Differential geometry 2023

Exercises 9

Let V be a vector space. Let $k \geq 1$. Let $A \in T^{(0,k)}(V)$ and let $\sigma \in S_k$. The tensor $\sigma \cdot A$ is defined by setting

$$(\sigma \cdot A)(v_1, \ldots, v_k) = A(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for all $v_1, \ldots, v_k \in V$.
If $\sigma \cdot A = A$ for all $\sigma \in S_k$, then A is *symmetric.*. The *space of symmetric* $(0,k)$*-tensors* on V is

$$S^k(V) = \left\{ \alpha \in T^{(0,k)}(V) : \forall \sigma \in S_k,\ \sigma \cdot \alpha = \alpha \right\}.$$

If $\sigma \cdot A = \operatorname{sign}(\sigma) A$ for all $\sigma \in S_k$, then A is *alternating.* The *space of alternating* $(0,k)$*-tensors* of V is

$$A^k(V) = \left\{ \alpha \in T^{(0,k)}(V) : \forall \sigma \in S_k,\ \sigma \cdot \alpha = \operatorname{sign}(\sigma)\alpha \right\}.$$

1. Let $\det \colon (\mathbb{R}^n)^n \to \mathbb{R}$,

$$\det(v_1, v_2, \ldots, v_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v^1_{\sigma(1)} v^2_{\sigma(2)} \cdots v^n_{\sigma(n)},$$

where $v_j = \sum_{k=1}^n v_j^k \mathbf{e}_k \in \mathbb{R}^n$ for all $1 \leq j \leq n$.
(1) Prove that det is an alternating $(0,n)$-tensor.
(2) Prove that

$$\det(v_1, v_2, \ldots, v_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v_1^{\sigma(1)} v_2^{\sigma(2)} \cdots v_n^{\sigma(n)}.$$

Solution. (1) Let $\tau \in S_n$ and $v_1, \ldots, v_n \in \mathbb{R}^n$. With the change of index $\sigma' = \sigma\tau$ (bijective from S_n to S_n), we compute

$$\begin{aligned}(\tau \cdot \det)(v_1, \ldots, v_n) = \det(v_{\tau(1)}, \ldots, v_\tau(n)) &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v^1_{\sigma(\tau(1))} \cdots v^n_{\sigma(\tau(n))} \\ &= \sum_{\sigma' \in S_n} \operatorname{sign}(\sigma'\tau^{-1}) v^1_{\sigma'(1)} \cdots v^n_{\sigma'(n)} \\ &= \operatorname{sign}(\tau^{-1}) \det(v_1, \ldots, v_n) = \operatorname{sign}(\tau) \det(v_1, \ldots, v_n).\end{aligned}$$

(2) Noticing the rearrangement of product $v^1_{\sigma(1)} \cdots v^n_{\sigma(n)} = v_1^{\sigma^{-1}(1)} \cdots v_n^{\sigma^{-1}(n)}$ and using this time the change of index $\sigma' = \sigma^{-1}$, we compute

$$\begin{aligned}\det(v_1, \ldots, v_n) &= \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) v_1^{\sigma^{-1}(1)} \cdots v_n^{\sigma^{-1}(n)} = \sum_{\sigma' \in S_n} \operatorname{sign}(\sigma'^{-1}) v_1^{\sigma'(1)} \cdots v_n^{\sigma'(n)} \\ &= \sum_{\sigma' \in S_n} \operatorname{sign}(\sigma') v_1^{\sigma'(1)} \cdots v_n^{\sigma'(n)}.\end{aligned}$$

2. Let V be a vector space. Let $A \in T^{(0,k)}(V)$. Prove that
(1) A is symmetric if and only if $\tau \cdot A = A$ for all transpositions $\tau \in S_k$ and
(2) A is alternating if and only if $\tau \cdot A = -A$ for all transpositions $\tau \in S_k$.

Solution. (1) The direct implication is trivial. For the converse, recall that the transpositions generate the whole symmetric group S_k and notice that we have the formula $(\sigma' \cdot \sigma) \cdot A = \sigma' \cdot (\sigma \cdot A)$ (in more elaborate words, the application $(\sigma, A) \mapsto \sigma \cdot A$ is an action of the group S_k on the set of tensors $T^{(0,k)}(V)$). Let $\sigma \in S_n$ and decompose it into a product of transpositions $\sigma = \tau_1 \cdots \tau_n$. Then applying recursively the latter formula and the invariance $\tau \cdot A = A$ for transpositions, we obtain

$$\sigma \cdot A = (\tau_1 \cdots \tau_n) \cdot A = (\tau_1 \cdots \tau_{n-1}) \cdot (\tau_n \cdot A) = (\tau_1 \cdots \tau_{n-1}) \cdot A = \ldots = A.$$

(2) The proof is similar, using in addition the fact that sign : $S_k \to \{0, 1\}$ is a group morphism.

Let V be a finite-dimensional vector space. The mapping $\mathrm{Sym}\colon T^{(0,k)}(V) \to T^{(0,k)}(V)$,

$$\mathrm{Sym}\, A = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot A$$

is *symmetrization*. The mapping $\mathrm{Alt}\colon T^{(0,k)}(V) \to T^{(0,k)}(V)$,

$$\mathrm{Alt}\, A = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\, \sigma \cdot A$$

is *alternation*.

3. (1) Prove that Alt and Sym are linear mappings
(2) Let $\alpha \in T^{(0,k)}(V)$. Prove that $\mathrm{Sym}\, A = A$ if and only if $A \in S^k(V)$.
(3) Let $A \in T^{(0,k)}(V)$. Prove that $\mathrm{Alt}\, A = A$ if and only if $A \in A^k(V)$.

Solution. (1) The map Sym (resp. Alt) is a linear combinations of mappings of the form $A \mapsto \sigma \cdot A$ (with $\sigma \in S_k$). The latter mappings are all linear, hence so is Sym (resp. Alt).
(2) First notice that the image of Sym is included in $S^k(V)$ (in fact, it is equal to $S^k(V)$ as shown by the converse) since, for all $\tau \in S_k$, using the change of index $\sigma' = \tau\sigma$ we have

$$\tau \cdot \mathrm{Sym}\, A = \frac{1}{k!} \sum_{\sigma \in S_k} (\tau\sigma) \cdot A = \frac{1}{k!} \sum_{\sigma' \in S_k} \sigma' \cdot A = \mathrm{Sym}\, A.$$

In particular, if $\mathrm{Sym}\, A = A$, then A is symmetric. For the converse, we assume that A is symmetric and compute

$$\mathrm{Sym}\, A = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot A = \frac{|S_k|}{k!} A = A.$$

(3) Similarly, the image of the mapping Alt is included in (in fact equal to) $A^k(V)$ giving us the first implication, and using the formula $\mathrm{sign}(\sigma)^2 = 1$ we obtain the converse.

4. Prove that the tensor $\mathbf{e}^1 \otimes \mathbf{e}^2 \otimes \mathbf{e}^3 \in T^{(0,3)}(\mathbb{R}^3)$ is not the sum of a symmetric and an alternating tensor.

Solution. By definition, we have the formula $e^1 \otimes e^2 \otimes e^3(x,y,z) = x_1 y_2 z_3$ for all vectors $x, y, z \in \mathbb{R}^3$. Denote this tensor product by u. By contradiction, assume that we can write $u = A + B$ where $A \in S^k(V)$ and $B \in A^k(V)$. Using Exercise 3, we obtain

$$\mathrm{Sym}(u) = \mathrm{Sym}(A) + \mathrm{Sym}(B) = A + \mathrm{Sym}(B) \text{ and } \mathrm{Alt}(u) = \mathrm{Alt}(A) + \mathrm{Alt}(B) = \mathrm{Alt}(A) + B.$$

Using the formula $|\{\sigma \in S_3 : \mathrm{sign}(\sigma) = 1\}| = |\operatorname{Ker} \mathrm{sign}| = \frac{3!}{2} = 3$ and hence $|\{\sigma \in S_3 : \mathrm{sign}(\sigma) = -1\}| = 3$ too, we obtain the equalities $\mathrm{Sym}(B) = 0$ and $\mathrm{Alt}(A) = 0$. All in all, we can explicitly compute A and B as follows: for all vectors $x, y, z \in \mathbb{R}^3$,

$$A(x,y,z) = \mathrm{Sym}(u) = \frac{1}{3!} \sum_{\sigma \in S_3} x_{\sigma(1)} y_{\sigma(2)} z_{\sigma(3)},$$

$$\text{and } B(x,y,z) = \mathrm{Alt}(u) = \frac{1}{3!} \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) x_{\sigma(1)} y_{\sigma(2)} z_{\sigma(3)}.$$

Thus

$$u(x,y,z) = \frac{2}{3!}(x_1 y_2 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2).$$

This is clearly not equal to $u = e^1 \otimes e^2 \otimes e^3(x,y,z)$ (for instance, they differ at the point $x = (0,1,0)$, $y = (0,0,1)$ and $z = (1,0,0)$).

Let M be a smooth manifold. The *space of smooth* (r,s)*-tensor fields* on M is denoted by $\Gamma\big(T^{(r,s)}M\big)$.

5. Let M and N be smooth manifolds and let $F \colon M \to N$ be a smooth mapping. Let $A \in \Gamma(T^{(0,k)}N)$ and $B \in \Gamma(T^{(0,m)}N)$. Prove that
(1) if $k = m$, then $F^*(A + B) = F^*(A) + F^*(B)$
(2) $F^*(A \otimes B) = F^*(A) \otimes F^*(B)$.

Solution. (1) Let $p \in M$ and let $v_1, \ldots, v_k \in T_pM$. Then

$$\begin{aligned} F^*(A+B)(v_1,\ldots,v_k) &= (A+B)(dF_p v_1, \ldots, dF_p v_k) \\ &= A(dF_p v_1, \ldots, dF_p v_k) + B(dF_p v_1, \ldots, dF_p v_k) \\ &= F^*A(v_1,\ldots,v_k) + F^*B(v_1,\ldots,v_k) \\ &= (F^*(A) + F^*(B))(v_1,\ldots,v_k). \end{aligned}$$

(2) Let $p \in M$ and let $v_1, \ldots, v_{k+m} \in T_pM$. Then

$$\begin{aligned} F^*(A \otimes B)(v_1,\ldots,v_{k+m}) &= (A \otimes B)(dF_p v_1, \ldots, dF_p v_{k+m}) \\ &= A(dF_p v_1, \ldots, dF_p v_k)\, B(dF_p v_{k+1}, \ldots, dF_p v_{k+m}) \\ &= F^*A(v_1,\ldots,v_k)\, F^*B(v_{k+1},\ldots,v_{k+m}) \\ &= (F^*(A) \otimes F^*(B))(v_1,\ldots,v_{k+m}). \end{aligned}$$

6. Let M be a smooth manifold. Let $A \in \Gamma\big(T^{(r,s)}M\big)$ and $B \in \Gamma\big(T^{(t,u)}M\big)$. Prove that $A \otimes B \in \Gamma\big(T^{(r+t,s+u)}M\big)$.

Solution. Set $n = \dim M$. We have already seen the purely algebraic part of this result: for every point $p \in M$, we have indeed $A_p \otimes B_p \in T^{(r+s,t+u)}(T_pM)$. It remains to check the smoothness of $A \otimes B : p \mapsto A_p \otimes B_p$.

Fix a chart (U,ϕ) of M and write the associated local coordinates $\frac{\partial}{\partial x^i}$ (vector fields) and dx^j (1-forms). Then we know there exists smooth real valued functions $A^{i_1,\ldots,i_r}_{j_1,\ldots j_s}$ and $B^{i'_1,\ldots,i'_t}_{j'_1,\ldots,j'_u}$ defined on U and such that, for all $p \in U$,

$$A_p = \sum_{1\le i_1,\ldots,i_r\le n} \sum_{1\le j_1,\ldots,j_s\le n} A^{i_1,\ldots,i_r}_{j_1,\ldots j_s}(p)\frac{\partial}{\partial x^{i_1}}|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}}|_p \otimes dx^{j_1}_p \otimes \ldots \otimes dx^{j_s}_p$$

$$\text{and } B_p = \sum_{1\le i'_1,\ldots,i'_t\le n} \sum_{1\le j'_1,\ldots,j'_u\le n} B^{i'_1,\ldots,i'_t}_{j'_1,\ldots j'_u}(p)\frac{\partial}{\partial x^{i'_1}}|_p \otimes \ldots \otimes \frac{\partial}{\partial x^{i'_t}}|_p \otimes dx^{j'_1}_p \otimes \ldots \otimes dx^{j'_u}_p.$$

Then, in the local coordinates associated to (U,ϕ), the map $A\otimes B$ is described by the functions $p \mapsto A^{i_1,\ldots,i_r}_{j_1,\ldots j_s}(p)B^{i'_1,\ldots,i'_t}_{j'_1,\ldots j'_u}(p)$ which are smooth (as a pointwise product of smooth real functions). By Proposition 8.3, the map $A\otimes B$ is smooth, and hence is a tensor field.