : g(0) = 0\},\|\cdot\|_\infty\big)$$

defined by

$$(\mathscr{I}f)(x) = \int_0^x f(t)\,dt\,,$$

is a bijection. Prove that $\mathscr{I}$ is bounded and that $\mathscr{I}^{-1}$is not bounded.

7. Let $V \neq \{0\}$. Prove that

$$\|T\| = \sup_{v \in V - \{0\}} \frac{\|Tv\|_W}{\|v\|_V} = \sup_{\|v\|_V = 1} \|Tv\|_W\,.$$

8. Let V_1, V_2, V_3 be normed spaces and let $S\colon V_1 \to V_2$ and $T\colon V_2 \to V_3$ be bounded linear mappings. Prove that $\|T \circ S\| \leq \|T\|\|S\|$. Give an example where the inequality is strict.

[1] If $T \subset S(0,1)$ is a countable dense subset, considet $\widetilde{T} = \{rt : r \in \mathbb{Q}, r > 0, t \in T\}$.