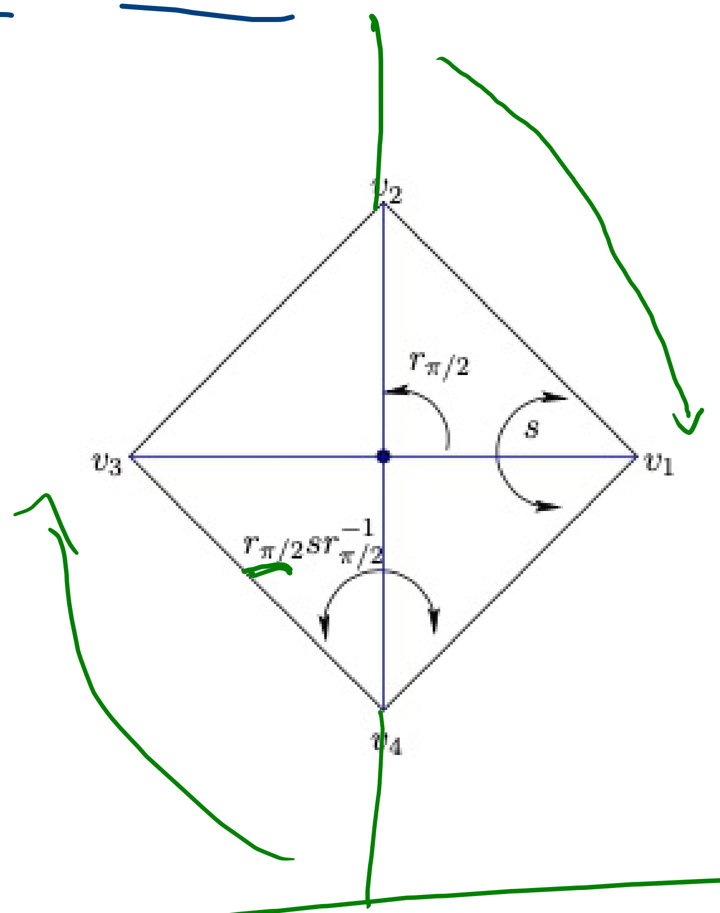
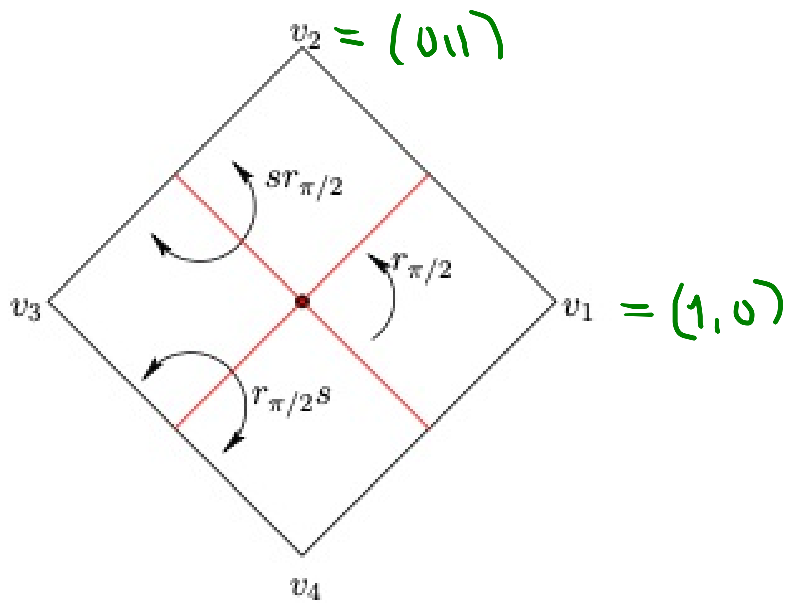
Ryhmät 4.5.2021

P_4





$D_4 = \{ A \in O(2) : A P_4 = P_4 \}$.

Lemma 13.5

1) $D_n < O(2)$

2) $D_n = \langle s, r_{\frac{2\pi}{n}} \rangle$.

3) $\# D_n = 2n$ alkiota.

Tod. 2) eilen.

1) Aliryhmätesti: $I_2 \in D_n$, sillä $I_2 x = x \ \forall x \in \mathbb{R}^2 \Rightarrow I_2 P_n = P_n$.

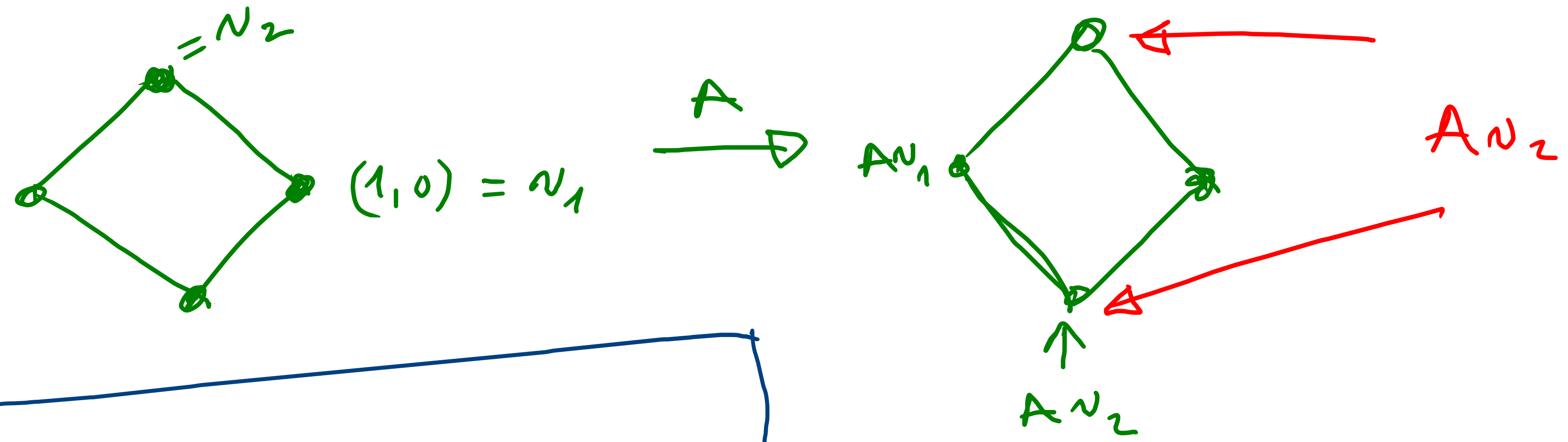
Olk. $A, B \in D_n$. Tällöin $(AB)P_n = A(\underbrace{BP_n}_{P_n}) = P_n \Rightarrow AB \in D_n$

$AP_n = P_n$ $\quad A^{-1}\underbrace{(AP_n)}_{"P_n"} = A^{-1}P_n \quad \Rightarrow A^{-1} \in D_n$.

A on kääntyvä

(1)

3) Olk. $A \in D_n$. Kärki $A\nu_1$ void. valita n. tavalla, $A\nu_2$ void. valita 2 tavalla senjälkeen $\Big\}$ $\#D_n = 2n$



$\rho(5) = (2\,4)$

Ryhmän D_n permutaatioesitys

Matriisia A vastaava kärkien permutaatio:

$\rho_n : D_n \to S_n$, $\rho_n(A)$ on matriisia A vastaava kärkien permutaatio:

$$\boxed{A\nu_k = \nu_{\rho_n(A)(k)}}$$

$$r_{\frac{2\pi}{n}} \nu_k = \begin{cases} \nu_{k+1}, & \text{jos } k \in \{1, \dots, n-1\} \\ \nu_1, & \text{jos } k = n \end{cases} \rightsquigarrow r_{\frac{2\pi}{n}} \nu_k = \nu_{(1\,2\cdots n)k}$$

$$\Rightarrow \rho_n(r_{\frac{2\pi}{n}}) = (1\,2 \cdots n)$$

Prop. $\varrho_n : D_n \to S_n$ on injektiivinen homomorfismi. (uskollinen esitys)

Tod. $\nu_{\varrho_n(AB)(k)} = AB\underbrace{\nu_k}_{\nu_{\varrho_n(B)(k)}} = \nu_{\varrho_n(A)\varrho_n(B)(k)}$. Siis ϱ_n on homomorfismi.

Os. että ϱ_n on injektio.

Olk. $A \in \ker \varrho_n$, niin $A\nu_1 = \nu_1$
$A\nu_2 = \nu_2$ — muod. $\mathbb{R}^2$:n kannan.

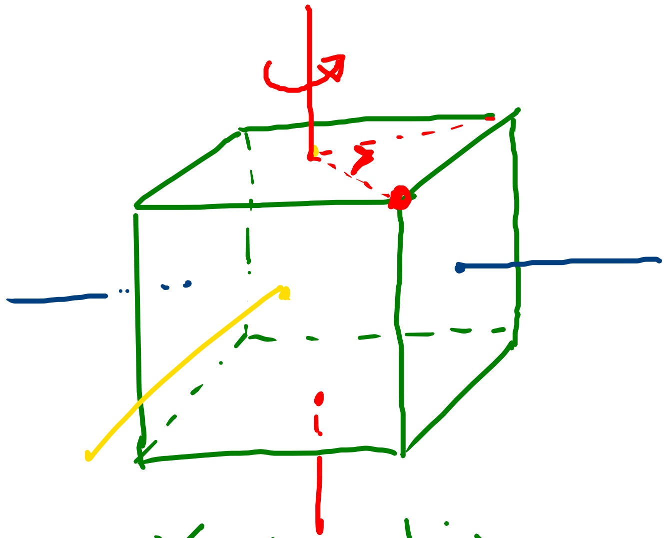
A lin. kuvaus $\Rightarrow A = I_2$. (LAG). Siis $\ker \varrho_n = \{I_2\}$, joten ϱ_n on injektio prop. 9.14 nojalla.

Huom. Jos $n \geq 4$, niin $\# D_n = 2n < n! = \# S_n$

Seuraus: $D_3 \cong S_3$.

Tod. $\# D_3 = 2 \cdot 3 = 3! = \# S_3$

$D_3 \cong \varrho_3(D_3) \leq S_3$. Koska $\# \varrho_3(D_3) = \# D_3$ (Prop. 13.6), niin $\varrho_3(D_3) = S_3$. $\Rightarrow \varrho_3$ surj. $\Rightarrow \varrho_3$ on isom.



$\Gamma_K = \{ A \in O(3) : AK = K \}$.

$\Gamma_K^+ = \{ A \in SO(3) : AK = K \}$.

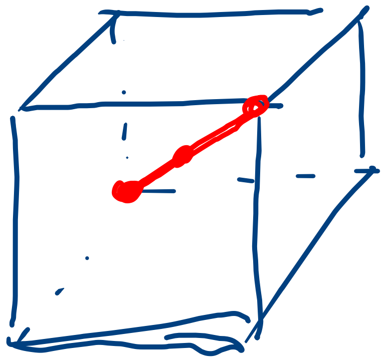
Kuution kärjet $V_K = \{ (\varepsilon_1, \varepsilon_2, \varepsilon_3) ; \varepsilon_i \in \{\pm 1\} \}$.

Symmetrioita : $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$ jne.
$\in \Gamma_K^+$

I_2, kiertoja koord. akselien ympäri (9 kpl)

$A \in \Gamma_K$ määrää kärkien permutation $\rightsquigarrow$ S_8:n alkion

$\boxed{\Gamma_K \cong S_4 \times (\mathbb{Z}/2\mathbb{Z})}$.



Prop. $\Gamma_K^+ \cong S_4$.

Tod. $H = \{ \{v, -v\} : v \in V_K \}$, $\# H = 4$.

Määr. $\rho_0 : \Gamma_K \to \mathrm{Perm}(H) \cong S_4$.

$\rho_0(g)(\{v, -v\}) = \{gv, -gv\}$ homomorfismi.

$\rho_0(g) = \mathrm{id} \iff gv = \pm v \quad \forall v \in V_K$

$\Rightarrow \ker \rho_0 = \{\pm I_3\}$.

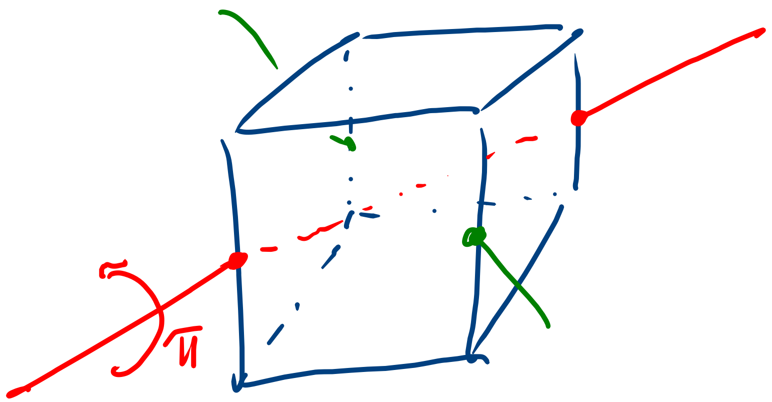
$\ker \rho_0 |_{\Gamma_K^+} = \{I_3\}$. $\rightsquigarrow$ $\underline{\text{homomorfismi}}$ (inj.) $\rho : \underline{\Gamma_K^+} \longrightarrow S_4 = \mathrm{Perm}(H)$.

$\underline{\# S_4 = 4! = 24}$.

$\Rightarrow \Gamma_K^+ \cong \rho(\Gamma_K^+) \leq S_4$

Lagrangen lause: $\# \rho(\Gamma_K^+) \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.

Edellä löydettiin 10 Γ_K^+:n alkiota.



6 lisää.

$\Rightarrow \# \Gamma_K^+ = \# \rho(\Gamma_K^+) \geq 16$

Lagrange $\Rightarrow \# \Gamma_K^+ = \# \rho(\Gamma_K^+) = 24$

$\Rightarrow \Gamma_K^+ \cong \rho(\Gamma_K^+) = \underline{S_4}$. $\square$

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Prop. 13.14. $\Gamma_K \cong S_4 \times (\mathbb{Z}/2\mathbb{Z})$.

Γ_K:n keskus.

Tod. $\rho : \Gamma_K \to S_4$, $\ker \rho = \{\pm I_3\} \leq Z(\Gamma_K) = \{B \in \Gamma_K : BA = AB \ \forall A \in \Gamma_K\}$.

$-I_3 \in \Gamma_K - \Gamma_K^+$, koska $\det(-I_3) = \det\begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} = -1$.

$[\Gamma_K : \Gamma_K^+] = 2$. $\quad \underline{\Gamma_K^+ \lhd \Gamma_K}$, koska $\Gamma_K^+ = \ker \det$

$\det : \Gamma_K \to \{\pm 1\}$

$\Rightarrow \Gamma_K^+ \langle -I_3 \rangle \leq \Gamma_K$ Prop 12.22 nojalla.

$\Gamma_K^+ <$

$\Rightarrow \Gamma_K^+ \langle -I_3 \rangle = \Gamma_K$. $\quad \Gamma_K^+ \cap \langle -I_3 \rangle = \{I_3\}$

$\{-I_3, I_3\}$

Siis Γ_K on aliryhmien Γ_K^+ ja $\langle -I_3 \rangle$ sis. suora tulo.

Prop. 9.30 $\Rightarrow \Gamma_K \cong \Gamma_K^+ \times \underbrace{\langle -I_3 \rangle}_{\cong \mathbb{Z}/2\mathbb{Z}} \overset{\text{P. 8.19}}{\cong} \Gamma_K^+ \times (\mathbb{Z}/2\mathbb{Z})$. $\square$