Differential geometry 2023

Exercises 7

Let V and W be vector spaces and let $L\colon V \to W$ be a linear mapping. The mapping $L^*\colon W^* \to V^*$,

$$L^*(\omega) = \omega \circ L$$

is the *dual mapping* or the *transpose* of L.

1. (1) Prove that the dual mapping of a linear mapping is a linear mapping.

(2) Let $L_1\colon V_1 \to V_2$ and $L_2\colon V_2 \to V_3$ be linear mappings. Prove that

$$(L_2 L_1)^* = L_1^* L_2^* \,.$$

Solution. (1) Let $L\colon V \to W$ be a linear mapping. Let $\omega_1, \omega_2 \in W^*$ and let $a_1, a_2 \in \mathbb{R}$. By the definitions of pullback and of linear combinations of linear transformations, for all $v \in V$, we have

$$\begin{aligned} L^*(a_1\omega_1 + a_2\omega_2)(v) &= (a_1\omega_1 + a_2\omega_2) \circ L(v) = a_1\omega_1 \circ L(v) + a_2\omega_2 \circ L(v) \\ &= (a_1 L^*\omega_1 + a_2 L^*\omega_2)(v)\,. \end{aligned}$$

(2) By the definition of the dual mapping and the usual rules of the composition of mappings

$$(L_2L_1)^*\omega = \omega(L_2L_1) = (\omega L_2)L_1 = (L^*\omega)L_1 = L_1^*(L_2^*\omega) = (L_1^*(L_2^*)\omega$$

for all $\omega \in V_3^*$.

Let $(v_1, v_2, \ldots v_n)$ be a basis of a vector space V. Let $\epsilon^1, \ldots, \epsilon^n \in V^*$ be a basis of V^* such that

$$\epsilon^i(E_j) = \delta^i_j = \begin{cases} 1\,, & \text{if } i = j\,, \\ 0\,, & \text{otherwise} \end{cases} \,.$$

Then $(\epsilon^1, \ldots, \epsilon^n)$ is *dual* to $(v_1, v_2, \ldots v_n)$.

2. Let $L\colon V \to W$ be a linear mapping. Let $(v_1, \ldots, v_n)$ be a basis of V and let $(w_1, \ldots, w_m)$ be a basis of W. Let $(\bar v^1, \ldots, \bar v^n)$ and $(\bar w^1, \ldots, \bar w^m)$ be bases of the dual spaces V^* and W^* that are dual to the bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$. Let (a^i_j) be the matrix of L with respect to the bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$.[1] Prove that[2]

$$L^*\bar w^i = \sum_{j=1}^n a^i_j \bar v^j \,.$$

---

[1] Recall that this means that

$$L v_j = \sum_{i=1}^m a^i_j w_i$$

for all $1 \le j \le n$.

[2] A shorter formulation of this exercise: Prove that the matrix of the transpose of a linear mapping is the transpose of the matrix of the linear mapping.

Solution. Let $L^*(\bar{w}^i) = \sum_{j=1}^n b^i_j \bar{v}^j$. First, using the duality of the bases of V and V^*, we get

$$L^*(\bar{w}^i)v_k = \Big(\sum_{j=1}^n b^i_j \bar{v}^j\Big)v_k = \sum_{j=1}^n b^i_j \delta^j_k = b^i_k$$

Second, starting with the definition of the dual mapping, we get

$$\begin{aligned} L^*(\bar{w}^i)v_k &= (\bar{w}^i \circ L)v_k = \bar{w}^i(Lv_k) = \bar{w}^i(\sum_{j=1}^m a^j_k w_j) \\ &= \sum_{j=1}^m a^j_k \bar{w}^i(w_j) = \sum_{j=1}^m a^j_k \delta^i_j = a^i_k \,. \end{aligned}$$

3. Let $X, Y \in \mathfrak{X}(\mathbb{E}^2 - \{0\})$,

$$X_x = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} \qquad \text{ja} \qquad Y_x = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} \,.$$

Find $\omega \in \mathfrak{X}^*(\mathbb{E}^2 - \{0\})$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

Solution. Let $\omega = a(x)dx^1 + b(x)dx^2$. Now, the equations $\omega_x X_x = 1$ and $\omega_x X_x = 0$ give the pair of equations

$$\begin{cases} -a(x)x^2 + b(x)x^1 = 1 \\ \ \ a(x)x^1 + b(x)x^2 = 0 \end{cases}$$

has the unique solution $a(x) = \frac{-x^2}{\|x\|^2}$ and $b(x) = \frac{x^1}{\|x\|^2}$. Thus,

$$\omega = \frac{-x^2 dx^1 + x^1 dx^2}{\|x\|^2} \,.$$

4. Let M be a smooth manifold and let $f, g \in \mathfrak{F}(M)$. Prove that
(1) $d(fg) = f\,dg + g\,df$.
(2) if $g(p) \neq 0$ for all $p \in M$, then

$$d\Big(\frac{f}{g}\Big) = \frac{g\,df - f\,dg}{g^2} \,.$$

Solution. (1) Let $X \in \mathfrak{X}(M)$. The Leibniz rule of vector fields gives

$$d(fg)X = X(fg) = fXg + gXf = fdgX + gdfX = (fdg + gdf)X \,.$$

(2) Observe first that by (1), we have $0 = d\,1 = d(\frac{1}{g}\,g) = \frac{1}{g}\,dg + g\,d(\frac{1}{g})$, which implies $d(\frac{1}{g}) = -\frac{dg}{g^2}$. Again by (1) we have

$$d\Big(\frac{f}{g}\Big) = d\Big(\frac{1}{g}\,f\Big) = \frac{1}{g}\,df + f\,d\Big(\frac{1}{g}\Big) = \frac{g\,df - f\,dg}{g^2} \,.$$

5. Let M and N be smooth manifolds and let $F\colon M \to N$ be a smooth mapping. Let $\omega, \tau \in \mathfrak{X}^*(N)$. Prove that $F^*(\omega + \tau) = F^*(\omega) + F^*(\tau)$.

Solution. Let X be a vector field. By definition of the pullback and the definition of the addition of forms, we have

$$F^*(\omega + \tau)X = (\omega + \tau)dFX = \omega dFX + \tau dFX = F^*\omega X + F^*\tau X = (F^*\omega + F^*\tau)X \,.$$